LECTURE 9

9. Normalized Solutions, Rescalings, and Blowups

9.1. **Local and globals solutions.** The further analysis of the free boundary is based on so-called blowup approach. Since the regularity of the free boundary is a local question, we may restruct ourselves to solutions in defined in balls, which are centered at free boundary points. We start with the definition of the appropriate classes of normalized solutions.

Definition 9.1. (Local solutions) For given $R, M > 0$, and $x_0 \in \mathbb{R}^n$ let $P_R(x_0, M)$ be the class of $C^{1,1}$ solutions u of Problems A, B, or C in $B_R(x_0)$ such that

- $\|D^2 u\|_{L^\infty(B_R(x_0))} \le M$,
- $x_0 \in \Gamma(u)$.

In the case $x_0 = 0$ we also set $P_R(M) = P_R(0, M)$.

Taking formally $R = \infty$ in the above definition, we obtain solutions in the entire space $\mathbb{R}^n$, which grow quadratically at infinity. Slightly abusing terminology, we call them *global solutions*.

Definition 9.2. (Global solutions) For given $M > 0$ and $x_0 \in \mathbb{R}^n$ let $P_\infty(x_0, M)$ be the class of $C^{1,1}_{\mathrm{loc}}$ solutions u of Problems A, B, or C, such that

- $\|D^2 u\|_{L^\infty(\mathbb{R}^n)} \le M$,
- $x_0 \in \Gamma(u)$.

We also set $P_\infty(M) = P_\infty(0, M)$.

9.2. **Rescalings and blowups.** The following scaling and translation properties are enjoyed by the solutions in the above classes. If $u \in P_R(x_0, M)$ and $\lambda > 0$, then the *rescaling* of u at x_0

$$u_\lambda(x) = u_{x_0,\lambda}(x) := \frac{u(x_0 + \lambda x) - u(x_0)}{\lambda^2}, \quad x \in B_{R/\lambda} \tag{9.1}$$

will be from class $P_{R/\lambda}(M)$. Using this simple obervation, we will often state the results for normalized classes $P_R(M)$ or even $P_1(M)$ as the corresponding statements for classes $P_R(M)$ can be easily recovered.

Observe that the rescalings satisfy the estimate $|D^2 u_\lambda| \le M$ in $B_{R/\lambda}$ for all $\lambda > 0$. For solutions of Problems A, B, as well as solutions of Problem C with $x_0 \in \Gamma'(u)$, after integration, we obtain the uniform estimates

$$|u_\lambda(x)| \le \frac{1}{2} M |x|^2, \quad x \in B_{R/\lambda}.$$

Hence, if we can find a sequence $\lambda = \lambda_j \to 0$ such that

$$u_{\lambda_j} \to u_0 \quad \text{in } C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n) \text{ for any } 0 < \alpha < 1$$

where $u_0 \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$. Such u_0 is called a *blowup of u with fixed center x_0* and Proposition 9.3 below implies that u_0 is a global solution; more precisely, $u_0 \in$

$P_\infty(M)$. An important remark is that it is apriori not clear if u_0 is unique, as different sequences $\lambda_j \to 0$ may lead to different limits u_0.

The construction above can be easily generalized to the case when we have a sequence of free boundary points $x_j \to x_0$ (variable centers) instead of a fixed center x_0. Namely, we consider the limits

$$u_{x_j,\lambda_j} \to u_0 \quad \text{in } C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n) \text{ for any } 0<\alpha<1.$$

for $\lambda_j \to 0$. We call such u_0 *blowups at* x_0 (emphasizing variable centers, if necessary). Furthermore, sometimes we may need to study the limits of $(u_j)_{x_j,\lambda_j}$ for different $u_j \in P_R(x_j, M)$ and we will still call such limits blowups.

In the case when $u \in P_\infty(M)$, we may also let $\lambda \to \infty$. The resulting limits are called *shrink-downs*, instead of blowups.

Proposition 9.3 (Limits of solutions). *Suppose that we have a sequence $u_j \in P_1(M)$, $j = 1, 2, \ldots$, and that*

$$u_j \to u_0 \quad \textit{in } C^{1,\alpha}(B_1)$$

for some $0 < \alpha < 1$, as $j \to \infty$. Then

(i) $u_0 \in P_1(M)$

(ii) *In the limit we have the inclusions*

$$\Omega(u_0) \subset \liminf_{j\to\infty} \Omega(u_j), \tag{9.2}$$

$$\limsup_{j\to\infty}\{|\nabla u_j| = 0\} \subset \{|\nabla u_0| = 0\}, \tag{9.3}$$

$$\mathrm{Int}(\{|\nabla u_0| = 0\}) \subset \liminf_{j\to\infty}\{|\nabla u_j| = 0\} \tag{9.4}$$

$$\limsup_{j\to\infty} \Gamma(u_j) \subset \Gamma(u_0) \tag{9.5}$$

(iii) $u_j \to u_0$ *strongly in $W^{2,p}_{\mathrm{loc}}(B_1)$ for any $1 < p < \infty$.*

In (ii), for a sequence of sets E_j, $\limsup$ denotes the limit points of all sequences $x_{j_k} \in E_{j_k}$, $j_k \to \infty$ and $\liminf$ is the set af all limits of sequenses $x_j \in E_j$, $j \to \infty$.

Proof. 1) We start by showing the part (ii). The inclusions (9.2) and (9.3) follow from the C^1 convergence $u_j \to u_0$. For (9.5) one also needs to use the nondegeneracy (Lemmas 8.1, 8.4, 8.5). Let us give a more detailly proof of (9.4).

If $|\nabla u_0| = 0$ in $B_r(x_0)$ then $|\nabla u_j| = 0$ in a smaller ball $B_\rho(x_0)$, $\rho < r$, for large j, otherwise by non-degeneracy of the gradient (see Corollary 8.6).

$$\sup_{B_r(x_0)} |\nabla u_j| > c(r-\rho),$$

which contradicts to the uniform convergence of $|\nabla u_j| \to 0$ in $B_r(x_0)$. As a consequence, we obtain

$$B_r(x_0) \subset \liminf_{j\to\infty}\{|\nabla u_j| = 0\}$$

and consequently (9.4).

2) To show that $u_0 \in P_1(M)$ observe that the condition (i) in Definition 9.1 follows easily, since $D^2u_j \to D^2u_0$ weakly in $L^p(B_r)$ for all $p < \infty$. The condition (ii) follows immediately from nondegeneracy.

It remains to show that u_0 satisfies the corresponding equation for Problems A, B, C. Consider first Problems A, B. Since $\nabla u_0 = 0$ off $\Omega(u_0)$, $\Delta u_0 = 0$ a.e. there. Therefore, for any $\phi \in C_0^\infty(B_1)$, we have

$$\begin{aligned}\int \Delta u_0\, \phi\, dx &= \int \chi_{\Omega(u_0)} \Delta u_0\, \phi\, dx \\ &= \lim_{j\to\infty} \int \chi_{\Omega(u_0)} \Delta u_j\, \phi\, dx = \lim_{j\to\infty} \int \chi_{\Omega(u_0)}\, \chi_{\Omega(u_j)} \phi\, dx \\ &= \int \chi_{\Omega(u_0)}\, \phi\, dx,\end{aligned}$$

where in the last step we have used the inclusion (9.2).

Finally, for Problem C, we may use the same argument, since the portion $\Gamma''(u_0)$ of the free boundary where $|\nabla u_0| \neq 0$ is a smooth surface. Hence, u_0 satisfies the desired equation.

3) To prove that $u_j \to u_0$ strongly in $W^{2,p}_{\mathrm{loc}}$, $1 < p < \infty$, it will siffice to show that

$$D^2 u_j \to D^2 u_0 \quad \text{a.e. on } B_1,$$

since $D^2 u_j$ are uniformly bounded. The pointwise convergence in $\Omega(u_0)$ follows from (9.2) and the interior Schauder estimates. The convergence on $\mathrm{Int}(\Omega(u_0)^c)$ follows from (9.4). The only remaining points are the ones on the free boundary $\Gamma(u_0)$ and since it has Lebesgue measure zero, we obtain the a.e. convergence of $D^2 u_j$ to $D^2 u_0$. $\square$