LECTURE 15

15. Global solutions

In this lecture we study the so-called global solutions, i.e. solutions defined in the whole space, with an additional assumption that they grow quadratically at infinity. More precisely, we consider elements of the class $P_{\infty}(x_0, M)$ which satisfy
- $\|D^2 u\|_{L^\infty(\mathbb{R}^n)} \leq M$,
- $x_0 \in \Gamma(u)$.

The global solutions may exist by their own, but most importantly they may appear as blowups of one or a sequence of functions with variable centers, i.e. limits of rescalings
$$u_{x_j, r_j}(x) = \frac{u(x_j + r_j x) - u(x_j)}{r_j^2}.$$ 

We will first study the global solution for the classical obstacle problem, then generalize the results for Problems A, B and at the end of this lecture we will study the case of Problem C.

15.1. Classical Obstacle Problem.

**Theorem 15.1.** Let $u \in P_{\infty}(M)$ be a global solution of Problem A and assume that $u \geq 0$ in $\mathbb{R}^n$. Then $u$ is a convex function in $\mathbb{R}^n$, i.e.
$$\partial_{ee} u(x) \geq 0, \quad \text{for any direction } e \text{ and } x \in \mathbb{R}^n$$

In particular, the set $\{u = 0\}$ is convex.

**Proof.** Fix any direction $e$. Without loss of generality suppose that $e = e_n = (0, \cdots, 0, 1)$. Assume, on the contrary, that
$$-m := \inf_{\Omega} \partial_{nn} u < 0,$$
and let $x_j \in \Omega$ be a minimizing sequence for the value $-m$, i.e.
$$\lim_{j \to -\infty} \partial_{nn} u(x_j) = -m.$$

Let $d_j = \text{dist}(x_j, \Gamma)$ and consider the rescalings
$$u_j(x) = u_{x_j, d_j}(x) = \frac{1}{d_j^2} u(x_j + d_j x).$$

Observe that $B_1 \subset \Omega(u_j)$ and the free boundary $\Gamma(u_j)$ contains at least one point on $\partial B_1$. Since also $\|D^2 u_j\|$ are uniformly bounded we have the uniform estimates
$$|u_j(x)| \leq \frac{M}{2} (R + 1)^2$$

for all $R > 0$ and therefore we can extract a subsequence converging in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n)$ to a global solution $u_0$ of Problem A. The assumption $u \geq 0$, implies that $u_0 \geq 0$ and therefore, $\Omega(u_0) = \{u_0 > 0\}$. Moreover, similarly to $u_j$, observe that since $B_1 \subset \Omega(u_0)$, and $\partial B_1$ contains at least one free boundary point.
Next observe, since all functions \( u_j \) satisfy \( \Delta u_j = 1 \) in \( B_1 \), the convergence to \( u_0 \) can be assumed to be at least in \( C^2_{\text{loc}}(B_1) \). Hence, the limit function \( u_0 \) satisfies
\[
\Delta u_0 = 1, \quad \partial_{nn} u_0 \geq -m \text{ in } B_1, \quad \partial_{nn} u_0(0) = -m.
\]
Since \( \partial_{nn} u_0 \) is harmonic in \( B_1 \), the minimum principle implies that \( \partial_{nn} u_0 \equiv -m \) in \( B_1 \). In fact we have even more, \( \partial_{nn} u_0 = -m \) in the connected component of \( \Omega(u_0) \) which contains \( B_1 \). Hence we obtain the representations
\[
\partial_n u_0(x) = g_1(x') - mx_n, \quad x' = (x_1, \ldots, x_{n-1})
\]
and
\[
u_0(x) = g_2(x') + g_1(x')x_n - \frac{m}{2} x_n^2,
\]
in \( B_1 \). Now let us choose a point \((x', 0) \in B_1 \) and start moving in the direction \( e_n \).
Observe that as long as we stay in \( \Omega(u_0) \), we still have \( \partial_{nn} u = -m \) and therefore still have the representations (15.1)–(15.2). However, sooner or later we will reach \( \partial \Omega(u_0) \), otherwise if \( x_n \) becomes very large (15.2) will imply \( u_0 < 0 \), contrary to our assumption. Since \( u_0 = |\nabla u_0| = 0 \) on \( \partial \Omega(u_0) \), from (15.1) we obtain that the first value \( \xi(x') \) of \( x_n \) for which we arrive at \( \partial \Omega(u_0) \) is given by
\[
\xi(x') = \frac{g_1(x')}{m}.
\]
Hence from (15.2) we deduce that
\[
g_2(x') = -\frac{g_1(x')^2}{2m}.
\]
Now, the representation (15.2) takes the form
\[
u_0(x) = -\frac{m}{2}(x_n - \xi(x'))^2,
\]
which is not possible since \( u_0 \geq 0 \). This concludes the proof. \( \square \)

15.2. Problems A, B. Next, our goal is to generalize Theorem 15.1 for global solutions of Problems A, B. We will consider two case: when the complement of \( \Omega \) is bounded and when it is unbounded.

15.2.1. The compact complement case. Assume now we have \( u \in P_\infty(x_0, M) \) for which \( \Omega^c \) is compact.

Lemma 15.2. Let \( u \in P_\infty(x_0, M) \) be a global solution of Problem A, B, such that \( \Omega^c \) is compact and \( \text{Int} \Omega^c \neq \emptyset \). Then \( x_0 \) is a low energy point.

Proof. Suppose, towards a contradiction, that \( x_0 \) is a high energy point. Consider then a so-called “shrink-down” of \( u \) with a fixed center at \( x_0 \), i.e. sequence of rescalings
\[
u_k(x) = u_{x_0, R_k}(x) = \frac{u(x_0 + R_k x) - u(x_0)}{R_k^2}
\]
for \( R_k \to \infty \) which converges to a global solution \( u_\infty \). Similarly to blowups with fixed centers (Theorem 10.3), it is not hard show that \( u_\infty \) is a homogeneous global solution, as a simple corollary of Weiss’s monotonicity formula (see Lecture 10). The same monotonicity formula implies
\[
\alpha_n = \omega(x_0) \leq W(r, u, x_0) \leq \lim_{R_k \to \infty} W(R_k, u, x_0) = \lim_{R_k \to \infty} W(1, u_k) = W(1, u_\infty).
\]
On the other hand, we know that for homogeneous global solutions $W$ can take only two values: $\alpha_n/2$ and $\alpha_n$, hence $W(1, u_\infty) = \alpha_n$. global solution and $W(1, u_\infty) \leq \alpha_n$. This, combined with Hence, from (15.3) implies that $W(r, u, x_0) = \alpha_n$ for any $r > 0$. Thus, by Theorem 10.2 $u$ must be homogeneous with respect to the point $x_0$ and the classification of homogeneous solutions implies that $u$ must be a polynomial solution. This contradicts the assumption $\text{Int } \Omega^c \neq \emptyset$. \hfill \Box

Lemma 15.3. Let $u$ be as in Lemma 15.2. Then $\Omega^c$ will consist of finite union of connected components $\Omega_i^c$, $i = 1, \ldots, N$ with $C^1$ boundaries and nonempty interiors such that $u$ is constant in $\Omega_i^c$.

Proof. Note that every point on $\partial\Omega$ is of low energy. Applying now Theorems 12.4 and 13.1 we obtain the desired structure for $\Omega^c$. \hfill \Box

Lemma 15.4. Let $u$ be as in Lemma 15.2 and suppose that
\begin{equation}
\sup_{\Omega^c} u = 0
\end{equation}
Then, for a suitable choice of the origin in $\Omega^c$, the function
$$r \mapsto \frac{u(rx)}{r^2}$$
is nondecreasing, for any fixed $x$.

Proof. We will give the proof for $n \geq 3$. Denote by $V$ the Newtonian potential of $\Omega^c$, i.e.
$$V(x) = \int_{\Omega^c} \frac{c_n}{|x - y|^{n-2}} dy.$$ Then $V$ is bounded and superharmonic in $\mathbb{R}^n$ and harmonic in $\Omega$. By the maximum principle, there is at least one point $\zeta_0 \in \Omega^c$ such that
$$V(\zeta_0) \geq V(x) \quad \text{for all } x \in \mathbb{R}^n.$$ Set the origin at $\zeta_0$.
Since
$$\Delta(u - V) = 1$$in the sense of distributions and all second order partial derivatives of $u - V$ are bounded harmonic functions, the Hessian of $u - V$ is a constant matrix, by Liouville’s theorem. Hence $u - V$ is a polynomial of degree two. Set
$$P(x) = u(x) - V(x) - u(0) + V(0).$$Note that $|\nabla V(0)| = |\nabla u(0)| = 0$. Hence $P(0) = |\nabla P(0)| = 0$, this implies that $P$ is homogeneous. Now consider the function
$$h(x) = x \cdot \nabla u(x) - 2u(x).$$
h is continuous in $\mathbb{R}^n$ and for all $x \neq 0$ fixed,
$$\frac{d}{dr} \left( \frac{u(rx)}{r^2} \right) = \frac{1}{r^3} h(rx).$$
We will show that $h$ is non-negative in $\mathbb{R}^n$. In fact
$$h(x) = -2u(x) \geq 0, \quad \forall x \in \Omega^c.$$
On the other hand, by the homogeneity of $P$,
\[ h(x) = x \cdot \nabla V(x) - 2V(x) + 2V(0) - 2u(0) \]
then
\[ \lim_{|x| \to \infty} h(x) = 2V(0) - 2u(0) \geq 0. \]
Since $h$ harmonic in $\Omega$, by the minimum principle, $h$ is positive in $\Omega$. \qed

Now we can prove the main result of this section.

**Theorem 15.5.** Let $u \in P_\infty(M)$ be such that $\Omega^c$ is compact and $\text{Int } \Omega^c \neq \emptyset$. Suppose also that (15.4) holds. Then $u \geq 0$ in $\mathbb{R}^n$ and $u$ is a convex function. In particular $\{u = 0\}$ is a convex set.

**Proof.** Choose the origin as in Lemma 15.4.

Consider first the case of Problem A. We claim that there exists small $\rho > 0$ such that $u \geq 0$ in $B_\rho$. Indeed, if $0 \in \text{Int } \Omega^c$, this is immediate. If $0 \in \Gamma$, then it is a low energy point by Lemma 15.2 and therefore the statement follows from Lemma 12.3. Now, invoking Lemma 15.4, we conclude
\[ 0 \leq u(\rho x) \leq \frac{\rho^2}{R^2} u(Rx), \quad x \in B_1, \quad R > \rho, \]
i.e. $u \geq 0$ everywhere in $\mathbb{R}^n$. Then we invoke Theorem 15.1.

In the case of Problem B, we observe that the set $\{u \leq 0\}$ is star-like and therefore connected. Let now use the structure of $\Omega^c$. If $\Omega^c_i$, $i = 1, \ldots, N$ are the components as in Lemma 15.3 and $u = c_i$ there, then $c_i \leq 0$ by the assumption (15.4). On the other hand since, $u$ is subharmonic, we must have either $u = 0$ in the interior of $\{u \leq 0\}$ or $u < 0$. The latter is impossible, since it will imply that $c_i < 0$ for all $i = 1, \ldots, N$ (recall that $\Omega^c_i$ have nonempty interiors), which contradicts (15.4). Therefore we must have
\[ \{u \leq 0\} = \{u = 0\}. \]
and we arrive at the situation of Problem A. \qed

15.2.2. **Global solutions with unbounded $\Omega^c$.**

**Theorem 15.6.** Let $u \in P_\infty(M)$ such that $\Omega^c$ is unbounded and has nonempty interior. Then, there is $a \in \mathbb{R}$ such that $u \geq a$ and $\Omega^c = \{u = a\}$.

**In particular, by Theorem 15.1 $\Omega^c$ is convex.**

**Proof.** Suppose that some shrink-down $u_\infty$ of $u$ at 0 is a half space solution. Then, arguing as in the proof of Lemma 15.2 we will have that $u - u(0)$ is a half space solution. Hence the theorem follows in this case.

Now, if no shrink-down is a half-space solution, we may assume $u_\infty$ is a polynomial. The assumption $\text{Int } \Omega^c \neq \emptyset$ prevents $u$ from being a polynomial.

Since $\Omega^c$ is unbounded, there exists a sequence $x_j \in \partial \Omega$ tending to $\infty$. In this case we may scale by $R_j = |x_j|$ so as to obtain, in the limit, a global solution with a free boundary point $e$ on the unit sphere. By homogeneity then the ray $\{re: \ r > 0\}$, must lie in the free boundary. Since $u_\infty$ is a homogeneous quadratic polynomial, this is possible only if $\partial_e u_\infty \equiv 0$. Consider now the Alt-Caffarelli-Friedman monotonicity functional
\[ \phi_e(r, u) := \Phi(r, (\partial_e u)^+, (\partial_e u)^-). \]
Since \((\partial_e u)\pm\) are subharmonic by Lemma 6.2, from ACF monotonicity formula we have that
\[0 \leq \phi_e(r, u) \leq \phi_e(\infty, u) = \phi_e(1, u_\infty) = 0.\]
Hence, either \((\partial_e u)^+\) or \((\partial_e u)^-\) must vanish identically and we may assume without loss of generality that \(\partial_e u \geq 0\) (otherwise we replace \(e\) by \(-e\)).

Next, without loss of generality assume \(e = e_n\) and (after changing the origin)
\[B_r(0) \subset \Omega_c(u).\]
Then \(u \equiv u(0) \text{ in } B_r(0).\) Moreover, by monotonicity in the direction \(e_n\) we have that \(u \leq u(0) \text{ in the half-infinite cylinder } B'_r(0) \times (-\infty, 0),\) where \(B'_r(0)\) stands for a ball in \(\mathbb{R}^{n-1}.\) Since \(u\) is subharmonic, the maximum principle implies now that \(u(x', x_n) = u(0)\) for \(x' \in B'_r(0),\ x_n \leq 0.\)

Define now a \((n-1)\)-dimensional solution \(\hat{u}(x') = \lim_{x_n \to -\infty} u(x', x_n)\)
First, we notice that the limit exists by the monotonicity in the direction \(e_n.\) Next, the limit is finite, since \(B'_r(0) \times (-\infty, 0) \subset \Omega^c\) which gives the estimate
\[|u(x) - u(0)| \leq M \frac{|x'|^2}{2}.\]
Thus, \(\hat{u}\) is a \((n-1)\)-dimensional solution with a quadratic growth at infinity. Also note that
\[B'_r(0) \subset \Omega^c(\hat{u}).\]
First, suppose that \(\hat{u}\) is either a half space solution, or falls into the hypotheses of Theorem 15.5. Then \(\hat{u}\) is convex and non-negative. Since \(u(x', x_n) - u(0) \geq \hat{u}(x') \geq 0\) we conclude the proof by applying Theorem 15.1 to \(u(x) - u(0).\)

Next, if the lower dimensional solution \(\hat{u}\) is neither of the above it must fall into the third category analyzed above. Hence we repeat our argument and translate \(\hat{u}\) again in a new direction and reduce the dimension further. Finally, by induction, we need to classify the one dimensional solutions. However, the only one-dimensional solutions are \(x^1_1/2, (x^1_1)^2/2,\) or two separated solutions of the latter, which are all nonnegative. \(\square\)

15.3. Problem C.

**Theorem 15.7.** Let \(u \in P_\infty(M)\) be a solution of Problem C such that the origin is a branching point, i.e. \(0 \in \partial\{u > 0\} \cap \partial\{u < 0\}\) and \(|\nabla u(0)| = 0.\) Then \(u\) is a two-plane solution
\[u(x) = \frac{\lambda_+}{2}(x \cdot e)^2_+ - \frac{\lambda_-}{2}(x \cdot e)^2_-\]
for a certain direction \(e.\)

**Proof.** The proof follows from the classification of homogeneous global solutions in Theorem 11.1 and the following shrink-down argument.
Consider a limit
\[u_\infty(x) = \lim_{R_j \to \infty} \frac{u(R_j x)}{R_j^2}\]
over a certain sequence \(R_j \to \infty.\) Then Weiss's monotonicity formula implies that \(u_\infty\) is a homogeneous global solution. Since we still have that \(0\) is a branching
free-boundary point for $u_\infty$. Theorem 11.1 implies that $u_\infty$ is a two-plane solution for a certain direction $e$.

In particular, for any direction $\nu$, $\partial_\nu u_\infty$ does not change sign in $\mathbb{R}^n$ and therefore

$$\phi_\nu(r, u_\infty) = \Phi(r; (\partial_\nu u_\infty)^+, (\partial_\nu u_\infty)^-) = 0.$$ 

On the other hand, by the ACF monotonicity formula we have

$$0 \leq \phi_\nu(r, u) \leq \phi_\nu(\infty, u) = \phi_\nu(1, u_\infty) = 0,$$

implying $\partial_\nu u$ does not change sign. Since this holds for all directions $\nu$ we conclude $u$ is one dimensional and hence can be computed as earlier. \qed