LECTURE 1

1. Catalog of Problems

1.1. **The classical obstacle problem.**

1.1.1. *The Dirichlet principle.* The well-known variational principle of Dirichlet says that the solution of the boundary value problem

$$\Delta u = 0 \quad \text{in } D, \qquad u = g \quad \text{on } \partial D$$

can be found as the minimizer of the (Dirichlet) functional

$$J_0(u) = \int_D |\nabla u|^2 dx$$

among all u such that $u = g$ on ∂D. More precisely (and slightly more generally), if D is a bounded open set in $\mathbb{R}^n$, $g \in W^{1,2}(D)$ and $f \in L^\infty(D)$, then the minimizer of

$$J(u) = \int_D (|\nabla u|^2 + 2fu)dx \tag{1}$$

on the set

$$K_g = \{u \in W^{1,2}(D) : u - g \in W_0^{1,2}(D)\},$$

solves the equation

$$-\Delta u + f = 0 \quad \text{in } D, \qquad u = g \quad \text{on } \partial D$$

in the sense of distributions, i.e.

$$\int_D (\nabla u \nabla \eta + f\eta)dx = 0$$

for all test functions $\eta \in C_0^\infty(D)$ (and more generally for all $\eta \in W_0^{1,2}(D)$). One can think of the graph of u as the membrane attached to the thin wire (the graph of g over ∂D).

1.1.2. *The obstacle problem.* Now, suppose that we are given additionally a function $\psi \in C^2(D)$, which we will call the *obstacle*, such that $g \geq \psi$ on ∂D in the sense that $(g - \psi)_- \in W_0^{1,2}(D)$. Consider then the minimizers of (1) as before, but on the constrained set

$$K_{g,\psi} = \{u \in W^{1,2}(D) : u - g \in W_0^{1,2}(D), u \geq \psi \text{ a.e. in } D\}.$$

Since J is continuous and strictly convex on a convex subset $K_{g,\psi}$ of the Hilbert space $W^{1,2}(D)$, it has a unique minimizer on $K_{g,\psi}$.

If as before we think of the graph of u as the membrane attached to a fixed wire, it must stay above the graph of ψ. The new feature in this problem is that the membrane actually can touch the obstacle, i.e. the set

$$\Lambda = \{u = \psi\},$$

known as the *coincidence set*, may be nonempty. We also denote

$$\Omega = D \setminus \Lambda.$$

The boundary

$$\Gamma = \partial\Lambda \cap D = \partial\Omega \cap D$$

is called the *free boundary*, as it is not known apriori. The study of the free boundary in this and related problems is the main objective in this course.

To obtain the conditions satisfied by the minimizer, we note that using the so-called method of penalization (or regularization, we discuss this later) one can show that the minimizer is not only in $W^{1,2}(D)$, but actually is in $W^{2,p}_{\mathrm{loc}}(D)$ for any $p < \infty$ and consequently (by the Sobolev embedding theorem) are in $C^{1,\alpha}_{\mathrm{loc}}(D)$ for any $0 < \alpha < 1$.

Then, it is straightforward to show that

$$-\Delta u + f = 0 \quad \text{in } \Omega = \{u > \psi\}, \qquad \Delta u = \Delta\psi \quad \text{a.e. on } \Lambda = \{u = \psi\}.$$

Besides,

$$-\Delta u + f \geq 0 \quad \text{in } D$$

in the sense of distributions, i.e.

$$\int_D (\nabla u \nabla \eta + f\eta) dx \geq 0$$

for any nonnegative $\eta \in W^{1,2}_0(D)$, which follows by passing to the limit $\epsilon \to 0+$ in the inequality

$$\frac{J(u + \epsilon\eta) - J(u)}{\epsilon} \geq 0.$$

Combining the properties above, we obtain that the solution of the obstacle problem is a function $u \in W^{2,p}(D)$ for any $p < \infty$, which satisfies

$$-\Delta u + f \geq 0, \quad u \geq \psi, \quad (-\Delta u + f)(u - \psi) = 0 \quad \text{a.e. in } D \tag{2}$$

$$u - g \in W^{1,2}_0(D) \tag{3}$$

These are known as the *complementary conditions* and uniquely characterize the minimizers of J over $K_{g,\psi}$.

Reduction to the case of zero obstacle. Since the governing operator (Δ) is linear it is possible to reduce the problem to the case when the obstacle is 0. Indeed, if u is the solution of the obstacle problem as above, consider the difference $v = u - \psi$. It is straightforward to see that v is the minimizer of the functional

$$J_1(v) = \int_D (|\nabla v|^2 + 2 f_1 v) dx$$

on the set $K_{g_1,0}$, where

$$f_1 = f - \Delta\psi, \quad g_1 = g - \psi.$$

Moreover, v will satisfy

$$\Delta v = f_1 \chi_{\{v>0\}} \quad \text{in } D$$

in the sense of distributions.

1.2. **Problem from Potential Theory.** Let Ω be a bounded open set in $\mathbb{R}^n$ and f a certain bounded measurable function on Ω. Consider then the Newtonian potential of the distribution of mass $f\chi_\Omega$, i.e.

$$U(x) = \Phi_n * (f\chi_\Omega)(x) = \int_\Omega \Phi_n(x-y)f(y)dy$$

where Φ_n is the fundamental solution of the Laplacian in $\mathbb{R}^n$, i.e. $\Delta\Phi_n = \delta$ in the sense of distributions. It can be shown that the potential U is in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$ for any and satisfies

$$\Delta U = f\chi_\Omega \quad \text{in } \mathbb{R}^n$$

in the sense of distributions (or a.e., which is the same in this case). In particular, U is harmonic in $\mathbb{R}^n \setminus \overline{\Omega}$.

Let $x_0 \in \partial\Omega$ and suppose for some small $r > 0$ there is a harmonic function h in the ball $B_r(x_0)$ such that $h = U$ on $B_r \setminus \Omega$. We say in this case that h is a harmonic continuation of U into Ω at x_0. If such continuation exists, the difference $u = U - h$ satisfies

$$\Delta u = f\chi_\Omega \quad \text{in } B_r(x_0), \qquad u = |\nabla u| = 0 \quad \text{on } B_r(x_0) \setminus \Omega. \tag{4}$$

Using the Cauchy-Kowalevskaya theorem, it is straightforward to show that the harmonic continuation exists if $\partial\Omega$ and f are real-analytic in a neighborhood of x_0. The converse to this property would be, given that the solution to (4) exist for some $r > 0$, what can be said about the regularity of $\partial\Omega$.

1.3. **Pompeiu Problem.** A nonempty bounded open set $\Omega \subset \mathbb{R}^n$ is said to have the *Pompeiu property* if the only continuous function such that

$$\int_{\sigma(\Omega)} f(x)dx = 0$$

for all rigid motions $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is the identically zero function. A ball of any radius fails this property: take $f(x) = \sin(ax_1)$ for $a > 0$ satisfying $J_{n/2}(aR) = 0$, where J_ν is the Bessel function of order ν. Furthermore, any finite disjoint union of balls of the same radius again fail the Pompeiu property, with the same function f.

A long standing conjectured in integral geometry says that if Ω fails the Pompeiu property and has a sufficiently regular (Lipschitz) boundary $\partial\Omega$ homeomorphic to the unit sphere, then Ω must be a ball. It is known (Williams) that for such Ω there exists a solution to the problem

$$\Delta u + \lambda u = \chi_\Omega \quad \text{in } \mathbb{R}^n, \qquad u = |\nabla u| = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega$$

for some $\lambda > 0$. An open conjecture of Schiffer says that any Ω admitting solutions of the overdetermined problem above must be a ball.

1.4. **A problem from superconductivity.** In analyzing the evolution of vortices arising in the mean-field model of penetration of the magnetic field into superconducting bodies, one ends up with a degenerate parabolic-elliptic system. A simplified stationary model of this problem (in a local setting), where the scalar stream function admits a functional dependence on the scalar magnetic potential, reduces to finding u such that

$$\Delta u = u\chi_{\{|\nabla u|>0\}}, \quad u \geq 0, \quad \text{in } B_r(x_0), \tag{5}$$

the equation is in the sense of distribution, and appropriate boundary data are fulfilled.

One can actually consider a more general problem of the form

$$\Delta u = f(x,u)\chi_{\{|\nabla u|>0\}}, \qquad \text{in } B_r(x_0), \tag{6}$$

with $f > 0$, and $f(\cdot, t) \in C^\alpha$.

Obviously this problem is slightly more general than the above problem in potential theory, as the solution function can have different constant values on the set $|\nabla u| = 0$.

To give an example of a solution to such a problem, one considers a dumbbell shape region $D \subset R^2$,

$$D := B_1(x^1) \cup B_1(x^2) \cup \{x : |x_2| < \epsilon, \quad |x_1| < 2\}$$

with $x^1 = (2,0)$, $x^2 = (-2,0)$, and ϵ very small positive number. The solution to

$$\Delta v = f \quad \text{in } D, \qquad v = 0 \quad \text{on } \partial D.$$

forms a shape of hanging graphs over D, and of course symmetric in x_1-axis.

Now solve the obstacle problem $\Delta u = f\chi_{\{u>\psi\}})$ in D with zero boundary values and the obstacle ψ which is smooth and equals $\min v + \delta_i$ on each ball $B_{1/2}(x^i)$. Here $\delta_1 > \delta_2 > 0$ are small constants. This gives us an example of a solution to (5).

1.5. **Two-phase obstacle problem.** Given a bounded open set D in $\mathbb{R}^n$, $g \in W^{1,2}(D)$ and bounded measurable functions λ_+ and λ_- in D consider the problem of minimization of the functional

$$J(u) = \int_D (|\nabla u|^2 + 2\lambda_+ u^+ + 2\lambda_- u^-)dx \tag{7}$$

over the set

$$K_g = \{u \in W^{1,2}(D) : u - g \in W_0^{1,2}(D)\}.$$

Here

$$u^+ = \max\{u, 0\}, \qquad u^- = \max\{-u, 0\}$$

The case $\lambda_- = 0$ and $g \geq 0$ the problem is equivalent to the obstacle problem with zero obstacle, see above.

Possible applications of this functional may come in several problems when the external force is a function of u itself, in this case the external force is

$$\lambda_+ H(u) - \lambda_- H(-u).$$

As a specific example, imagine a membrane in $\mathbb{R}^{n+1}$ under the influence of an electric or a magnetic field of the form

$$F = -\lambda_+ \chi_{\{x_{n+1}>0\}} e_{n+1} + \lambda_- \chi_{\{x_{n+1}<0\}} e_{n+1},$$

where e_{n+1} is $(n+1)$-th vector in the standard basis in $\mathbb{R}^{n+1}$. If we assume the membrane to be modeled by a graph in the x_{n+1}-direction and to be clamped in at the boundary, then the equilibrium state would correspond to the minimizer of our functional.

Another physical interpretation of this problem is the consideration of a thin membrane (film) which is fixed on the boundary of a given domain, and some part of the boundary data of this film is below the surface of a thick liquid (heavier than the film itself). Now the weight of the film produces a force downwards (call it λ_+) on that part of the film which is above the liquid surface. On the other side the part in the liquid is pushed upwards by a force λ_-, since the liquid is heavier than the film. Obviously the equilibrium state of the film is given by a minimization of the above mentioned functional.

One of the difficulties one confronts in this problem is that the interface $\{u = 0\}$ consists in general of two parts – one where the gradient of u is nonzero and one where the gradient of u vanishes. Close to points of the latter part we expect the gradient of u to have linear growth. However, because of the decomposition into two different types of growth, it is not possible to derive a growth estimate by classical techniques.

1.6. **A problem in Optimal Control theory.** Yet another application of the two membrane problem, coming from optimal control theory, can be given as follows. Let D we an open set with regular boundary (Lipschitz) in $\mathbb{R}^n$ and consider the following problem:

$$\Delta u = f \quad \text{in } D \qquad \frac{\partial u}{\partial \nu} = h \quad \text{on } \partial D.$$

Here ν is outward normal to ∂D, $h \in L^\infty(\partial\Omega)$ is a given function, $f \in L^\infty(D)$ is a control function such that

$$f \in U_{\mathrm{ad}} := \left\{ \sup_D |f| \le 1, \int_D f = \int_{\partial D} h \right\}.$$

The solution is understood in the weak sense, i.e.

$$\int_D (\nabla u \nabla \eta + f\eta) dx = \int_{\partial D} h\eta$$

for any $\eta \in W^{1,2}(D)$.

It is required to minimize the functional

$$I(u) := \int_D |\nabla u|^2 + |u| - \int_{\partial D} hu$$

for all solutions with $f \in U_{\mathrm{ad}}$.

It is easy to calculate that

$$I(u) = \int_D |u|(1 - f \operatorname{sign} u) \ge 0,$$

and $I(u) = 0$ iff $f = f(u) = \operatorname{sign} u$.

1.7. **Composite membrane.** Build a body of prescribed shape out of given materials (of varying densities) in such a way that the body has a prescribed mass and so that the basic frequency of the resulting membrane (with fixed boundary) is as small as possible. Let us consider a more general problem: Given a domain $D \subset \mathbb{R}^n$ (bounded, connected, with Lipschitz boundary) and numbers $\alpha > 0$, $A \in [0, |D|]$ (with $|.|$ denoting volume). For any measurable subset $\Omega \subset D$ let $\lambda_D(\alpha, \Omega)$ denote the lowest eigenvalue λ of the problem

$$-\Delta v + \alpha \chi_\Omega v = \lambda v \quad \text{in } D, \qquad v = 0 \quad \text{on } \partial D \tag{8}$$

(Chanillo, Grieser, Imai, Kurata, Ohnishi). Define

$$\Lambda_D(\alpha, A) = \inf_{\Omega \subset D,\ |\Omega| = A} \lambda_D(\alpha, \Omega)$$

Any minimizer D in the latter equation will be called an optimal configuration for the data. If Ω is an optimal configuration and v satisfies (8) then (v, Ω) will be called an optimal pair (or solution). It is known that $\Omega = \{v \le t\}$ for some t such that $|A| = \{v \leq t\}$. Now upon rewriting $u = v - t$ we can rephrase the above equation as

$$\Delta u = (\alpha \chi_{\{u \leq 0\}} - \lambda)(u + t),$$

and with yet another rewriting

$$\Delta u = \big((\alpha - \lambda)\chi_{\{u \leq 0\}} - \lambda \chi_{\{u > 0\}}\big)\,(u + t).$$

The particular case $\alpha < \lambda$ is of special interest, as the problem does not fall under general theory.

This problem, by rewriting, can still be seen as a special case of minimizing the functional

$$J(u) = \int_D \big(|\nabla u|^2 + \lambda_+ u^+ + \lambda_- v^- \ \big)\, dx,$$

with $\lambda_\pm$ smooth functions, and possibly varying signs.