7.18 Theorem *There exists a real continuous function on the real line which is nowhere differentiable.*

Proof Define

$$\varphi(x) = |x| \qquad (-1 \le x \le 1) \tag{34}$$

and extend the definition of $\varphi(x)$ to all real x by requiring that

$$\varphi(x+2) = \varphi(x). \tag{35}$$

Then, for all s and t,

$$|\varphi(s) - \varphi(t)| \le |s - t|. \tag{36}$$

In particular, φ is continuous on R^1. Define

$$f(x) = \sum_{n=0}^{\infty} (\tfrac{3}{4})^n \varphi(4^n x). \tag{37}$$

Since $0 \le \varphi \le 1$, Theorem 7.10 shows that the series (37) converges uniformly on R^1. By Theorem 7.12, f is continuous on R^1.

Now fix a real number x and a positive integer m. Put

$$\delta_m = \pm \tfrac{1}{2} \cdot 4^{-m} \tag{38}$$

where the sign is so chosen that no integer lies between $4^m x$ and $4^m(x + \delta_m)$. This can be done, since $4^m |\delta_m| = \frac{1}{2}$. Define

$$\gamma_n = \frac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m}. \tag{39}$$

When $n > m$, then $4^n \delta_m$ is an even integer, so that $\gamma_n = 0$. When $0 \le n \le m$, (36) implies that $|\gamma_n| \le 4^n$.

Since $|\gamma_m| = 4^m$, we conclude that

$$\begin{aligned} \left| \frac{f(x+\delta_m) - f(x)}{\delta_m} \right| &= \left| \sum_{n=0}^{m} \left(\frac{3}{4}\right)^n \gamma_n \right| \\ &\ge 3^m - \sum_{n=0}^{m-1} 3^n \\ &= \tfrac{1}{2}(3^m + 1). \end{aligned}$$

As $m \to \infty$, $\delta_m \to 0$. It follows that f is not differentiable at x.

EQUICONTINUOUS FAMILIES OF FUNCTIONS

In Theorem 3.6 we saw that every bounded sequence of complex numbers contains a convergent subsequence, and the question arises whether something similar is true for sequences of functions. To make the question more precise, we shall define two kinds of boundedness.