LECTURE 15

15. Global solutions

In this lecture we study the so-called global solutions, i.e. solutions defined in the whole space, with an additional assumption that they grow quadratically at infinity. More precisely, we consider elements of the class $P_\infty(x_0, M)$ which satisfy

- $\|D^2 u\|_{L^\infty(\mathbb{R}^n)} \le M$,
- $x_0 \in \Gamma(u)$.

The global solutions may exist by their own, but most importantly they may appear as blowups of one or a sequence of functions with variable centers, i.e. limits of rescalings

$$u^j_{x_j,r_j}(x) = \frac{u^j(x_j + r_j x) - u^j(x_j)}{r_j^2}.$$

We will first study the global solution for the classical obstacle problem, then generalize the results for Problems A, B and at the end of this lecture we will study the case of Problem C.

15.1. Classical Obstacle Problem.

Theorem 15.1. *Let $u \in P_\infty(M)$ be a global solution of Problem A and assume that $u \ge 0$ in $\mathbb{R}^n$. Then u is a convex function in $\mathbb{R}^n$, i.e.*

$$\partial_{ee} u(x) \ge 0, \quad \textit{for any direction } e \textit{ and } x \in \mathbb{R}^n$$

In particular, the set $\{u = 0\}$ is convex.

Proof. Fix any direction e. Without loss of generality suppose that $e = e_n = (0, \cdots, 0, 1)$. Assume, on the contrary, that

$$-m := \inf_\Omega \partial_{nn} u < 0,$$

and let $x_j \in \Omega$ be a minimizing sequence for the value $-m$, i.e.

$$\lim_{j\to\infty} \partial_{nn} u(x_j) = -m.$$

Let $d_j = \operatorname{dist}(x_j, \Gamma)$ and consider the rescalings

$$u_j(x) = u_{x_j,d_j}(x) = \frac{1}{d_j^2} u(x_j + d_j x).$$

Observe that $B_1 \subset \Omega(u_j)$ and the free boundary $\Gamma(u_j)$ contains at leas one point on ∂B_1. Since also $\|D^2 u_j\|$ are uniformly bounded we have the uniform estimates

$$|u_j(x)| \le \frac{M}{2}(R+1)^2$$

for all $R > 0$ and therefore we can extract a subsequence converging in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n)$ to a global solution u_0 of Problem A. The assumption $u \ge 0$, implies that $u_0 \ge 0$ and therefore, $\Omega(u_0) = \{u_0 > 0\}$. Moreover, similarly to u_j, observe that since $B_1 \subset \Omega(u_0)$, and ∂B_1 contains at least one free boundary point.

Next observe, since all functions u_j satisfy $\Delta u_j = 1$ in B_1, the convergence to u_0 can be assumed to be at least in $C^2_{\mathrm{loc}}(B_1)$. Hence, the limit function u_0 satisfies

$$\Delta u_0 = 1, \qquad \partial_{nn} u_0 \geq -m \quad \text{in } B_1, \qquad \partial_{nn} u_0(0) = -m.$$

Since $\partial_{nn} u_0$ is harmonic in B_1, the minimum principle implies that $\partial_{nn} u_0 \equiv -m$ in B_1. In fact we have even more, $\partial_{nn} u_0 = -m$ in the connected component of $\Omega(u_0)$ which contains B_1. Hence we obtain the representations

$$\partial_n u_0(x) = g_1(x') - m x_n, \quad x' = (x_1, \dots, x_{n-1}) \tag{15.1}$$

and

$$u_0(x) = g_2(x') + g_1(x') x_n - \frac{m}{2} x_n^2, \tag{15.2}$$

in B_1. Now let us choose a point $(x', 0) \in B_1$ and start moving in the direction e_n. Observe that as long as we stay in $\Omega(u_0)$, we still have $\partial_{nn} u = -m$ and therefore still have the representations (15.1)–(15.2). However, sooner or later we will reach $\partial\Omega(u_0)$, otherwise if x_n becomes very large (15.2) will imply $u_0 < 0$, contrary to or assumption. Since $u_0 = |\nabla u_0| = 0$ on $\partial\Omega(u_0)$, from (15.1) we obtain that the first value $\xi(x')$ of x_n for which we arrive at $\partial\Omega(u_0)$ is given by

$$\xi(x') = \frac{g_1(x')}{m}.$$

Hence from (15.2) we deduce that

$$g_2(x') = -\frac{g_1(x')^2}{2m}.$$

Now, the representation (15.2) takes the form

$$u_0(x) = -\frac{m}{2}(x_n - \xi(x'))^2,$$

which is not possible since $u_0 \geq 0$. This concludes the proof. □

15.2. **Problems A, B.** Next, our goal is to generalize Theorem 15.1 for global solutions of Problems A, B. We will consider two case: when the complement of Ω is bounded and when it is unbounded.

15.2.1. *The compact complement case.* Assume now we have $u \in P_\infty(x_0, M)$ for which Ω^c is compact.

Lemma 15.2. *Let $u \in P_\infty(x_0, M)$ be a global solution of Problem A, B, such that Ω^c is compact and $\operatorname{Int}\Omega^c \neq \emptyset$. Then x_0 is a low energy point.*

Proof. Suppose, towards a contradiction, that x_0 is a high energy point. Consider then a so-called "shrink-down" of u with a fixed center at x_0, i.e. sequence of rescalings

$$u_k(x) = u_{x_0, R_k}(x) = \frac{u(x_0 + R_k x) - u(x_0)}{R_k^2}$$

for $R_k \to \infty$ which converges to a global solution u_∞. Similarly to blowups with fixed centers (Theorem 10.3), it is not hard show that u_∞ is a homogeneous global solution, as a simple corollary of Weiss's monotonicity formula (see Lecture 10). The same monotonicity formula implies

$$\begin{aligned} \alpha_n = \omega(x_0) \leq W(r, u, x_0) &\leq \lim_{R_k \to \infty} W(R_k, u, x_0) \\ &= \lim_{R_k \to \infty} W(1, u_k) = W(1, u_\infty). \end{aligned} \tag{15.3}$$

On the other hand, we know that for homogeneous global solutions W can take only two values: $\alpha_n/2$ and α_n, hence $W(1, u_\infty) = \alpha_n$. global solution and $W(1, u_\infty) \leq \alpha_n$. This, combined with Hence, from (15.3) implies that $W(r, u, x_0) = \alpha_n$ for any $r > 0$. Thus, by Theorem 10.2 u must be homogeneous with respect to the point x_0 and the classification of homogeneous solutions implies that u must be a polynomial solution. This contradicts the assumption $\operatorname{Int}\Omega^c \neq \emptyset$. □

Lemma 15.3. *Let u be as in Lemma 15.2 . Then Ω^c will consist of finite union of connected components Ω^c_i, $i = 1, \ldots, N$ with C^1 boundaries and nonempty interiors such that u is constant in Ω^c_i.*

Proof. Note that every point on $\partial\Omega$ is of low energy. Applying now Theorems 12.4 and 13.1 we obtain the desired structure for Ω^c. □

Lemma 15.4. *Let u be as in Lemma 15.2 and suppose that*

$$\sup_{\Omega^c} u = 0 \tag{15.4}$$

Then, for a suitable choice of the origin in Ω^c, the function

$$r \mapsto \frac{u(rx)}{r^2}$$

is nondecreasing, for any fixed x.

Proof. We will give the proof for $n \geq 3$. Denote by V the Newtonian potential of Ω^c, i.e.

$$V(x) = \int_{\Omega^c} \frac{c_n}{|x - y|^{n-2}} dy.$$

Then V is bounded and superharmonic in $\mathbb{R}^n$ and harmonic in Ω. By the maximum principle, there is at least one point $\zeta_0 \in \Omega^c$ such that

$$V(\zeta_0) \geq V(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Set the origin at ζ_0.

Since

$$\Delta(u - V) = 1$$

in the sense of distributions and all second order partial derivatives of $u - V$ are bounded harmonic functions, the Hessian of $u - V$ is a constant matrix, by Liouville's theorem. Hence $u - V$ is a polynomial of degree two. Set

$$P(x) = u(x) - V(x) - u(0) + V(0).$$

Note that $|\nabla V(0)| = |\nabla u(0)| = 0$. Hence $P(0) = |\nabla P(0)| = 0$, this implies that P is homogeneous. Now consider the function

$$h(x) = x \cdot \nabla u(x) - 2u(x)\,.$$

h is continuous in $\mathbb{R}^n$ and for all $x \neq 0$ fixed,

$$\frac{d}{dr}\left(\frac{u(rx)}{r^2}\right) = \frac{1}{r^3} h(rx).$$

We will show that h is non-negative in $\mathbb{R}^n$. In fact

$$h(x) = -2u(x) \geq 0\,, \quad \forall x \in \Omega^c\,.$$

On the other hand, by the homogeneity of P,

$$h(x) = x \cdot \nabla V(x) - 2V(x) + 2V(0) - 2u(0)$$

then

$$\lim_{|x|\to\infty} h(x) = 2V(0) - 2u(0) \geq 0\,.$$

Since h harmonic in Ω, by the minimum principle, h is positive in Ω. □

Now we can prove the main result of this section.

Theorem 15.5. *Let $u \in P_\infty(M)$ be such that Ω^c is compact and $\operatorname{Int}\Omega^c \neq \emptyset$. Suppose also that (15.4) holds. Then $u \geq 0$ in $\mathbb{R}^n$ and u is a convex function. In particular $\{u = 0\}$ is a convex set.*

Proof. Choose the origin as in Lemma 15.4.

Consider first the case of Problem A. We claim that there exists small $\rho > 0$ such that $u \geq 0$ in B_ρ. Indeed, if $0 \in \operatorname{Int}\Omega^c$, this is immediate. If $0 \in \Gamma$, then it is a low energy point by Lemma 15.2 and therefore the statement follows from Lemma 12.3. Now, invoking Lemma 15.4, we conclude

$$0 \leq u(\rho x) \leq \frac{\rho^2}{R^2} u(Rx), \quad x \in B_1,\ R > \rho,$$

i.e. $u \geq 0$ everywhere in $\mathbb{R}^n$. Then we invoke Theorem 15.1.

In the case of Problem B, we observe that the set $\{u \leq 0\}$ is star-like and therefore connected. Let now use the structure of Ω^c. If Ω^c_i, $i = 1, \dots, N$ are the components as in Lemma 15.3 and $u = c_i$ there, then $c_i \leq 0$ by the assumption (15.4). On the other hand since, u is subharmonic, we must have either $u = 0$ in the interior of $\{u \leq 0\}$ or $u < 0$. The latter is impossible, since it will imply that $c_i < 0$ for all $i = 1, \dots, N$ (recall that Ω^c_i have nonempty interiors), which contradicts (15.4). Therefore we must have

$$\{u \leq 0\} = \{u = 0\}.$$

and we arrive at the situation of Problem A. □

15.2.2. *Global solutions with unbounded Ω^c.*

Theorem 15.6. *Let $u \in P_\infty(M)$ such that Ω^c is unbounded and has nonempty interior. Then, there is $a \in \mathbb{R}$ such that $u \geq a$ and $\Omega^c = \{u = a\}$.*

In particular, by Theorem 15.1 Ω^c is convex.

Proof. Suppose that some shrink-down u_∞ of u at 0 is a half space solution. Then, arguing as in the proof of Lemma 15.2 we will have that $u - u(0)$ is a half space solution. Hence the theorem follows in this case.

Now, if no shrink-down is a half-space solution, we may assume u_∞ is a polynomial. The assumption $\operatorname{Int}\Omega^c \neq \emptyset$ prevents u from being a polynomial.

Since Ω^c is unbounded, there exists a sequence $x_j \in \partial\Omega$ tending to ∞. In this case we may scale by $R_j = |x_j|$ so as to obtain, in the limit, a global solution with a free boundary point e on the unit sphere. By homogeneity then the ray $\{re\colon r > 0\}$, must lie in the free boundary. Since u_∞ is a homogeneous quadratic polynomial, this is possible only if $\partial_e u_\infty \equiv 0$. Consider now the Alt-Caffarelli-Friedman monotonicity functional

$$\phi_e(r, u) := \Phi(r, (\partial_e u)^+, (\partial_e u)^-).$$

Since $(\partial_e u)^\pm$ are subharmonic by Lemma 6.2, from ACF monotonicity formula we have that

$$0 \le \phi_e(r,u) \le \phi_e(\infty,u) = \phi_e(1,u_\infty) = 0.$$

Hence, either $(\partial_e u)^+$ or $(\partial_e u)^-$ must vanish identically and we may assume without loss of generality that $\partial_e u \ge 0$ (otherwise we replace e by $-e$).

Next, without loss of generality assume $e = e_n$ and (after changing the origin) that

$$B_r(0) \subset \Omega^c(u).$$

Then $u \equiv u(0)$ in $B_r(0)$. Moreover, by monotonicity in the direction e_n we have that $u \le u(0)$ in the half-infinite cylinder $B'_r(0) \times (-\infty, 0)$, where $B'_r(0)$ stands for a ball in $\mathbb{R}^{n-1}$. Since u is subharmonic, the maximum principle implies now that

$$u(x', x_n) = u(0) \quad \text{for } x' \in B'_r(0),\ x_n \le 0.$$

Define now a $(n-1)$-dimensional solution

$$\hat{u}(x') = \lim_{x_n \to -\infty} u(x', x_n)$$

First, we notice that the limit exists by the monotonicity in the direction e_n. Next, the limit is finite, since $B'_r(0) \times (-\infty, 0] \subset \Omega^c$ which gives the estimate

$$|u(x) - u(0)| \le \frac{M}{2}|x'|^2.$$

Thus, $\hat{u}$ is a $(n-1)$-dimensional solution with a quadratic growth at infinity. Also note that

$$B'_r(0) \subset \Omega^c(\hat{u}).$$

First, suppose that $\hat{u}$ is either a half space solution, or falls into the hypotheses of Theorem 15.5. Then $\hat{u}$ is convex and non-negative. Since $u(x', x_n) - u(0) \ge \hat{u}(x') \ge 0$ we conclude the proof by applying Theorem 15.1 to $u(x) - u(0)$.

Next, if the lower dimensional solution $\hat{u}$ is neither of the above it must fall into the third category analyzed above. Hence we repeat our argument and translate $\hat{u}$ again in a new direction and reduce the dimension further. Finally, by induction, we need to classify the one dimensional solutions. However, the only one-dimensional solutions are $x_1^2/2$, $(x_1^+)^2/2$, or two separated solutions of the latter, which are all nonnegative. □

15.3. **Problem C.**

Theorem 15.7. *Let $u \in P_\infty(M)$ be a solution of Problem C such that the origin is a branching point, i.e. $0 \in \partial\{u > 0\} \cap \partial\{u < 0\}$ and $|\nabla u(0)| = 0$. Then u is a two-plane solution*

$$u(x) = \frac{\lambda_+}{2}(x \cdot e)_+^2 - \frac{\lambda_-}{2}(x \cdot e)_-^2$$

for a certain direction e.

Proof. The proof follows from the classification of homogeneous global solutions in Theorem 11.1 and the following shrink-down argument.

Consider a limit

$$u_\infty(x) = \lim_{R_j \to \infty} \frac{u(R_j x)}{R_j^2}$$

over a certain sequence $R_j \to \infty$. Then Weiss's monotonicity formula implies that u_∞ is a homogeneous global solution. Since we still have that 0 is a branching free-boundary point for u_∞, Theorem 11.1 implies that u_∞ is a two-plane solution for a certain direction e.

In particular, for any direction ν, $\partial_\nu u_\infty$ does not change sign in $\mathbb{R}^n$ and therefore

$$\phi_\nu(r, u_\infty) = \Phi(r, (\partial_\nu u_\infty)^+, (\partial_\nu u_\infty)^-) = 0.$$

On the other hand, by the ACF monotonicity formula we have

$$0 \le \phi_\nu(r, u) \le \phi_\nu(\infty, u) = \phi_\nu(1, u_\infty) = 0,$$

implying $\partial_\nu u$ does not change sign. Since this holds for all directions ν we conclude u is one dimensional and hence can be computed as earlier. $\square$