LECTURE 19

19. C^1 Regularity of the Free Boundary: Problem C

19.1. C^1 regularity.

Theorem 19.1. *Let $u \in P_1(M)$ be a solution of Problem C. Then there are constants $\sigma_0 > 0$ and $r_0 > 0$ such that if*

$$|\nabla u(0)| \le \sigma, \quad \Omega^\pm(u) \cap B_\sigma \ne \emptyset, \tag{19.1}$$

then $\Gamma^\pm(u) \cap B_{r_0}$ are C^1-surfaces. The constants σ, r_0 and the modulus of continuity of the normal vectors to these surfaces depend only on $\lambda_\pm$, M and the space dimension n.

Remark 19.2. The C^1-regularity is optimal in the sense that the graphs are in general not of class $C^{1,\mathrm{Dini}}$. This means that the normal of the free boundary may not be Dini continuous, i.e. if ω is the modulus of continuity of the normal vector then

$$\int_0^1 \frac{\omega(t)}{t} dt = \infty.$$

Corollary 19.3. *Let $u \in P_1(M)$ and suppose that $0 \in \Gamma'(u)$ is a two-phase point. Then there is a constant $r_0 > 0$ such that $\Gamma^\pm(u) \cap B_{r_0}$ are C^1-surfaces. The constant r_0 and the modulus of continuity of the normal vectors to these surfaces depend only on $\lambda_\pm$, M and the space dimension n.*

Proof of Therem 19.1. From Theorem 17.7 we know that $\Gamma^\pm \cap B_{r_0}$ are given as Lipschitz graphs (after a suitable rotation of coordinate axes)

$$x_n = f_\pm(x')$$

with Lipschitz continuous $f_\pm$ satisfying $|\nabla f_\pm(x')| \le L < 1$ for $(x', f_\pm(x')) \in \Gamma^\pm \cap B_{r_0}$. Moreover, we know that $f_\pm$ are differentiable and even C^1. So, it will suffice to show that the normals are equicontinuous on $\Gamma^\pm(u) \cap B_{r_0/2}$ for u in the class of solutions specified in the statement of the theorem.

We claim that for $\epsilon > 0$ there is $\delta_\epsilon > 0$ depending only on the parameters in the statement such that for any pair of free boundary points $y_1, y_2 \in \Gamma^+ \cap B_{r_0/2}$,

$$|y_1 - y_2| \le \delta_\epsilon \quad \Rightarrow \quad |\nu(y_1) - \nu(y_2)| \le 2\epsilon. \tag{19.2}$$

Fix $\epsilon > 0$. Let σ_ϵ and r_ϵ denote the constants σ_0 and r_0 respectively in Theorem 17.7 for $L = \epsilon$. In what follows $\rho_\epsilon := \min\{r_\epsilon, \sigma_\epsilon\} r_0/4$.

Suppose first that u is non-negative in $B_{\rho_\epsilon}(y_1)$. Then we can apply the $C^{1,\alpha}$ regularity result the scaled function

$$u_{y_1,\rho_\epsilon}(x) := u(y_1 + \rho_\epsilon x)/\rho_\epsilon^2.$$

Since the $C^{1,\alpha}$-norm of the normal on $B_{c_0} \cap \partial\{u_{y_1,\rho_\epsilon} > 0\}$ bounded by a constant C_0, where c_0 and C_0 depend only on the parameters in the statement, we may choose

$$\delta_\epsilon := \min\{(\epsilon/C_0)^{1/\alpha}, c_0\}\, \rho_\epsilon$$

to obtain (19.2).

Next, suppose that u changes its sign in $B_{\rho_\epsilon}(y_1)$. This means $B_{\rho_\epsilon}(y_1)$ intersects both $\{\pm u > 0\}$. If there is a point $y \in B_{\rho_\epsilon}(y_1) \cap \partial\{u > 0\}$ such that $|\nabla u(y)| \le \rho_\epsilon$ then the rescaling $u_{y,r_0/2}$ satisfies the conditions of Theorem 17.7 with $L = \epsilon$. Namely,

$$|\nabla u_{y,r_0/2}(0)| \le \sigma_\epsilon, \quad B_{\sigma_\epsilon} \cap \{\pm u_{y,r_0/2} > 0\} \ne \emptyset.$$

Hence, the free boundary $\partial\{u > 0\} \cap B_{r_\epsilon r_0/2}(y) \supset \partial\{u > 0\} \cap B_{\rho_\epsilon}(y_1)$ is Lipschitz with Lipschitz norm not greater than ϵ. Hence (19.2) follows in this case with $\delta_\epsilon := \rho_\epsilon$.

Finally, if $|\nabla u| \ge \rho_\epsilon$ for all points $y \in B_{\rho_\epsilon}(y_1) \cap \partial\{u > 0\}$, we proceed as follows: from the equation $u(x', f_+(x')) = 0$ we infer that $\nabla' u + \partial_n u \nabla' f_+ = 0$ on $\partial\{u > 0\} \cap B_{r_0/2}$. Hence we obtain

$$|\nabla f_+(y_1) - \nabla f_+(y_2)| \le \frac{4M}{\rho_\epsilon}|y_1 - y_2|.$$

(Here M is such that $|D^2 u| \le M$ in B_1.) In particular we may choose

$$\delta_\epsilon := \frac{\epsilon\rho_\epsilon}{4M}$$

to arrive at (19.2).

□

19.2. **Optimality of C^1 regularity.** Let us now show that the free boundaries $\Gamma^\pm$ are not generally $C^{1,\mathrm{Dini}}$.

Lemma 19.4. *If $v \in W^{1,2}(D)$ is a solution of the one-phase obstacle problem*

$$\Delta v = \chi_{\{v>0\}} \quad \text{in } D$$

such that $v = 0$ on $\Sigma \subset \partial D$, then for any $B_r(x_0) \subset \mathbb{R}^n$ satisfying $B_r(x_0) \cap \partial D \subset \Sigma$,

$$\sup_{D\cap B_r(x_0)} v \le r^2/(8n) \quad \Rightarrow \quad v \equiv 0 \quad \text{in } D \cap B_{r/2}(x_0)\ .$$

Proof. Comparison of v in $D \cap B_{r/2}(y)$ to $w_y(x) = |x-y|^2/(2n)$ for $y \in B_{r/2}(x_0) \cap D$. □

Let now $\zeta \in C^\infty(\mathbb{R})$ be such that $\zeta = 0$ in $[-1/2, +\infty)$, $\zeta = 1/16$ in $(-\infty, -1]$ and ζ is strictly decreasing in $(-1, -1/2)$. Moreover define for $M \in [0, 1]$ the function u_M as the solution of the one-phase obstacle problem

$$\Delta u_M = \chi_{\{u_M>0\}} \text{ in } Q := \{x \in \mathbb{R}^2 : x_1 \in (0,1), x_2 \in (-1,0)\}\ ,$$
$$u_M(x_1, x_2) = M\zeta(x_2) \text{ on } \{x_1 = 0\} \cap \partial Q,$$
$$u_M(x_1, x_2) = M/2 \text{ on } \{x_1 = 1\} \cap \partial Q,$$
$$\partial_2 u_M = 0 \text{ on } (\{x_2 = -1\} \cup \{x_2 = 0\}) \cap \partial Q\ .$$

For $M = 1$ we may compare u_M to the function $x_1^2/2$ to deduce that

$$u_1 > 0 \text{ in } Q\ .$$

For $M = 0$, clearly $u_0 \equiv 0$.

On the other hand, as $\partial_2 u_M$ is harmonic in the set $Q \cap \{\partial_2 u_M > 0\}$ and nonpositive on $\partial(Q \cap \{\partial_2 u_M > 0\})$, we obtain from the maximum principle that $\partial_2 u_M \le 0$ in Q. Thus the free boundary of u_M is a graph of the x_1-variable.

Suppose now towards a contradiction that $\{0\} \times (-1/4, 0) \subset \partial\{u_M = 0\}^\circ$ for all $M \in (0, 1)$. Then, as $M \to 1$, we obtain $u_1 = |\nabla u_1| = 0$ on $\{0\} \times [-1/4, 0]$, implying by the fact that $u_1 > 0$ in Q and by the Cauchy-Kovalevskaya theorem (applied repeatedly to $w = u_1 - x_1^2/2$) that $u_1 \equiv x_1^2/2$ in Q; this is a contradiction in view of the boundary data of u_1.

From the continuous dependence of u_M on the boundary data as well as Lemma 19.4 we infer therefore the existence of an $M_0 \in (0,1)$ as well as $\bar{x} = (\bar{x}_1, \bar{x}_2) \in (\{0\} \times [-1/4, 0]) \cap \partial\{u_{M_0} = 0\}^\circ \cap \partial\{u_{M_0} > 0\}$. Note that Hopf's principle, applied at the line segment $\{0\} \times (-1/2, \bar{x}_2)$, yields $\nabla u_{M_0} \neq 0$ on $\{0\} \times (-1/2, \bar{x}_2)$.

Now we may extend u_{M_0} by odd reflection at the line $\{x_1 = 0\}$ to a solution u of Problem C in an open neighborhood of $\bar{x}$; here $\lambda_+ = \lambda_- = 1$. The point $\bar{x}$ is a branch point, so we may apply Theorem 19.1 to obtain that the free boundary is the union of two C^1-graphs in a neighborhood of $\bar{x}$.

Suppose now towards a contradiction that $\partial\{u > 0\}$ is of class $C^{1,\mathrm{Dini}}$ in a neighborhood of $\bar{x}$. Then by a theorem of Widman, the Hopf principle holds at $\bar{x}$ and tells us that

$$\liminf_{x_1 \to 0} \frac{\partial_2 u_{M_0}(x_1, \bar{x}_2)}{x_1} < 0.$$

But that contradicts Lemma 16. which, applied to the rescalings of solution u at $y = \bar{x}$, shows that

$$\liminf_{x_1 \to 0} \frac{\partial_2 u_{M_0}(x_1, \bar{x}_2)}{x_1} = 0.$$

Consequently $\partial\{u > 0\}$ and $\partial\{u < 0\}$ are not of class $C^{1,\mathrm{Dini}}$.