LECTURE 11

11. Classification of Free Boundary points

11.1. Homogeneous Global Solutions. By Theorem 10.3, blowups of solutions of Problems A and B at fixed \( x_0 \in \Gamma \) and of Problem C at \( x_0 \in \Gamma' \) are homogeneous (of degree two) global solutions, i.e. a global solution \( u \) satisfying
\[
    u(\lambda x) = \lambda^2 u(x), \quad x \in \mathbb{R}^n, \; \lambda > 0.
\]

Another way to express the homogeneity is by the identity
\[
    \partial^{(2)} u(x) := x \cdot \nabla u(x) - 2u(x) = 0 \quad \text{in} \; \mathbb{R}^n.
\]

In this section we give a complete description of such solutions.

Theorem 11.1 (Classification of homogeneous global solution). Let \( u \) be a homogeneous global solution of Problem A, B, or C. Then \( u \) is of one of the following forms.

- In Problems A, B:
  - Polynomial solution \( u(x) = \frac{1}{2}(x \cdot Ax) \), \( x \in \mathbb{R}^n \). where \( A \) is an \( n \times n \) symmetric matrix with \( \text{Tr} A = 1 \).
  - Halfplane solutions \( u(x) = \frac{1}{2}(x \cdot e)^2 \), \( x \in \mathbb{R}^n \), where \( e \) is a unit vector.

- In Problem C:
  - Polynomial solutions (positive or negative) \( u(x) = \frac{\lambda}{2}(x \cdot Ax) \) or \( u(x) = -\frac{\lambda}{2}(x \cdot Ax) \), \( x \in \mathbb{R}^n \), where \( A \) is an \( n \times n \) nonnegative symmetric matrix with \( \text{Tr} A = 1 \).
  - Halfplane solutions (positive or negative) \( u(x) = \frac{\lambda}{2}(x \cdot e)^2 \) or \( u(x) = -\frac{\lambda}{2}(x \cdot e)^2 \), \( x \in \mathbb{R}^n \), for a unit vector \( e \).
  - Two-plane solution \( u(x) = \frac{\lambda}{2}(x \cdot e)^2 - \frac{\lambda}{2}(x \cdot e)^2 \), \( x \in \mathbb{R}^n \), for a unit vector \( e \).

Proof.

Problems A, B. Observe that \( u \in P_\infty(M) \) and
\[
    u(0) = |\nabla u(0)| = 0.
\]

Consider two possibilities. First suppose that \( \text{Int} \; \Omega_u = \emptyset \). Then, since \( \partial \Omega(u) \) has zero Lebesgue measure (see Corollary 8.10) the function \( u \) satisfies the equation \( \Delta u = \text{const} \; \text{a.e. in} \; \mathbb{R}^n \), and \( \| D^2 u_0 \|_{L_\infty(\mathbb{R}^n)} \leq M \). By Liouville’s theorem \( u \) is a degree two polynomial and homogeneity comes from (11.2).

Next, suppose \( \text{Int} \; \Omega_u \neq \emptyset \). Then we apply the ACF Monotonicity Formula (see Lecture 5) to the positive and the negative parts of \( \partial_e u \), for different directions \( e \). Recall that we have shown earlier that \( (\partial_e u)^\pm \) are subharmonic, so the ACF Monotonicity Formula is applicable (see Lemma 6.2 applied with \( M_1 = M_2 = 0 \)).

The homogeneity of \( u \) readily implies that for any direction \( e \)
\[
    \phi_e(r) := \Phi(r, (\partial_e u)^+, (\partial_e u)^-) = \frac{1}{r^2} \int_{B_r} |\nabla(\partial_e u)^+|^2 \frac{1}{|x|^{n-2}} \int_{B_r} |\nabla(\partial_e u)^-|^2 \frac{1}{|x|^{n-2}},
\]
By continuity for all directions. As before, this implies that \( C \) are either halfplane or two-plane solutions for which \( \Gamma \). But it is easy to show that all one-dimensional homogeneous solutions of Problem \( \nu \) hold for directions \( e \).  

\[ \frac{\lambda_+ + \lambda_-}{2} \int_{\{u = 0\} \cap B_r(y_0)} |e \cdot \nu| \, dH^{n-1} > 0, \]

where \( \nu = \nabla u/|\nabla u| \) is the normal to the surface \( u = 0 \). Thus the case (ii) cannot hold for directions \( e \) nonorthogonal to \( \nu_0 \). Thus, (i) holds for all such directions and by continuity for all directions. As before, this implies that \( u \) is one-dimensional. But it is easy to show that all one-dimensional homogeneous solutions of Problem \( C \) are either halfplane or two-plane solutions for which \( \Gamma'' = 0 \). This contradicts to assumption that \( \Gamma''(u) \) is non-empty.

Once we have that \( \Gamma''(u) = \emptyset \), renormalized positive and negative parts of \( u, \frac{1}{\lambda_\pm} u^\pm \), will solve Problem \( A \). Thus \( u \) has one of the forms described in the statement of the theorem. \( \square \)

### 11.2. Classification of Free Boundary Points

Since the blowups with fixed centers are homogeneous global solution by Theorem 11.1, this leads to a classification of free boundary points according to their blowup. But first we need to show any two blowups at a given point are of the same type.

We start by finding Weiss’s energy of homogeneous global solutions described in Theorem 11.1.

**Problem A.** If \( u \) is a homogeneous global solution and we integrate by parts in the expression for \( W \), using that \( \Delta u = 1 \) in \( \Omega \), we arrive at

\[
W(r, u, 0) = \int_{B_1} (|\nabla u|^2 + 2u) \, dx - 2 \int_{\partial B_1} u^2 \, dH^{n-1} = \int_{B_1} (-\Delta u + 2u) \, dx - \int_{\partial B_1} \partial^2 u u \, dH^{n-1} = \int_{B_1} u \, dx.
\]
Hence, for polynomial solutions $u(x) = \frac{1}{2} (x \cdot Ax)$, we have

$$W(r, u, 0) = \frac{1}{2} \int_{B_1} x \cdot Ax \, dx = \alpha_n$$

and for halfplane solutions $u(x) = \frac{1}{2} (x \cdot e)^2_+$

$$W(r, u, 0) = \frac{1}{2} \int_{B_1} (x \cdot e)^2_+ \, dx = \frac{\alpha_n}{2},$$

where

$$\alpha_n = \frac{1}{2} \int_{B_1} x^2_+ \, dx.$$

So, the only values taken by $W$ on homogeneous solutions are

$$\alpha_n, \quad \frac{\alpha_n}{2}.$$

**Problem B.** As it follows from Theorem 11.1, the homogeneous solutions of Problems A and B are identical and so are their Weiss functionals.

**Problem C.** Arguing similarly, we obtain that the homogeneous solutions in this case have energies $W$

$$\lambda \pm \alpha_n, \quad \frac{(\lambda_+ + \lambda_-)\alpha_n}{2}, \quad \frac{\lambda \pm \alpha_n}{2}.$$

The computation above and Weiss’s Monotonicity Formula lead us to the following definition.

**Definition 11.2** (Balanced Energy). Let $u \in P^R(x_0, M)$ be a solution of Problem A, B, or C and assume additionally that $x_0 \in \Gamma(u)$ in the case of Problem C. Then the limit

$$\omega(x_0) := \lim_{r \to 0} W(r, u, x_0),$$

which exists by Theorem 10.2, is called the balanced energy of $u$ at $x_0$.

If $u_0 = \lim_{j \to \infty} u_{x_0, \lambda_j}$ for $\lambda_j \to 0$ is a blowup of $u$ at a fixed center $x_0$ as in Theorem 10.3 then

$$\omega(x_0) = \lim_{j \to \infty} W(\lambda_j, u, x_0) = \lim_{j \to 0} W(1, u_{x_0, \lambda_j}, 0) = W(1, u_0, 0).$$

Thus, the balanced energy at a point coincides with the Weiss energy of any of blowups with fixed center $x_0$. This has two consequences: first that the balanced energy can take only a limited number of values and second that all blowups at $x_0$ are of the same type.

**Proposition 11.3.** Problems A, B. The balanced energy is an upper semicontinuous function of $x_0 \in \Gamma(u)$ and

$$\omega(x_0) \in \left\{ \alpha_n, \frac{\alpha_n}{2} \right\}.$$

Problem C. The balanced energy is an upper semicontinuous function of $x_0 \in \Gamma(u)$ and

$$\omega(x_0) \in \left\{ \lambda \pm \alpha_n, \frac{(\lambda_+ + \lambda_-)\alpha_n}{2}, \frac{\lambda \pm \alpha_n}{2} \right\}.$$
Proof. The upper semicontinuity follows from the fact that

\[ W(r, u, \cdot) =: \omega_r(\cdot) \downarrow \omega(\cdot), \quad \text{as } r \downarrow 0, \]

and the functions \( \omega_r(\cdot) \) are continuous for any \( r > 0 \). The values taken by the balanced energy are obtained from the analysis immediately before and after Definition 11.2.

\[ \square \]

**Proposition 11.4** (Unique type of the blowup). *Let \( x_0 \in \Gamma(u) \) in Problems A, B, or \( x_0 \in \Gamma'(u) \) in Problem C. Then any two blowups of solution \( u \) with fixed center \( x_0 \) have the same type.*

We leave the proof to the reader as an exercise.

**Definition 11.5** (Classification of Free Boundary Points).

- In Problems A, B for \( x_0 \in \Gamma(u) \) we will use the following terminology:
  - \( x_0 \) is a high-energy point, if \( \omega(x_0) = \alpha_n \)
  - \( x_0 \) is a low-energy point, if \( \omega(x_0) = \frac{\alpha_n}{2} \)

  Equivalently, \( x_0 \) is high-energy if blowups with fixed center \( x_0 \) are polynomial and low-energy if blowups are halfplane solutions.

- In Problem C, for \( x_0 \in \Gamma'(u) \), we say
  - \( x_0 \) is a two-phase point, if \( x_0 \in \partial\{u > 0\} \cap \partial\{u < 0\} \)
  - \( x_0 \) is an one-phase point, otherwise

One can show that the solution \( u \) of Problem C does not change sign in a neighborhood of an one-phase point. Similarly to Problems A, B we distinguish high-energy and low-energy one-phase points, depending on their balanced energy.