Large-time geometric properties of solutions of the evolution p-Laplacian equation

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Abstract

We establish the behavior of the solutions of the degenerate parabolic equation

$$u_t = \nabla \cdot \left(|\nabla u|^{p-2}\nabla u\right), \qquad p > 2,$$

posed in the whole space with nonnegative, continuous and compactly supported initial data. We prove a nonlinear concavity estimate for the pressure away from the maximum point. The estimate has important geometric consequences: it implies that the support of the solution becomes convex for large times and converges to a ball. In dimension one, we know also that the pressure itself eventually becomes p-concave. In several dimensions we prove concavity but for a small neighborhood of the maximum point.

1 Introduction

In this paper we establish the large time behavior of the solutions of the degenerate parabolic equation

$$u_t = \nabla \cdot \left(|\nabla u|^{p-2}\nabla u\right), \tag{1.1}$$

usually called the evolution p-Laplacian equation (PLE for short). We consider the initial value problem for this equation posed in $Q = \mathbf{R}^N \times (0, \infty)$, with initial data

$$u(x, 0) = u_0(x) \qquad \text{on} \ \ \mathbf{R}^N, \tag{1.2}$$

where u_0 is a nonnegative integrable function in $\mathbf{R}^N$ with support contained in the ball B_R centered at 0 and having radius R. Our main concern are the geometric properties of the solutions corresponding to nonnegative, integrable and compactly supported data as time goes to infinity, so-called eventual properties. In particular, we prove convexity of level lines for large times, even if the

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property was not true for the initial data. This property derives from a non-standard concavity property, that we call p-concavity, that sets in eventually for general solution in the considered class.

Nonlinear diffusion equations with gradient-dependent diffusivity. The evolution p-Laplacian equation is one of most widely researched equations in the class of nonlinear degenerate parabolic equations. The particular feature of equation (1.1) is its gradient-dependent diffusivity. Such equations, and their stationary counterparts, appear in different models in non-Newtonian fluids, turbulent flows in porous media, certain diffusion or heat transfer processes, and recently in image processing. See in that last respect [23, 7, 3]. Equation (1.1) is the most representative model in that class, already studied by Raviart [24], 1970 (along with the extension called doubly-nonlinear parabolic equation).

For exponent $p = 2$ the equation reduces to the classical heat equation, the theory of which is well known. Among its features we find C^∞ smoothness of solutions, infinite speed of propagation of disturbances and the strong maximum principle. These properties generalize to a number of related evolution equations, notably those which are linear and uniformly parabolic.

A marked departure occurs in (1.1) when the exponent p is larger than 2. The equation is degenerate parabolic and finite propagation holds. It is well known that there exists a unique nonnegative weak solution of (1.1)–(1.2) and for each t it has compact support that increases with t. Hence, there exists an *interface* or *free boundary* separating regions where $u > 0$ from regions where $u = 0$. It is remarkable that the interface might not be a smooth surface if u_0 is topologically complicated, as the focusing solutions studied by Gil and Vázquez [17] show, see also [2]. However, the solutions are known to have locally Hölder continuous first derivatives, see [8, 11].

Though p-Laplacian equations are better studied in the stationary case due to their interest for elasticity and the Calculus of Variations, the evolution equation we study here offers the big mathematical extra feature of the evolution of the interface (and the level lines) and their eventual stabilization. These matters are of concern in this paper. In particular, our work contributes to the task of showing the essentially simple structure of the asymptotic behavior of solutions, free boundaries and level lines for equations whose mathematical theory is not at all simple and departs strongly from the standard heat flow.

Large-time behavior. Kamin and Vázquez [18] studied the uniqueness and asymptotic behavior of nonnegative solutions with finite mass. i.e., with $u_0 \in L^1(\mathbf{R}^N)$. They proved that the explicit solutions

$$U_M(x,t) = t^{-k}\left(C - q\left(\frac{|x|}{t^{k/N}}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}$$

found by Barenblatt in 1952, are essentially the only positive solutions to the

Cauchy problem with initial data

$$u(x,0) = M\delta(x), \qquad M > 0.$$

Here

$$k = \left(p - 2 + \frac{p}{N}\right)^{-1}, \qquad q = \frac{p-2}{p}\left(\frac{k}{N}\right)^{\frac{1}{p-1}}$$

and C is related to the mass M by $C = c\,M^{\alpha}$, where $\alpha = p(p-2)k/(N(p-1))$ and $c = c(p,N)$ is determined from the condition $\int U_M(x,t)dx = M$. Using the idea of asymptotic radial symmetry, [18] establishes that any nonnegative solution with globally integrable initial values is asymptotically equal to the Barenblatt solution as $t \to \infty$.

Eventual geometric properties. The main result of this paper is the property of *asymptotic concavity* that can be best expressed in terms of the convenient variable,

$$v = \frac{p-1}{p-2}\, u^{\frac{p-2}{p-1}}, \tag{1.3}$$

known as the *generalized pressure* (while u is known as the *density*) and which satisfies

$$v_t = \frac{p-2}{p-1}\, v\, \Delta_p v + |\nabla v|^p. \tag{1.4}$$

We remark that this pressure variable, introduced in [15, 16], is appropriate to study properties related to interface behavior and geometry, while u is better suited for existence and uniqueness questions. We show that when t is sufficiently large $v(x,t)$ becomes concave in x away from its maximum point and inside the positivity set

$$P(t;v) = \{x : v(x,t) > 0\} = \{x : u(x,t) > 0\},$$

which tends to a ball of prescribed radius (see Theorem 3.2 for a precise formulation). In particular, we obtain that the positivity set $P(t;v)$ eventually becomes convex. This result is the counterpart of the asymptotic concavity of the pressure in porous medium equation $u_t = \Delta u^m$ obtained by two of the authors in [22].

We recall that convexity properties of level sets or log-concavity of solutions are well-known subjects in the theory of the heat equation, but standard results deal with conservation in time of such properties, while in the present case we show *convexity properties that arise from non-convex data as a consequence of the asymptotic stabilization process.*

The rough idea of the proof is the following concavity property of the pressure V_M of the Barenblatt solutions observed in [16]: $V_M(\cdot,t)$ is concave in its support and satisfies

$$\partial_e(|\nabla V_M|^{p-2}\partial_e V_M) = -\frac{K}{t}, \qquad K = \left(\frac{p-2}{p-1}\right)^{2(p-1)} \frac{k}{N} \tag{1.5}$$

in the set $\{V_M(\cdot,t)>0\}$ for every spatial direction e. Contrary to what happened in the porous medium case, we now have to involve a strange-looking second-order operator in the right-hand side of (1.5), which reduces to the directional second derivative for $p=2$. However, it is exact and neat on V and that is what counts. Actually, it reflects the peculiar geometry associated to the p-Laplacian. We call an estimate of the form $\partial_e(|\nabla f(x)|^{p-2}\partial_e f(x))\le -c$, for a function $f(x)$, $x\in\Omega$, a uniform p-concavity estimate for f in Ω.

As a consequence of that formula, if we show that the appropriate rescalings of v converge to V_M at least in C^2_x norm near the interface (more generally, away from the maximum point of V_M), we will obtain that

$$\partial_e(|\nabla v|^{p-2}\partial_e v)<0, \tag{1.6}$$

which implies the concavity of v (in the usual sense, see Lemma 5.1.) An important technical step is to prove the above mentioned estimates near the interface, which is done by generalizing the method developed originally by Koch [20] for porous medium equation (see Theorem 3.1 and Section 4.). In combination with the rates of asymptotic convergence and regularity and compactness, we obtain the final result.

Moreover, in one dimension we can tell a bit more (see Theorem 3.2′.) The pressure eventually becomes concave in the whole set $\{v(\cdot,t)>0\}$. In particular $v(\cdot,t)$ has only one point of maximum, which we denote $\gamma(t)$. We show that $\gamma(t)$ becomes $C^{1,\alpha}$ regular for large t as a consequence of the work of Bertsch and Hilhorst [4] on the regularity of the interface in one-dimensional two-phase porous medium equation.

Outline of the paper:

- Section 2 contains definitions and preliminary results and in Section 3 we state our main results.

- Section 4 contains the proof of C^∞ regularity near the interface for $p>2$.

- Section 5 deals with convergence to the Barenblatt solution.

In the next three sections we work in one dimension.

- Section 6 contains the proof of eventual concavity for $1<p<2$ and Section 7 for $p>2$.

- The study of the curve of maxima is done in Section 8.

2 Definitions and preliminary results

The Cauchy problem (1.1)–(1.2) (or (CP) for short) does not possess classical solutions for general data in the class $u_0\in L^1(\mathbf{R}^N)$, $u_0\ge 0$ (or even in a smaller class, like the set of smooth nonnegative and rapidly decaying initial data). This is due to the fact that the equation is parabolic only where $|\nabla u|>0$, but degenerate where $|\nabla u|=0$. Therefore, we need to introduce a concept of generalized solution and make sure that the problem is well-posed in that class.

By a *weak solution* of the equation (1.1) we will mean a nonnegative measurable function $u(x,t)$, defined for $(x,t) \in Q$ such that: (i) viewed as a map

$$t \to u(\cdot,t) = u(t). \tag{2.1}$$

we have $u \in C((0,\infty); L^1(\mathbf{R}^N))$; (ii) the functions u and $|\nabla u|^{p-2}\nabla u$ belong to $L^1(t_1,t_2; L^1(\mathbf{R}^N))$ for all $0 < t_1 < t_2$; and (iii) the equation (1.1) is satisfied in the weak sense

$$\iint \{u\varphi_t - |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\}\, dxdt = 0$$

for every smooth test function $\varphi \geq 0$ with compact support in Q.

By a *solution* of (CP) we mean a weak solution of (1.1) such that the initial data (1.2) are taken in the following sense:

$$u(t) \to u_0 \quad \text{in } L^1(\mathbf{R}^N) \quad \text{as } t \to 0. \tag{2.2}$$

In other words, $u \in C([0,\infty); L^1(\mathbf{R}^N))$ and $u(0) = u_0$.

The existence and uniqueness of solutions of (CP) in Q for the general class of initial data $u_0 \in L^1_{\text{loc}}(\mathbf{R}^N)$ with an optimal growth condition at infinity when $p > 2$

$$\sup_{\rho \geq \rho_0} \rho^{-\lambda} \int_{B_\rho(0)} u_0(x)dx < \infty, \qquad \lambda = N + \frac{p}{p-2}$$

is shown by DiBenedetto and Herrero [12, 13].

Next, we list some important properties of solutions.

Property 1 The solutions of (CP) satisfy the law of mass conservation

$$\int_{\mathbf{R}^N} u(x,t)\, dx = \int_{\mathbf{R}^N} u_0(x)\, dx, \tag{2.3}$$

i.e., $||u(t)||_{L^1(\mathbf{R}^N)} = ||u_0||_{L^1(\mathbf{R}^N)}$ for all $t > 0$.

The proof is standard in this type of problems. The proof of the following estimate can be found in [28].

Property 2 The solutions are bounded for $t \geq \tau > 0$. More precisely,

$$|u(x,t)| \leq U_M(0,t) = c_*(p,N)M^{pk/N}t^{-k}, \tag{2.4}$$

where $M = ||u_0||_{L^1(\mathbf{R}^N)}$ and $k = (p-2+p/N)^{-1}$.

Property 3 The weak solutions $u(x,t)$ and their spatial gradients $\nabla u(x,t)$ are uniformly Hölder continuous for $0 < \tau \leq t \leq T < \infty$.

See e.g. [8, 11] for detailed regularity results. The next semi-convexity estimate is due to Esteban-Vázquez [16].

Property 4 For $p > 2N/(N+1)$ there exist a constant $C = C(p, N)$ such that for any nonnegative solution u of the Cauchy problem (CP), the pressure v satisfies the estimate

$$\Delta_p v \geq -\frac{C}{t}. \tag{2.5}$$

in the sense of distributions.

Property 5 (Finite propagation property) If the initial function u_0 is compactly supported, so are the functions $u(\cdot, t)$ for every $t > 0$. Under these conditions there exists a free boundary or interface which separates the regions $\{(x,t) \in Q : u(x,t) > 0\}$ and $\{(x,t) \in Q : u(x,t) = 0\}$. This interface is usually an N-dimensional hypersurface in $\mathbf{R}^{N+1}$.

Property 6 (Scaling) One of the critical properties of the p-Laplacian equation is the scaling invariance. Any solution $u(x,t)$ of (1.1) will produce a family of solutions

$$\left(\frac{B}{A^p}\right)^{\frac{1}{p-2}} u(Ax, Bt) \tag{2.6}$$

for any $A, B > 0$. In particular, choosing $A = \theta^{-k/N}$, $B = \theta^{-1}$ for $\theta > 0$, we obtain the scaling

$$\frac{1}{\theta^k}\, u\left(\frac{x}{\theta^{k/N}}, \frac{t}{\theta}\right) \tag{2.7}$$

which is the one that conserves the mass for the density u.

Next, we point out that the source-type solutions $U_M(x,t)$ are weak solutions of (1.1), but they are *not* solutions of problem (CP) as stated, since they do not take L^1 initial data. Indeed, it is easy to check that U_M converges to a Dirac mass

$$U_M(x,t) \to M\,\delta(x) \qquad \text{as } \ t \to 0, \tag{2.8}$$

This is the reason for the name *source-type solutions.* They are invariant under the scaling for the choice $A = B^{k/N}$.

The asymptotic behavior of any solution of the Cauchy problem is described in terms of the Barenblatt solution with the same mass.

Theorem 2.1 *Let $u(x,t)$ be the unique solution of problem* (CP) *with initial data $u_0 \in L^1(\mathbf{R}^N)$, let $M = \int u_0(x)\,dx$. If U_M is the Barenblatt solution with the same mass as u_0, then as $t \to \infty$ we have*

$$\lim_{t\to\infty} t^k \|u(t) - U_M(t)\|_{L^\infty(\mathbf{R}^n)} = 0. \tag{2.9}$$

The proof can be found in [16]. Let us finally recall that the functions $U_M(x,t)$ have the self-similar form

$$U_M(x,t) = t^{-k} F(xt^{-k/N}; C), \tag{2.10}$$

where $k = (p-2+p/N)^{-1}$ and $k/N = (N(p-2)+p)^{-1}$ are the similarity exponents and

(2.11) $$F(s) = (C - q s^{\frac{p}{p-1}})_+^{\frac{p-1}{p-2}}$$

is the profile, where $C = c\, M^{\alpha}$, with $\alpha = p(p-2)k/N(p-1)$, $c = c(p,N)$ and $q = (1-2/p)(k/N)^{1/(p-1)}$. For the pressure variable we can write

$$V_M(r,t) = \frac{1}{(Lt)^{k\frac{p-2}{p-1}}}\, G_L\left(\frac{|x|}{(Lt)^{k/N}}\right),$$

with $L = M^{p-2}$ and profile

$$G_L(s) = L^{\frac{1}{p-1}}\,(c - q\, s^{\frac{p}{p-1}})_+$$

for c, q depending only on p and N. The free boundary is given by the equation $|x| = (c/q)(Lt)^{k/N}$.

Property 7 (Asymptotic error for the support) Using Aleaksandrov's Reflection Principle, one can prove the following sharp estimates on the size of the positivity set $\Omega(t) = \{v(\cdot,t) > 0\}$, see e.g. [29]

$$B_{r(t)} \subset \Omega(t) \subset B_{R(t)}, \quad R(t) \le r(t) + 2R_0, \quad R(t) \sim (c/q)(Lt)^{k/N},$$

if the support of the initial data contained in the ball of radius R_0 (with center at 0).

3 Statement of the main results

We are going to impose the conditions on the initial pressure v_0, which have been used to get the long-time nondegenerate Lipschitz solutions in [9, 29] and $C^{1,\alpha}$ regularity of the interface in [19]. These conditions go back to the papers of Caffarelli, Vázquez and Wolanski [5] and Caffarelli and Wolanski [6] on the porous medium equation.

Conditions:

1. The support of v_0, $\overline{\Omega}_0 = \overline{\{v_0 \ge 0\}}$, is contained in a ball of radius $R > 0$,
2. Regularity: $\partial\Omega_0$ is C^1 regular and $v_0 \in C^1(\overline{\Omega}_0)$,
3. Non-degeneracy: $0 < 1/K < v_0 + |\nabla v_0| < K$ in $\overline{\Omega}_0$,
4. Semi-concavity: $\partial_{ee} v_0 \ge -K_0$ in a strip $S \subset \Omega_0$ near the boundary $\partial\Omega_0$ for any direction e.

The uniform convergence result (2.9) can be restated as

(3.1) $$\lim_{t\to\infty} t^{k\frac{p-2}{p-1}}\, \|v(t) - V_M(t)\|_{L^\infty(\mathbf{R}^N)} = 0.$$

Our goal in this paper is the improvement of the uniform convergence up to C^∞-convergence. This will imply the convexity of the positivity set, $\Omega(t) = \{v(\cdot,t) > 0\}$ (or the support, which is its closure) and the concavity of $v(x,t)$ in $\Omega(t)$ (away from the maximum of V_M) at large times $t \gg 1$.

Scaling (see Property 6 in the previous section) will play an important role in what follows. As the first application, we use it to reduce the problem to the case with the mass 1. Indeed, given a solution with mass $M > 0$ we can use the scaling

$$\tilde{v}(x,t) = \frac{1}{A^{\frac{p}{p-1}}}\, v(Ax,t)$$

with $A = M^{(k/N)(p-2)}$ to get another solution $\tilde{v}$ with mass 1. Therefore, we can take $M = L = 1$ in the sequel. We will write G instead of G_1 for the Barenblatt profile, and V instead of V_1 for the corresponding solution.

On a more fundamental level, given a solution $v = v(x,t)$ with mass M we will define the family

$$v_\lambda(x,t) = \lambda^{k\frac{p-2}{p-1}} v(\lambda^{k/N}x, \lambda t), \qquad \lambda > 0, \tag{3.2}$$

which are again solutions of the same equation with same mass, now normalized to 1. The long-time behavior can be captured through the uniform bound for the scaled solutions. Formula (3.1) can be stated equivalently as

$$|v_\lambda(x,t) - V(x,t)| \to 0 \qquad \text{as } \ \lambda \to \infty,$$

uniformly in $|x| < K$, $1 < t < 2$. Therefore, we will concentrate on the convergence of v_λ toward $V(x,t) = t^{-k\frac{p-2}{p-1}} G(xt^{-k/N})$.

First, we will show estimates for all possible derivatives of v_λ (for large $\lambda > 0$); they hold at times uniformly for $t \in [1,2]$ everywhere except in an arbitrary small ball B_ε centered at the origin.

Theorem 3.1 *For every $k > 0$ and $\varepsilon > 0$ there exists a value of the scaling parameter $\lambda_{k,\varepsilon}$ and a uniform constant $C_{k,\varepsilon} > 0$ such that*

$$||v_\lambda(x,t)||_{C^k_{x,t}(\overline{\Omega(v_\lambda)}\setminus B_\varepsilon(0)\times[1,2])} < C_{k,\varepsilon} \qquad \textit{for all } \lambda > \lambda_{k,\varepsilon}, \tag{3.3}$$

where

$$\Omega(v_\lambda) = \{(x,t) : v_\lambda(x,t) > 0,\ 1 < t < 2\}.$$

Let us translate these results into the asymptotic concavity statements. We remark that we have the following identity for $G(x)$

$$\partial_e(|\nabla G|^{p-2}\partial_e G) = -q^{p-1}$$

for every direction e and at all points x such that $G(x) > 0$. Then the C^2_x-convergence away from the origin implies

$$\partial_e(|\nabla v_\lambda|^{p-2}\partial_e v_\lambda) < 0 \qquad \text{on } \{v_\lambda(\cdot,t) > 0\} \setminus B_\varepsilon$$

for λ large.

Theorem 3.2 *There exists $t_0 > 0$ such that $\Omega(t) = \{x : v(x,t) > 0\}$ is a convex subset of $\mathbf{R}^N$ for $t \ge t_0$ and its curvature converges to the constant curvature of the free boundary of the Barenblatt solution*

$$\lim_{t\to\infty} t^{k/N}\, K(x,t) = C. \tag{3.4}$$

uniformly in $x \in \partial\Omega(t)$. Moreover, for every $\varepsilon > 0$ there exists $t_{0,\varepsilon}$ such that $v(x,t)$ is concave in $\Omega(t) \setminus B_{\varepsilon t^{k/N}}$ for $t \ge t_{0,\varepsilon}$. More precisely,

$$\lim_{t\to\infty} t\, \partial_e(|\nabla v|^{p-2}\partial_e v) = -q^{p-1} \tag{3.5}$$

for every direction e uniformly for $x \in \Omega(t) \setminus B_{\varepsilon t^{k/N}}$ for every $\varepsilon > 0$.

In the one-dimensional case, we can also establish the concavity near the origin.

Theorem 3.2′ *In dimension $N = 1$, the convergence* (3.5) *is uniform for $x \in \Omega(t)$. As a consequence, all level sets $\{x : v(x,t) \ge c\}$, $c > 0$, are convex (if not empty). The function $u(\cdot,t)$ has only one maximum point $\gamma(t)$. Moreover, the curve $x = \gamma(t)$ is $C^{1,\alpha}$-regular for $t \ge t_0$.*

In the next sections we perform the proofs of these results.

4 Regularity near the interface, $p > 2$

Let v be a solution of (1.4) for $p > 2$. Due to Esteban and Vázquez [16], we know that

$$\Delta_p v \ \ge\ -\frac{C}{t}, \qquad \text{for } C = C(p,N) > 0 \tag{4.1}$$

$$v_t \ \ge\ -\frac{p-1}{p-2}\cdot\frac{C}{t}\cdot v \tag{4.2}$$

and also

$$v_t,\ |\nabla v|^p = O\left(t^{-k\frac{p-2}{p-1}-1}\right), \tag{4.3}$$

which after the scaling (3.2) take the form

$$\Delta_p v_\lambda \ge -C, \qquad v_t \ge -\frac{p-1}{p-2}\, C\, v_\lambda, \qquad v_{\lambda,t},\ |\nabla v|^p \le C \tag{4.4}$$

with C independent of λ.

4.1 Nondegeneracy of ∇v_λ near the free boundary.

From the exact estimates on the growth of the domain $\Omega(t) = \{x : v(x,t) > 0\}$, see Property 7 in Section 2, after rescaling we can assume that

$$B_{\rho_0} \subset \Omega_\lambda(t) = \{x : v_\lambda(x,t) > 0\} \subset B_{\rho_1}, \quad \text{for } t \in [1,2] \tag{4.5}$$

for some $\rho_0,\, \rho_1 > 0$, independent of λ. We claim that there is $\delta_0 > 0$ and $c_0 > 0$ independent of λ such that

$$|\nabla v_\lambda| \geq c_0 \quad \text{in } \delta_0\text{-neighborhood of } \partial\Omega_\lambda(t)\ ,\ t \in [1,2]. \tag{4.6}$$

or, equivalently,

$$v_\lambda + |\nabla v_\lambda| \geq K_1(K, K_0) > 0 \quad \text{in } \{v_\lambda > 0\} \text{ for } t \in [1,2]. \tag{4.7}$$

The proof of this statement is based on the inequality

$$\frac{A-p}{p-1}\, v(x,t) + x \cdot \nabla v(x,t) + (At+B) v_t(x,t) \geq 0 \tag{4.8}$$

in $\mathbf{R}^N \times (0,\infty)$, which can be found in [9, 29]. Here $A,\ B > 0$ depend only on K and K_0. Using straightforward computations, one can show that after rescaling (3.2) the inequality (4.8) will take the form

$$\frac{A-p}{p-1}\, v_\lambda(x,t) + x \cdot \nabla v_\lambda(x,t) + (At + B/\lambda)\, v_{\lambda,t}(x,t) \geq 0. \tag{4.9}$$

Now, using ideas analogous to those in the proof of Lemma 3.3 in [5], and carried out in detail for p-Laplacian equation by Ko [19], Section 3, one can prove the estimate (4.7). An important observation is that even though $B/\lambda \to 0$ as $\lambda \to \infty$, the estimates in [19] depend only on A and $At + B/\lambda$, and when $t \in [1,2]$ and $\lambda > 1$, we have

$$A \leq At + B/\lambda \leq 2A + B,$$

which implies the uniformity of these estimates.

4.2 $C^{1,\alpha}$-regularity of the pressure

From now on, when there is no ambiguity, we will omit the index λ and will write v for the rescaled pressure v_λ.

From the result of Y. Ko [19], we know that the interface $\partial\Omega(t)$ will be $C^{1,\alpha}$ regular for $t \in [1,2]$. However, we need also $C^{1,\alpha}$ regularity of the pressure v in order to prove C^∞ regularity of the interface. We apply the method originally due to Koch [20], which was used to prove C^∞ regularity of the interface in the porous medium equation.

We know that the positivity set $\Omega(t)$ of $v(\cdot,t)$ contains a ball B_{ρ_0} for $t \in [1,2]$. Moreover, we may assume $\Omega(0)$ is contained in $B_{\rho_0/2}$ for large λ. Then, a simple reflection argument used in [1], Proposition 2.1, implies that there is a uniform cone of directions $\mathcal{C} = \{\alpha \in S^{N-1} : \text{angle}\,(\alpha, x/|x|) < \pi/2 - \eta_0\}$ such that the function $v(x,t)$ decreases in any direction from $\mathcal{C}$ for x with $|x| > \rho_0/2$. Here $\eta_0 > 0$ is a uniform constant, which can be made as small as we wish if we take λ sufficiently large. This, together with the uniform nondegeneracy of the gradient of $v(\cdot,t)$ near $\partial\Omega(t)$, implies that there exist uniform positive constants δ_0 and c_0 such that

$$\partial_e v(x,t) \le -c_0$$

for $x \in B_{\delta_0}(x_0)$, where $x_0 \in \partial\Omega(t)$ and $e = x_0/|x_0|$.

Let us now fix $(x_0,t_0) \in \partial\{v>0\}$ with $t_0 \in (1,2)$. Denote $e = x_0/|x_0|$ and without loss of generality assume that $\nabla v(x_0,t_0)$ is directed along $e_N = (0,0,\dots,1)$. Since e is the axis of the cone of monotonicity $\mathcal{C}$ with opening $\pi/2-\eta_0$, we have that angle$(e,-e_N) \le \eta_0$ and therefore, if λ is sufficiently large and η_0 is small, we will have

$$\partial_{e_N} v(x,t) \ge c_0$$

in $B_{\delta_0}(x_0)$ (with possibly different c_0, δ_0 than before.) Consider now the mapping $(x,t) \mapsto (y,t) = (x', v(x,t), t)$ defined in a small neighborhood

$$V_0 = B_{\delta_0}(x_0) \times (t_0-\delta_0, t_0+\delta_0) \cap \overline{\{v>0\}}$$

into a subset W_0 of $\{(y,t) : y_N > 0\}$ which is open in the relative topology of the half-space and contains the point $(0,t_0)$. The Jacobian of this mapping is $\partial_{e_N} v \ge c_0 > 0$ and hence by the implicit function theorem there exist an inverse mapping $(y,t) \mapsto (x,t) = (y', w(y,t), t)$, where the functions w and v are related through the identity

$$x_N = w(x', v(x,t), t). \tag{4.10}$$

Differentiating (4.10), we find

$$v_{x_N} = \frac{1}{w_{y_N}}, \quad v_{x_i} = -\frac{w_{y_i}}{w_{y_N}},\ i = 1,2,\dots,N-1, \quad v_t = -\frac{w_t}{w_{y_N}}. \tag{4.11}$$

and using the differentiation rules

$$\partial_{x_N} = \frac{1}{w_{y_N}}\,\partial_{y_N}, \quad \partial_{x_i} = \partial_{y_i} - \frac{w_{y_i}}{w_{y_N}}\,\partial_{y_N} \tag{4.12}$$

one can deduce an equation for w from the equation (1.4) for v:

$$\begin{aligned} -\frac{w_t}{w_{y_N}} = \frac{p-1}{p-2}\, y_N \Bigg[& \frac{1}{w_{y_N}} \left(a^{p-2}\,\frac{1}{w_{y_N}}\right)_{y_N} - \left(a^{p-2}\,\frac{w_{y_i}}{w_{y_N}}\right)_{y_i} + \\ & + \frac{w_{y_i}}{w_{y_N}} \left(a^{p-2}\,\frac{w_{y_i}}{w_{y_N}}\right)_{y_i} \Bigg] + a^p, \end{aligned} \tag{4.13}$$

where

$$a = a\left(\nabla_y w\right) = |\nabla_x v| = \frac{\sqrt{1+|\nabla_{y'} w|^2}}{w_{y_N}}. \tag{4.14}$$

After the simplification, the equation above can be rewritten in the form

$$w_t = c_p\, y_N\, (a^{p-2} w_{y_i})_{y_i} - c_p\, y_N^{-\sigma} \left(y_N^{1+\sigma} a^{p-2}\, \frac{1+|\nabla_{y'} w|^2}{w_{y_N}} \right)_{y_N}, \tag{4.15}$$

where

$$c_p = \frac{p-1}{p-2} \quad \text{and} \quad \sigma = -\frac{1}{p-1} > -1. \tag{4.16}$$

To the equations of type (4.15) one can apply the regularity theory of Koch [20]. What follows is mainly a modification of the proof of Theorem 5.6.1 in [20]. We show first that the derivatives w_{y_i} are C^α for $i = 1, \dots, N-1$ and then we prove C^α-regularity of w_{y_N}.

Let g be a difference quotient of w in a direction tangential to the boundary. Then it satisfies

$$g_t = y_N (\mathcal{A}^{ij} g_{y_j})_{y_i} + y_N^{-\sigma} \left(y_N^{1+\sigma} \mathcal{A}^{Nj} g_{y_j} \right)_{y_N} \tag{4.17}$$

where $\mathcal{A}^{kj}$ is uniformly elliptic. Then by [20], Theorem 4.5.5, g are uniformly C^α, hence so are the derivatives w_{y_i}, $i = 1, \dots, N-1$.

Next, the derivative $g = w_{y_N}$ satisfies an equation

$$g_t = y_N (\mathcal{B}^{ij} g_{y_j})_{y_i} + y_N^{-1-\sigma} \left(y_N^{2+\sigma} \mathcal{B}^{Nj} g_{y_j} \right)_{y_N} + c_p (a^{p-2} w_{y_i})_{y_i}. \tag{4.18}$$

Applying now [20], Theorem 4.5.6, we find constants C, $c > 0$ such that on any cube $Q_h = Q'_h(0) \times [0, 2h] \times [t^0 - h, t^0 + h]$ with closure contained in W_0 we have

$$||g||_{C^\alpha(Q_h)} \le C + c \sum_{i=1}^{N-1} ||a^{p-2} w_{y_i}||_{C^\alpha(Q_h)}, \tag{4.19}$$

where

$$a = a(\nabla_y w) = \frac{\sqrt{1 + |\nabla_{y'} w|^2}}{w_{y_N}}. \tag{4.20}$$

We can rewrite

$$a^{p-2} w_{y_i} = f^i(\nabla_{y'} w) w_{y_N}^{2-p}$$

where $f^i(\nabla_{y'} w) = \left(1 + |\nabla_{y'} w|^2\right)^{\frac{p-2}{2}} w_{y_i}$ will be C^α and moreover

$$f^i(\nabla_{y'} w)\big|_{(0,t^0)} = 0, \qquad i = 1, \dots, N-1. \tag{4.21}$$

Next, we can estimate

$$\begin{aligned} ||a^{p-2} w_{y_i}||_{C^\alpha(Q_h)} &= ||f^i(\nabla_{y'} w) w_{y_N}^{2-p}||_{C^\alpha(Q_h)} \\ &\le C_1 ||w_{y_N}||_{C^\alpha(Q_h)} ||f^i||_{L^\infty(Q_h)} + C_2 ||f^i||_{C^\alpha(Q_h)}. \end{aligned} \tag{4.22}$$

If we now take h sufficiently small, so that

$$||f^i||_{L^\infty(Q_h)} < \varepsilon_0$$

(which is possible by (4.21)) we will obtain

$$||a^{p-2} w_{y_i}||_{C^\alpha(Q_h)} \le C_1 \varepsilon_0 ||w_{y_N}||_{C^\alpha(Q_h)} + C_3. \tag{4.23}$$

Substituting this estimate into (4.19), we obtain

$$||w_{y_n}||_{C^\alpha(Q_h)} \le C_4 + C_5\varepsilon_0 ||w_{y_n}||_{C^\alpha(Q_h)} \tag{4.24}$$

and taking h small enough so that $C_5\varepsilon_0 < 1/2$ we find

$$||w_{y_N}||_{C^\alpha(Q_h)} \le C_6. \tag{4.25}$$

which proves that w, and consequently v, is $C^{1,\alpha}$.

4.3 C^∞-regularity of the pressure

To prove the C^∞-regularity of v we should basically iterate the argument for the $C^{1,\alpha}$-regularity. That is, we should take successive derivatives of the equation (4.15) first in the directions e_i, $i = 1, 2, \ldots, N-1$ and then in e_N. The new terms, that will appear in the equation, will be of the form $f + \sum \partial_j (y_N f^j)$ with f and f^j already known to be C^α. For more details we refer to Koch's paper [20], proof of Theorem 5.6.1. The result that can be proved is as follows.

Proposition 4.1 *There exist a uniform neighborhood U of $(0, t_0)$ in $\mathbf{R}^{N-1} \times [0, \infty) \times \mathbf{R}$ such that $w_\lambda \in C^\infty(U)$ and $||w_\lambda||_{C^\ell_{x,t}(U)} < C_\ell$ for $\ell > 0$ and $\lambda > \lambda_0$.*

We also find that our $C^{1,\alpha}$-estimate is enough to use the Schauder-type estimates in Daskalopoulos–Hamilton [10] for higher regularity. In [10], they assumed weighted $C^{2,\alpha}_\delta$ regularity of the initial data to get a degenerate equation with Hölder coefficient in a fixed domain after a global change of coordinates. On the other hand those assumptions are not necessary in our case, since we just make a local argument. In other words, $C^{1,\alpha}$-regularity of v gives us the same type of degenerate equation (4.17) with Hölder coefficient.

5 Convergence to the Barenblatt solution

In this section we prove Theorems 3.1 and 3.2.

After the inverse change of variables, the C^∞ estimate in Proposition 4.1 implies that the uniform convergence of $v_\lambda(x,t)$ as $\lambda \to \infty$ to the self-similar $V(x,t) = t^{-k\frac{p-2}{p-1}} G(xt^{-k/N})$, $t \in [1,2]$, is in fact a convergence in C^ℓ norm for every $\ell > 0$ near the interface $\partial\{G(xt^{-k/N}) > 0\}$. The latter is understood in the sense that for large λ there exists a C^∞ function $h_\lambda(x,t)$ such that

$$v_\lambda(x,t) = t^{-k\frac{p-2}{p-1}} G\left(xt^{-k/N} + h_\lambda(x,t)x/|x|\right) \tag{5.1}$$

in the δ-neighborhood of interface $\mathcal{N}_\delta = \{(x,t) : \mathrm{dist}(x, \partial\Omega_\lambda(t)) < \delta,\ t \in [1,2]\}$, where $\Omega_\lambda(t) = \{v_\lambda(\cdot, t) > 0\}$, and

$$||h_\lambda(x,t)||_{C^\ell_{x,t}(\mathcal{N}_\delta)} \to 0 \qquad \text{as } \lambda \to \infty \tag{5.2}$$

for every $\ell > 0$. As a consequence of this convergence (in fact $\ell = 2$ is enough) and the identity

$$\partial_e \left(|\nabla G|^{p-2} \partial_e G \right) = -q^{p-1}, \qquad q = q(N,p) > 0 \tag{5.3}$$

we will have that

$$\partial_e \left(|\nabla v_\lambda|^{p-2} \partial_e v_\lambda \right) < 0 \tag{5.4}$$

in $\mathcal{N}_\delta$ for large λ. Moreover, we claim that the inequality (5.4) is true in $\Omega_\lambda(t) \setminus B_\varepsilon$ for any small ball B_ε centered at the origin, if we take λ sufficiently large. Indeed, if $\operatorname{dist}(x, \partial\Omega_\lambda(t)) \geq \delta$, we will have $v_\lambda(x,t) \geq \eta > 0$ and the uniform $C^{1,\alpha}$ regularity of the density $u_\lambda(\cdot,t)$ (see [11]) will imply the uniform $C^{1,\alpha}$ regularity of $v_\lambda(\cdot,t)$ in $\{v_\lambda(\cdot,t) \geq \eta\}$. In particular, $v_\lambda(\cdot,t)$ will converge to $V(\cdot,t)$ in $C^{1,\beta}$ norm. Observe now that for the Barenblatt solution we have $|\nabla V(x,t)| > 0$ for $|x| > 0$, hence the equation (1.4) for v_λ is uniformly parabolic on $\{(x,t) : v_\lambda(x,t) \geq \eta, 1 \leq t \leq 2\} \setminus B_\varepsilon \times [1,2]$ for sufficiently large λ and therefore

$$||v_\lambda(\cdot,t) - V(\cdot,t)||_{C^\ell(\{v_\lambda \geq \eta\} \setminus B_\varepsilon)} \to 0 \qquad \text{as } \lambda \to \infty \tag{5.5}$$

for every $\ell > 0$, see e.g. [21]. As a consequence, we obtain that (5.4) holds in $\Omega(t) \setminus B_\varepsilon$ for sufficiently large λ.

Proof of Theorem 3.1. The proof follows from (5.1)–(5.2) and (5.5). □

Proof of Theorem 3.2. The proof follows from (5.1)–(5.2), (5.3), (5.4) in $\Omega(t) \setminus B_\varepsilon$, and the Lemma 5.1 below by rescaling v_λ back to v. □

Lemma 5.1 *Let $w(x)$ be a C^2 function in an open set U of $\mathbf{R}^n$ such that*

$$Z_e := \partial_e(|\nabla w|^{p-2} \partial_e w) < 0$$

for any spatial direction e. Then $w(x)$ is locally concave in U.

Proof. For a spatial unit vector e define

$$Z_e = |\nabla w|^{p-2} w_{ee} + (p-2)|\nabla w|^{p-4} (\nabla w \cdot \nabla w_e) w_e.$$

Let now $x_0 \in U$ and choose the spatial coordinate system so that the matrix $D^2 w(x_0)$ is diagonal and let e be directed along one of the coordinate axes. Then, at x_0 we have

$$\begin{aligned} Z_e &= |\nabla w|^{p-2} w_{ee} + (p-2)|\nabla w|^{p-4} w_e^2 w_{ee} \\ &= (|\nabla w|^{p-2} + (p-2)|\nabla w|^{p-4} w_e^2) w_{ee}. \end{aligned}$$

Now, since $|\nabla w|^{p-2} + (p-2)|\nabla w|^{p-4} w_e^2 \geq (p-1)|\nabla w|^{p-4} w_e^2$ is always nonnegative, $Z_e < 0$ implies $w_{ee} < 0$. This proves that the eigenvalues of $D^2 w(x_0)$ are nonpositive and the lemma follows. □

6 Convexity in fast diffusion, $1 < p < 2$, $N = 1$

In this section we work in dimension one and for $p \in (1, 2)$. The equation is called in this exponent range the fast diffusion p-Laplacian evolution equation in analogy with the porous medium equation with $m \in (0, 1)$. In contrast to the case $p > 2$ the equation does not have the finite propagation property and the density becomes positive everywhere for $t > 0$.

In this case there is a problem with the definition (1.3) of the pressure v, since it becomes negative. We prefer therefore redefine it as

$$v = \frac{p-1}{2-p}\, u^{-\frac{2-p}{p-1}}.$$

Now it is positive and in dimension $N = 1$ satisfies

$$v_t = c_p v\, (|v_x|^{p-2} v_x)_x - |v_x|^p, \qquad c_p = \frac{2-p}{p-1} \tag{6.1}$$

Next, we know that v is $C^{1,\alpha}$ and that it is close to the Barenblatt profile after we pass to the rescaled solutions v_λ. The convergence is uniform away from $x = 0$ so we assume that v is close in C^2 (hence, convex) in any compact set except a small neighborhood of 0.

To prove the convexity of v in a small neighborhood of the origin, it is enough to prove that $Z = (|v_x|^{p-2} v_x)_x > 0$, as one can see from an obvious generalization of Lemma 5.1. As a starting point we mention the following estimate by Esteban and Vázquez [16]

$$-\frac{K_1}{t} \le Z \le \frac{K_2}{t} \tag{6.2}$$

for some positive constants K_1 and K_2 depending only on p.

We introduce an auxiliary function $U = |v_x|^{p-2} v_x$ so that we have $Z = U_x$. We are going to derive equations for U and Z, but the problem is that these quantities are not generally smooth, so we have to use a regularization. It can be done as in [16], or as we do below.

For a given $\varepsilon > 0$ consider the solutions v^ε of the approximating equation

$$v_t = c_p v\, (f^\varepsilon(v_x))_x - g^\varepsilon(v_x), \tag{6.3}$$

where

$$f^\varepsilon(s) = (s^2 + \varepsilon)^{\frac{p-2}{2}} s \tag{6.4}$$

$$g^\varepsilon(s) = (s^2 + \varepsilon)^{\frac{p-2}{2}} \left(s^2 + (2 - 1/(p-1))\, \varepsilon\right). \tag{6.5}$$

Since the equation (6.3) is locally uniformly parabolic, the solutions v^ε are C^∞ and taking ε small enough we can assume that v^ε are sufficiently close to the pressure v in $C^{1,\alpha}$ norm on compact subsets of Q. Next, we introduce

$$U^\varepsilon = f^\varepsilon(v^\varepsilon_x), \qquad Z^\varepsilon = U^\varepsilon_x. \tag{6.6}$$

Differentiating (6.3) with respect to x and multiplying by $(f^\varepsilon)'(v^\varepsilon)$ we find the equation for U^ε

$$\begin{aligned} U_t^\varepsilon &= c_p v^\varepsilon (f^\varepsilon)'(v_x^\varepsilon)\, U_{xx}^\varepsilon + [c_p v_x^\varepsilon (f^\varepsilon)'(v_x^\varepsilon) - (g^\varepsilon)'(v_x^\varepsilon)]\, U_x^\varepsilon \\ &= a^\varepsilon(x,t)\, U_{xx}^\varepsilon + b^\varepsilon(x,t)\, U_x^\varepsilon \end{aligned} \tag{6.7}$$

$$a^\varepsilon(x,t) = c_p v (v_x^2+\varepsilon)^{\frac{p-4}{2}}((p-1)v_x^2+\varepsilon) \tag{6.8}$$

$$b^\varepsilon(x,t) = -2(p-1)v_x(v_x^2+\varepsilon)^{\frac{p-2}{2}}. \tag{6.9}$$

Differentiating now (6.7), we obtain the equation for Z^ε:

$$Z_t^\varepsilon = a^\varepsilon\, Z_{xx}^\varepsilon + \tilde{b}^\varepsilon\, Z_x^\varepsilon - 2(p-1)(Z^\varepsilon)^2, \tag{6.10}$$

where a^ε as above, $\tilde{b}^\varepsilon = 2c_p v_x (f^\varepsilon)'(v_x) + c_p v v_{xx}(f^\varepsilon)''(v_x) - (g^\varepsilon)'(v_x)$. In computation we used the identity

$$c_p(f^\varepsilon)''(s) - (g^\varepsilon)''(s) = C_p(f^\varepsilon)'(s), \quad C_p = 2-2p-c_p.$$

Consider now Z^ε in a rectangle $\mathcal{R} = (-r,r)\times(1,2)$ and assume that $v = v_\lambda$ is the rescaled pressure. From the C^∞ convergence of v_λ to the Barenblatt solution on every compact K separated from 0, we have that $Z_\lambda \geq 2\delta_0 > 0$ on $\{-r,r\}\times[1,2]$ for large λ. But then, taking $\varepsilon < \varepsilon(\lambda)$, we can make $Z_\lambda^\varepsilon \geq \delta_0$ on $\{-r,r\}\times[1,2]$. For simplicity we will omit the indexes ε and λ in what follows, if there is no ambiguity. Also, if it is not stated otherwise, the constants that appear below are uniform in ε and λ.

Lemma 6.1 *Suppose that $Z \geq \delta_0 > 0$ on the parabolic boundary of a rectangle $\mathcal{R} = (-r,r)\times(t_1,t_2)$, i.e. on $[-r,r]\times\{t_1\}\cup\{-r,r\}\times[t_1,t_2]$. Then $Z^\varepsilon \geq \delta_1$ in $\mathcal{R}$, where $\delta_1 > 0$ depends only on δ_0, t_1 and t_2.*

Proof. The proof is rather standard and uses the comparison with the stationary solutions of (6.10), i.e. functions $\zeta(t)$ satisfying

$$\zeta' = -2(p-1)\zeta^2.$$

The solutions of this ODE have the form

$$\zeta(t) = \frac{c}{t+t_0}, \qquad c = \frac{1}{2(p-1)} \tag{6.11}$$

and we can choose t_0 very large, so that $\zeta(t) < \delta_0/2$ on $[t_1,t_2]$. Then we claim $Z(x,t) > \zeta(t)$ in $\mathcal{R}$. Indeed, assuming the contrary, let t^* be the minimal $t \in [t_1,t_2]$ such that $Z(x,t) = \zeta(t)$ for some $x \in [-r,r]$. It is clear that $t^* > t_1$ since $Z(x,t_1) \geq \delta_0 > \zeta(t_1)$. Next, let $x^* \in [-r,r]$ be such that $Z(x^*,t^*) = \zeta(t^*)$. Then x^* is an interior point, since for $Z(x,t) \geq \delta_0 > \zeta(t)$ on the lateral boundary $\{-r,r\}\times[t_1,t_2]$. It is easily follows now that

$$Z_x(x^*,t^*) \geq 0, \quad Z_x(x^*,t^*) = 0, \quad Z_t(x^*,t^*) \leq \zeta'(t^*).$$

Here, we actually need to modify $\zeta(t)$ a little bit if we wish to arrive at a contradiction. Let everywhere above $\zeta(t)$ be given by (6.11) but with $c < \frac{1}{2(p-1)}$, so that we have

$$\zeta'(t) = -\frac{1}{c}\zeta^2(t) < -2(p-1)\zeta^2(t).$$

But then the contradiction is immediate:

$$-2(p-1)\zeta^2(t^*) > \zeta'(t^*) \geq Z_t(x^*, t^*) \geq -2(p-1)Z^2(x^*, t^*),$$

where in the last inequality we have used the equation (6.10) for Z. Hence $Z(x,t) > \zeta(t)$ in $\mathcal{R}$ and the lemma follows. $\square$

We are thus left with the proof of the strict p-convexity (i.e. the positivity of Z) at some time. We make a second-order estimate for U, namely an estimate for

$$I = \iint Z_x^2 \, dxdt = \iint U_{xx}^2 \, dxdt.$$

We multiply the equation (6.7) by U_{xx} and integrate by parts in a rectangle $\mathcal{R} = (-r, r) \times (1, 2)$ with $r > 0$ small to get

$$\iint a\, U_{xx}^2 \, dxdt = \iint U_t \, U_{xx} \, dxdt - \iint b\, U_x U_{xx} \, dxdt = I_1 + I_2. \tag{6.12}$$

Since b is small, $b = O(r)$, we have

$$|I_2| \leq Cr \left(\iint U_x^2 \, dxdt \right)^{1/2} \left(\iint U_{xx}^2 \, dxdt \right)^{1/2} \leq Cr\, I^{1/2}, \tag{6.13}$$

since $U_x = Z$ is bounded. We estimate the other term as follows

$$\begin{aligned} I_1 = &- \iint U_x U_{xt} \, dxdt + \int_S U_t \, U_x \, dt = \\ &\frac{1}{2} \int U_x(x,1) \, dx - \frac{1}{2} \int U_x(x,2) \, dx + \int_S U_t \, U_x \, dt. \end{aligned} \tag{6.14}$$

Now, the first terms are bounded uniformly as $O(r)$ and the last term is very small when $\lambda \gg 1$ because of the uniform convergence away from $x = 0$ of the rescaled solutions. Summing up, we get

$$Cr^{-(2-p)/(p-1)} \iint U_{xx}^2 \, dxdt \leq \iint a\, U_{xx}^2 \, dxdt \leq C + Cr\, I^{1/2}, \tag{6.15}$$

which means that I is bounded and small. But as an iterated integral it means that for some $t = t_1 \in (1, 2)$ the integral $\int Z_x^2 \, dx$ is small. At that t we obtain

$$|Z - K_1/t| \leq \varepsilon + \int_{-r}^{x} |Z_x| \, dx \leq \varepsilon + r^{1/2} \left(\int Z_x^2 \, dx \right)^{1/2} \leq 2\varepsilon, \tag{6.16}$$

hence $Z \geq c_0 > 0$. Observe that we may assume $t_1 \in (1, \frac{3}{2})$. But then, by Lemma 6.1 we will have that $Z \geq c_0 > 0$ on $[-r, r] \times [\frac{3}{2}, 2]$. In particular, we obtain that $v_\lambda^\varepsilon\left(\cdot, \frac{3}{2}\right)$ is convex in $\mathbf{R}$ for λ very large and $0 < \varepsilon < \varepsilon(\lambda)$, and therefore v_λ is convex everywhere in $\mathbf{R}$. But then, taking $\lambda = \frac{2}{3}t$ this precisely means $v(\cdot, t)$ is convex in $\mathbf{R}$ for large t.

7 Concavity near the origin for $p > 2$, $N = 1$

We now perform the concavity analysis in the dimension $N = 1$ for the slow diffusion case, $p > 2$, and prove the first part of Theorem 3.2′ that the rescaled solutions v_λ are concave near the origin for $\lambda \gg 1$.

As before, concavity of v will follow if we prove that the quantity $Z = (|v_x|^{p-2}v_x)_x$ is nonpositive. In this case we only have a bound from below for Z by Esteban and Vázquez [16].

The proof is similar to the convexity proof in the case of the fast diffusion from the previous section. We consider an auxiliary variable $U = |v_x|^{p-2}v_x$, so that $U_x = Z$. All computations below are formal, but can be justified precisely as we did for the fast diffusion by considering regularizations v^ε, U^ε, and Z^ε.

From the pressure equation

$$v_t = c_p\, v\,(|v_x|^{p-2}v_x)_x + |v_x|^p, \qquad c_p = \frac{p-2}{p-1} \tag{7.1}$$

we obtain that U satisfies

$$\begin{aligned} U_t &= (p-2)\, v\,|v_x|^{p-2}U_{xx} + (2p-2)|v_x|^{p-2}v_x U_x \\ &= a(x,t)\, U_{xx} + b(x,t)\, U_x. \end{aligned} \tag{7.2}$$

Lemma 7.1 *There is a second-order estimate for U of the form*

$$I = \iint a\, Z_x^2\, dxdt = \iint a\, U_{xx}^2\, dxdt \le C. \tag{7.3}$$

Proof. We multiply by U_{xx} and integrate by parts in the rectangle $\mathcal{R} = (-r, r) \times (1, 2)$ with small $r > 0$ to get

$$\iint a\, U_{xx}^2\, dxdt = \iint U_t\, U_{xx}\, dxdt - \iint b\, U_x U_{xx}\, dxdt = I_1 + I_2. \tag{7.4}$$

First, we estimate I_2

$$|I_2| \le 2 \iint \frac{b^2}{a} U_x^2\, dxdt + \frac{1}{2} \iint a U_{xx}^2\, dxdt. \tag{7.5}$$

The quantity b^2/a above equals $C(p)|v_x|^p$, hence $b^2/a = O(r^{\frac{p}{p-1}})$. Also, we know that U_x is L^2 integrable, see [11], Chap. VIII, Proposition 3.1, which implies that

$$|I_2| \le C r^{\frac{p}{p-1}} + \frac{1}{2} I. \tag{7.6}$$

Next, to estimate, I_1 we integrate by parts.

$$\begin{aligned} I_1 = &- \iint U_x U_{xt}\, dxdt + \int_S U_t\, U_x\, dt = \\ &\frac{1}{2} \int U_x(x,1)\, dx - \frac{1}{2} \int U_x(x,2)\, dx + \int_S U_t\, U_x\, dt. \end{aligned} \tag{7.7}$$

Again, since U_x is spatially L^2 integrable (see the reference above), the first two integrals are bounded. The last integral will be bounded since $U_t U_x$ converges uniformly to the corresponding quantity for the Barenblatt solution on S. Hence we obtain that

$$|I_1| \le C \tag{7.8}$$

with C independent of r. Combining the estimates above, we obtain that

$$\iint a\, Z_x^2\, dxdt \le C. \tag{7.9}$$

Lemma is proved. □

Proof of Theorem 3.2′. We should start with a remark that as everywhere else in this section we must work with approximations of U and Z (as well as of a and b) as in the previous section, but for simplicity of the presentation we do formal computations with U and Z. From Theorem 3.2 it follows that for a given $r > 0$ and $\varepsilon > 0$ and large λ we have

$$|Z(x,t) + K/t| < \varepsilon \qquad \text{for } x \in \operatorname{supp} v(\cdot,t) \setminus (-r,r),\ t \in [1,2],$$

where $K = K(p) > 0$. From Lemma 7.1 above it follows that the integral $\int aZ_x^2\, dx$ is bounded for some $t = t_1 \in (1,2)$. At that particular t we have

$$\begin{aligned} |Z(x,t) + K/t| &\le \varepsilon + \int_{-r}^{x} |Z_x|\, dx \\ &\le \varepsilon + \left(\int_{-r}^{x} \frac{1}{a}\, dx\right)^{1/2} \left(\int a\, Z_x^2\, dx\right)^{1/2} \end{aligned} \tag{7.10}$$

and the negativity of Z will follow once we show that

$$\int_{-r}^{x} \frac{1}{a}\, dx = \int_{r}^{x} |v_x|^{2-p}\, dx \to 0 \qquad \text{as } r \to 0. \tag{7.11}$$

An indication that this might work is the fact that $a = |v_x|^{2-p} \simeq |x|^{-(p-2)(p-1)}$, which also suggests that the above quantity should actually be $O(r^{1/(p-1)})$. However, we need to make this precise. We start from a small distance $x = -r$ where the difference is less than ε (sufficiently small) and we integrate in the interval $[-r, x']$ where $x' \in (-r,r)$, for instance, is the first point at $Z = -K/(2t)$. Then $U_x = Z$ will be bounded away from zero and this implies that even if U vanishes at a point $x_0 \in [-r, x']$ (in the worst case) we will still have

$$|U(x,t)| \ge C|x - x_0| \quad \text{ in } I = [-r, x'].$$

Since $a = (p-2)v|U|^{(p-2)/(p-1)} < c(p)|x - x_0|^{(p-2)/(p-1)}$, the above formula (7.11) holds at $x = x'$ and leads to contradiction in the preceding estimate for Z. Indeed, we will have

$$K/(2t) = |Z(x',t) + K/t| \le \varepsilon + Cr^{1/(p-1)}$$

with C depending only on p, which is impossible if r and ε are sufficiently small. Therefore, Z never reaches the level $-K/2t$, and in fact stays near $-K/t$, for this particular $t = t_1 \in (1,2)$. Observe, however, that we may assume $t_1 \in (1, \frac{3}{2})$ and then apply an analogue of Lemma 6.1, which says that if $Z \le -\delta_0 <$ on a parabolic boundary of $(-r,r) \times (t_1,t_2)$ then if fact $Z \le -\delta_1 < 0$ everywhere in $[-r,r] \times [t_1,t_2]$. In our case we obtain $Z \le -c_0 < 0$ in $[-r,r] \times [\frac{3}{2},2]$ and in particular that $v_\lambda\left(\cdot,\frac{3}{2}\right)$ is strictly concave in its positivity set. But then, taking $\lambda = \frac{2}{3}t$, we find that $v(\cdot,t)$ is strictly concave in $\Omega(t) = \{v(\cdot,t) > 0\}$.

The second part of Theorem 3.2′ on the regularity of the curve of maxima is the content of the next section, where we conclude the proof of the theorem.

8 Regularity of the curve of maxima

As we have seen in the previous section, when the space dimension is one, after some moment the pressure $v(\cdot,t)$ will become strictly concave in its positivity set $\Omega(t)$. As a consequence, the function $v(\cdot,t)$ will have only one maximum point. We will denote this point by $\gamma(t)$. Below we show that the result of M. Bertsch and D. Hilhorst [4] on the regularity of the interface in one-dimensional two-phase porous medium equation implies that the curve $x = \gamma(t)$ is $C^{1,\alpha}$ regular. The connection with the porous medium equation is as follows. It is clear that $\gamma(t)$ is also the only maximum point of $u(\cdot,t)$. Moreover, $\gamma(t)$ is the only point, where the derivative u_x crosses the value 0. In other words, the curve $x = \gamma(t)$ separates the regions $\{u_x < 0\}$ from $\{u_x > 0\}$. Finally, the function

$$w(x,t) = u_x(x,t) \tag{8.1}$$

satisfies

$$w_t = (|w|^{p-2}w)_{xx}, \tag{8.2}$$

which is precisely the two-phase the porous medium equation with the parameter $m = p-1$.

Before we proceed, we remark that the C^α regularity of the curve of maxima was proved earlier by Sakaguchi [25]. However, the eventual concavity of the pressure was unknown at that time, and that did not allow to prove $C^{1,\alpha}$ regularity. A related result for the fast diffusion can be found in [26].

Proposition 8.1 *Let w be a solution of* (8.2) *on $(-L,L) \times (t_0,\infty)$ with the assumptions that $w(\cdot,t_0)$ is nonincreasing on $(-L,L)$ and $w(-L,t) = a$, $w(L,t) = -b$ for $t \ge t_0$ for some positive constants a and b. Then the null-set $\mathcal{N}(t) = \{x : w(x,t) = 0\}$ can be described as follows. There exist Lipschitz functions $\gamma_-(t)$ and $\gamma_+(t)$ such that*

$$\mathcal{N}(t) = [\gamma_-(t), \gamma_+(t)], \quad \textit{for } t \ge t_0.$$

and there is $t^* \ge t_0$ such that

(i) $\gamma_-(t) = \gamma_+(t) =: \gamma(t)$ *for* $t \geq t^*$*;*

(ii) $(|w|^{p-2}w)_x = 0$ *on* $\mathcal{N}(t)$ *for* $t \in [t_0, t^*]$ *and* $(|w|^{p-2}w)_x < 0$ *for* $t > t^*$.

Moreover, $\gamma \in C^{1,\alpha}((t^*, \infty))$ *for some* $\alpha \in (0,1)$.

Proof. This is a particular case of [4], Theorem 1.3 (see also Lemma 4.1.) □

Proof of Theorem 3.2′ (continuation.) In order to use Proposition 8.1 for $w = u_x$ we must prove that $w(\cdot, t_0)$ is nonincreasing on $(-L, L)$ for small L and large t_0. We actually consider the rescaled solutions $u_\lambda(x,t)$ for on $\mathcal{R} = (-r, r) \times (1,2)$ and respectively defined $w_\lambda = (u_\lambda)_x$. Then (omitting λ)

$$(|w|^{p-2}w)_x = C_p \left(v^{\frac{p-1}{p-2}} |v_x|^{p-2} v_x \right)_x = C_p v^{\frac{1}{p-2}} \left(c_p |v_x|^p + v(|v_x|^{p-2}v_x)_x \right)$$

and therefore for small r and large λ we have

$$(|w_\lambda|^{p-2} w_\lambda)_x < 0 \quad \text{on } \mathcal{R} = (-r, r) \times (1,2).$$

Indeed, this simply follows from the fact that for large λ we have $(|v_x|^{p-2}v_x)_x < -C(p) < 0$ and for small $r > 0$ v_x is small and v is like a positive constant. As a consequence, we obtain also that $w_\lambda(\cdot, 1)$ is nonincreasing on $(-r, r)$. Also for $t \in [1,2]$ $w_\lambda(-r, t) > 0$ and $w_\lambda(r, t) < 0$. Even though $w_\lambda(-r, t)$ and $w_\lambda(r, t)$ are not constants, (but separated) from 0, the conclusion of Proposition 8.1 above holds, since this condition is not essential for the proof. Moreover, we can take $t^* = 1$, since we proved that $(|w_\lambda|^{p-2}w_\lambda)_x < 0$ in $\mathcal{R}$. Scaling w_λ back to w we obtain that the curve $x = \gamma(t)$ is $C^{1,\alpha}$ regular, where $\gamma(t)$ is the only maximum point of the pressure v at time t, for $t \geq t_0$. This finishes the proof of Theorem 3.2′. □

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