ON EXISTENCE AND UNIQUENESS IN A FREE BOUNDARY PROBLEM FROM COMBUSTION

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ABSTRACT. We study a free boundary problem for the heat equation describing the propagation of laminar flames under certain geometric assumptions on the initial data. The problem arises as the limit of a singular perturbation problem, and generally no uniqueness of limit solutions can be expected. However, if the initial data is starshaped, we show that the limit solution is unique and coincides with the minimal classical supersolution. Under certain convexity assumption on the data, we prove first that the limit solution is a classical solution of the free boundary problem for a short time interval, and then that the solution, in fact, stays classical as long as it does not vanish identically.

1. INTRODUCTION

In this paper we consider a free boundary problem for the heat equation, which consists of finding a nonnegative continuous function u in $Q_T = \mathbf{R}^n \times (0,T)$, $T > 0$, such that

$$(P) \qquad \begin{cases} \Delta u - u_t = 0 \quad \text{in } \Omega = \{u > 0\} \\ |\nabla u| = 1 \quad \text{on } \partial\Omega \cap Q_T, \text{ and} \\ u(\cdot, 0) = u_0, \end{cases}$$

with given nonnegative initial function $u_0 \in C_0(\mathbf{R}^n)$. (Here $\Delta = \Delta_x$ and $\nabla = \nabla_x$.) The problem P arises in modeling the propagation of laminar flames as the limit of the singular perturbation problem (see [1] and, for further detail in combustion theory, [2])

$$(P_\varepsilon) \qquad \begin{cases} \Delta u^\varepsilon - u^\varepsilon_t = \beta_\varepsilon(u^\varepsilon) \\ u^\varepsilon(\cdot, 0) = u_{0,\varepsilon}, \end{cases}$$

as $\varepsilon \to 0+$, where $u_{0,\varepsilon}$ approximate u_0 in a proper way, $\beta_\varepsilon \geq 0$, $\beta_\varepsilon(s) = (1/\varepsilon)\beta(s/\varepsilon)$, with β a Lipschitz function, support $\beta = [0,1]$, and

$$(1.1) \qquad \int_0^1 \beta(s)ds = \frac{1}{2}.$$

The family of solutions $\{u^\varepsilon\}$ is uniformly bounded in $C^{1,1/2}_{x,t}$-norm on compact subsets of Q_T and every locally uniform limit $u = \lim_{j\to\infty} u^{\varepsilon_j}$ in Q_T of its subsequence with $\varepsilon_j \to 0$ is a solution of P in a certain weak sense, see [1]. We will refer to these limits as *limit solutions of* P.

In the two-phase case, i.e. when there is no sign restriction on u, problems P_ε approximate a free boundary problem, where the fixed gradient condition $|\nabla u| = 1$

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is replaced by the gradient jump condition $|\nabla u^+|^2 - |\nabla u^-|^2 = 1$. Limit solutions of this problem were studied detailly from the local point of view in [3, 4]. It was shown that a limit solution u of P is a *viscosity solution* in a domain $\mathcal{D}$ if $\{u = 0\}^\circ \cap \mathcal{D} = \emptyset$. For the one-phase problem, and recently for the two-phase problem as well, three concepts of solutions, limit, viscosity and classical, were shown to agree with each other to produce a unique solution under certain conditions on u_0 (see [5]), that guarantee the existence of the classical solutions. In this paper we provide another uniqueness theorem (Theorem 2.7) for limit solutions for starshaped u_0, see (S) from Section 2. In fact, we prove that the unique limit solution coincides with the minimal classical supersolution of P (see Definition 2.1) in this case.

The existence and analyticity of classical solutions of P, among other things, were proved in [6] in the case when the initial data is radially symmetric, by using the elliptic-parabolic approach. The classical solutions to P (at least to the problems similar to P), in special settings, were constructed in [7] and [8].

The most part of this paper is devoted to the proof of a short time existence theorem of classical solutions of P under certain convexity assumptions (Theorem 8.1), namely, when $\Omega_0 = \{u_0 > 0\}$ is convex and u_0 is log-concave and superharmonic in Ω_0. We consider the minimal element among the classical supersolutions of P that have the following geometric property, expected from the classical solution: the time sections $\Omega(t) = \{u(\cdot, t) > 0\}$ are convex domains shrinking in time; see [9], [1]. We prove that the minimal supersolution with this property has Lipschitz (in space and in time) lateral boundary for the short time interval, which enables us to apply a technique due to A. Henrot and H. Shahgholian [10, 11], to show that it is a classical solution of P.

Finally, in the last section (see Theorem 9.3) we prove that the unique limit solution stays classical up to the *extinction time*, which in combustion model corresponds to the time when the unburnt zone collapses. In this regard, we also want to mention a recent paper by P. Daskalopoulos and K. Lee [12].

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2. Uniqueness in the starshaped case

Throughout the paper we will assume that u_0 is a nonnegative continuous function in $\mathbf{R}^n$ with a compact support. In particular, $\Omega_0 = \{u_0 > 0\}$ is bounded.

Definition 2.1. A pair (u, Ω), where u is a compactly supported nonnegative continuous function in $\overline{Q}_T = \mathbf{R}^n \times [0, T]$, $T > 0$, and $\Omega = \{u > 0\}$, is called a *(classical) supersolution* of P if

(i) $\Delta u - u_t = 0 \quad \text{in } \Omega$;
(ii) $\limsup_{\Omega \ni (y,s) \to (x,t)} |\nabla u(y, s)| \le 1 \quad \text{for every } (x, t) \in \partial\Omega \cap Q_T$;
(iii) $u(\cdot, 0) \ge u_0$.

Respectively, a pair (u, Ω) is a *subsolution* of P if conditions (ii) and (iii) are satisfied with opposite inequality signs and $\liminf$ instead of $\limsup$ in (ii).

A pair (u, Ω) is a *classical solution* of P if it is sub- and supersolution of P at the same time.

Next, a supersolution (u, Ω) of P is a *strict supersolution* of P if there is a $\delta > 0$ such that the stronger inequalities

(ii') $\limsup_{\Omega\ni(y,s)\to(x,t)} |\nabla u(y,s)| \le 1-\delta$ for every $(x,t)\in\partial\Omega\cap Q_T$;
(iii') $u(\cdot,0)\ge u_0+\delta$ on $\Omega_0=\{u_0>0\}$

hold. Analogously the *strict subsolutions* are defined.

Remark 2.2. According to Theorem 6.1 in [3], every limit solution $u=\lim_{j\to\infty} u^{\varepsilon_j}$ of P is its classical supersolution in the sense of Definition 2.1. Generally, for the two-phase problem we have that if u is a limit solution of P then ($u^+=\max\{u,0\}$ and $u^-=\min\{u,0\}$)

$$\limsup_{(y,s)\to(x,t)} |\nabla u^-(y,s)| \le \gamma$$

implies

$$\limsup_{(y,s)\to(x,t)} |\nabla u^+(y,s)| \le \sqrt{1+\gamma^2}$$

for every free boundary point $(x,t)\in\partial\{u>0\}\cap Q_T$. When $u\ge 0$, we have $u^-\equiv 0$, hence one can take $\gamma=0$.

Remark 2.3. Suppose that the initial function u_0 is *starshaped* with respect to a point x_0 in the following sense:

$$\text{(S)} \qquad u_0(\lambda x+x_0)\ge u_0(x+x_0) \qquad \text{for every } \lambda\in(0,1) \text{ and } x\in\mathbf{R}^n,$$

or, equivalently, all the level sets $\mathcal{L}_s(u_0)=\{u_0>s\}$ are starshaped with respect to the same point x_0. In the sequel, we will always assume $x_0=0$.

Let (u,Ω) be a supersolution of P. Let λ and λ' be two real numbers with $0<\lambda<\lambda'<1$. Define

$$u_\lambda(x,t)=(1/\lambda')u(\lambda x,\lambda^2 t) \tag{2.1}$$

in Q_{T/λ^2}. The rescaling of variables is taken so that u_λ, like u, satisfies the heat equation in its positivity set

$$\Omega_\lambda=\{(x,t):(\lambda x,\lambda^2 t)\in\Omega\}. \tag{2.2}$$

Moreover, the selection $\lambda<\lambda'<1$ makes the pair $(u_\lambda,\Omega_\lambda)$ not only a supersolution of P, but also a *strict* supersolution.

Lemma 2.4. *Let the initial function u_0 satisfy condition (S). Then every subsolution of P is smaller than every supersolution of P.*

In this lemma and further in the paper we say that a pair (u',Ω') is smaller than (u,Ω) if $\Omega'\subset\Omega$ and $u'\le u$.

Proof. Let (u,Ω) be a supersolution and (u',Ω') a subsolution of P in Q_T. We need to prove only that $\Omega'\subset\Omega$; the inequality $u'\le u$ will follow from this inclusion by the maximum principle.

In the case when $u\in C^1(\overline{\Omega})$ and $u'\in C^1(\overline{\Omega'})$, the statement can be proved by the Lavrent'ev rescaling method as follows. Suppose

$$\lambda_0=\sup\{\lambda\in(0,1):\Omega'\subset\Omega_\lambda\}<1, \tag{2.3}$$

where Ω_λ as in (2.2). Then $\Omega'\subset\Omega_{\lambda_0}$ and there is a common point $(x_0,t_0)\in\partial\Omega'\cap\partial\Omega_{\lambda_0}\cap Q_T$. Let $\lambda_0<\lambda_0'<1$ and u_{λ_0} be as in (2.1). Then $u'\le u_{\lambda_0}$ in Ω'. At the common point (x_0,t_0) this inequality implies $\partial_\nu u'(x_0,t_0)\le\partial_\nu u_{\lambda_0}(x_0,t_0)$, where ν is the inward spatial normal vector for both $\partial\Omega'$ and $\partial\Omega_{\lambda_0}$ at (x_0,t_0) (recall that we are in C^1 case.) This leads to a contradiction, since $\partial_\nu u'(x_0,t_0)=|\nabla u'(x_0,t_0)|\ge 1$ and $\partial_\nu u_{\lambda_0}(x_0,t_0)=|\nabla u_{\lambda_0}(x_0,t_0)|=\frac{\lambda_0}{\lambda_0'}<1$. Therefore $\lambda_0=1$ and $\Omega'\subset\Omega$.

The general case can be reduced to the considered regular case by the following procedure. Let $(\widetilde{u},\widetilde{\Omega})$ be a supersolution. Choose $0<\lambda<\lambda'<1$ close to 1 and regularize $\widetilde{u}$ by setting

$$u(x,t)=(1/\lambda')(\widetilde{u}(\lambda x,\lambda^2 t+h)-\eta)^+$$

for small $h,\ \eta>0$. Analogously regularize a subsolution $(\widetilde{u'},\widetilde{\Omega'})$ (with possibly different parameters, in order to keep the inequality for $t=0$.) Then we will arrive in the considered regular case and can finish the proof by letting first $h,\ \eta\to 0+$ and then $\lambda\to 1-$. □

The following proposition opens a way to prove the uniqueness results for limit solutions of P.

Proposition 2.5. *Let $\widetilde{u}$ be a strict supersolution of P in Q_T, $T>0$, and u^ϵ solutions of P_ε, where $u_{0,\varepsilon}$ are nonnegative uniform approximations of u_0 and* support $u_{0,\varepsilon}\to$ support u_0. *Then*

$$\limsup_{\varepsilon\to 0+} u^\varepsilon(x,t)\le \widetilde{u}(x,t) \tag{2.4}$$

for every $(x,t)\in Q_T$.

Proof. Consider the ordinary differential equation

$$\phi^\varepsilon_{ss}(s)=\gamma_\varepsilon(\phi^\varepsilon(s)), \tag{2.5}$$

where $\gamma_\varepsilon(s)=(1/\varepsilon)\gamma(s/\varepsilon)$ and γ is obtained from β in P_ε by

$$\gamma(s)=\begin{cases} c\beta(s) & s\in[a,1]\\ 0 & s\notin[a,1].\end{cases} \tag{2.6}$$

Here $a\in(0,1)$ (to be specified later) and the constant $c>1$ is chosen so that

$$\int_a^1\gamma(s)ds=\frac{1}{2}, \tag{2.7}$$

cf. (1.1). Let us consider the solution ϕ^ε of (2.5), normalized by

$$\phi^\varepsilon(s)=a\varepsilon\quad\text{for } s\le 0,\quad\text{and}\quad \phi^\varepsilon(s)>a\varepsilon\quad\text{for } s>0. \tag{2.8}$$

The family $\{\phi^\varepsilon\}$ is recovered from a single function ϕ, a solution of

$$\phi_{ss}=\gamma(\phi)$$

with the appropriate normalization, through the relation $\phi^\varepsilon(s)=\varepsilon\phi(s/\varepsilon)$. Using

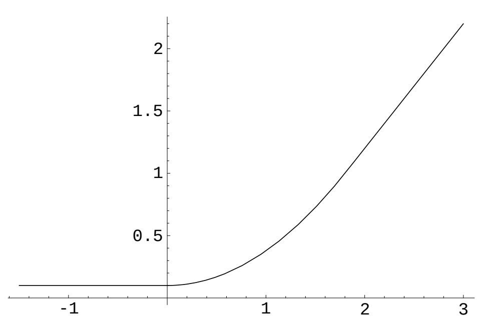

FIGURE 1. Profile of $\phi(s)$

this, we can find a constant $C > 0$ such that

$$\phi^\varepsilon(s) = (s - C\varepsilon) + \varepsilon \quad \text{for } s \geq C\varepsilon; \tag{2.9}$$

or in other words, that ϕ^ε is a linear function with slope 1, for $s \geq C\varepsilon$. Indeed, multiply both sides of (2.5) by ϕ^ε_s and integrate from $-\infty$ to s. We will obtain

$$(\phi^\varepsilon_s)^2 = 2\Gamma_\varepsilon(\phi^\varepsilon), \tag{2.10}$$

where $\Gamma_\varepsilon(s) = \int_a^s \gamma_\varepsilon(\sigma)d\sigma$. By (2.7) $\Gamma_\varepsilon(s) = 1/2$ for $s \geq \varepsilon$ and therefore ϕ^ε will become linear function with slope 1 as it reaches ε. Now (2.9) follows with C satisfying $\phi(C) = 1$. Moreover, (2.8)–(2.10) imply that

$$(s - C\varepsilon) + \varepsilon \leq \phi^\varepsilon(s) \leq s + a\varepsilon \quad \text{for } s \geq 0. \tag{2.11}$$

Let now $\widetilde{u}$ be a strict supersolution of P and consider the following regularization

$$u(x,t) = (\widetilde{u}(x, t + h) - \eta)^+,$$

for $h, \eta > 0$ small. Without loss of generality we may assume that there is $\delta \in (0, 1)$ such that

$$|\nabla u| \leq 1 - \delta \quad \text{if } u < \eta \qquad \text{and } u(\cdot, 0) > u_0 + \delta. \tag{2.12}$$

Consider now the compositions

$$w^\varepsilon(x,t) = \phi^\varepsilon(u(x,t))$$

for sufficiently small $\varepsilon > 0$. Since $\phi^\varepsilon(s)$ is constant for $s \leq 0$, we can rewrite $w^\varepsilon = \phi^\varepsilon(\tilde{u}(x, t + h) - \eta)$. Therefore w^ε is well-defined $C^{1,1}(\overline{Q}_{T-h})$ function, and satisfies

$$\Delta w^\varepsilon - w^\varepsilon_t = \phi^\varepsilon_s(u)[\Delta u - u_t] + \phi^\varepsilon_{ss}(u)|\nabla u|^2$$

in Q_{T-h}. The first term in the right-hand side is 0 everywhere. If now the value of a in the definition (2.6) is chosen so that

$$\int_a^1 \beta(s)ds = \frac{1}{2}(1 - \delta)^2,$$

then the constant c in the same definition equals $(1 - \delta)^{-2}$ and we obtain

$$\Delta w^\varepsilon - w^\varepsilon_t = \phi^\varepsilon_{ss}(u)|\nabla u|^2 \leq \gamma_\varepsilon(\phi^\varepsilon(u))(1 - \delta)^2 \leq \beta_\varepsilon(w^\varepsilon),$$

if $\varepsilon > 0$ is so small that $\phi^\varepsilon(s) \in \operatorname{support} \gamma_\varepsilon = [a\varepsilon, \varepsilon]$ implies $s < \eta$, or explicitly $\varepsilon < \eta/C$. Besides, from (2.11), for small values of ε,

$$w^\varepsilon(\cdot, 0) = \phi^\varepsilon(u(\cdot, 0)) \geq u(\cdot, 0) - C\varepsilon \geq u_{0,\varepsilon}$$

and therefore w^ε is a $C^{1,1}(\overline{Q}_{T-h})$ supersolution of P_ε. Using the comparison principle, we conclude that $u^\varepsilon(x,t) \leq w^\varepsilon(x,t)$ for every $(x,t) \in Q_{T-h}$. Passing to the limit as $\varepsilon \to 0+$, we obtain

$$\limsup_{\varepsilon \to 0+} u^\varepsilon(x,t) \leq \lim_{\varepsilon \to 0+} w^\varepsilon(x,t) = u(x,t)$$

for every $(x,t) \in Q_T$. Letting $h, \eta \to 0$, we complete the proof of the proposition. □

Remark 2.6. One can formulate and prove a result analogous to Proposition 2.5 for strict subsolutions of P. Details are left to the reader.

From Proposition 2.5 we derive now a uniqueness theorem in the starshaped case (S).

Theorem 2.7. *Let the initial function u_0 satisfy condition (S) for some $x_0 \in \mathbf{R}^n$. If nonnegative $u_{0,\varepsilon}$ approximate u_0 uniformly and* $\operatorname{support} u_{0,\varepsilon} \to \operatorname{support} u_0$*, then the limit solution of P is unique and coincides with the minimal supersolution of P.*

For the existence of limit solutions and their local properties we refer to [3], [4] and [1].

Proof. Without loss of generality we may assume $x_0 = 0$. Suppose first that u is a classical supersolution of P. As was noted in Remark 2.3,

$$u_\lambda(x,t) = (1/\lambda')u(\lambda x, \lambda^2 t) \qquad 0 < \lambda < \lambda' < 1$$

is a strict supersolution in $Q_{T/\lambda^2} \supset Q_T$, so that we can apply Proposition 2.5. Then, letting $\lambda \to 1-$, we will arrive at

$$\limsup_{\varepsilon \to 0+} u^\varepsilon(x,t) \le u(x,t). \tag{2.13}$$

Let now u be a limit solution of P. By Remark 2.2 u is a classical supersolution of P. Therefore (2.13) holds again. It is not difficult to understand that this completes the proof of the theorem. □

3. The convex case

The next sections of this paper will be devoted to the proof of the existence of classical solutions of P under the following *convexity* assumptions on data:

(C1) u_0 is superharmonic and log-concave in bounded convex $\Omega_0 = \{u_0 > 0\}$.

It is easily seen that (C1) implies (S) with x_0 the maximum point of u_0. In the sequel we will always assume that $x_0 = 0$.

As it follows from Lemma 2.4 and Theorem 2.7, a classical solution of P, if exists, coincides with the minimal supersolution, and therefore with the only limit solution of P. Next, from condition (C1) we may expect that the time sections $\Omega(t) = \{x : (x,t) \in \Omega\}$ of a classical solution (u, Ω) enjoy the following property:

$$\Omega(t) \text{ is convex and shrinks in time for } t \in [0,T], \tag{3.1}$$

cf. Theorem 1 in [9], and [1].

Definition 3.1. A supersolution (u, Ω) of P in Q_T is said to be in class $\mathcal{B}$ if Ω satisfies (3.1) and moreover $\partial\Omega \cap Q_T$ is Lipschitz regular in time.

The Lipschitz regularity in time is understood in the following sense: for every $(x_0, t_0) \in \partial\Omega \cap Q_T$ there exists a neighborhood V such that

$$V \cap \Omega = \{x_n > f(x_1, \dots, x_{n-1}, t)\} \cap V, \tag{3.2}$$

for a suitable spatial coordinate system and where f is a globally defined function, uniformly Lipschitz in time. We point out that in spatial coordinates f can be chosen to be convex, if time sections $\Omega(t)$ are convex.

If the class $\mathcal{B}$ just defined has a minimal element, then it is a good candidate for a classical solution of P. This idea goes back to Beurling's celebrated paper [13]. We set

$$\Omega^* = \bigcap_{(u,\Omega) \in \mathcal{B}} \Omega \tag{3.3}$$

and let also u^* be a solution to the Dirichlet problem

$$\Delta u^* - (u^*)_t = 0 \quad \text{in } \Omega^*; \qquad u^* = 0 \quad \text{on } \partial\Omega^* \cap Q_T; \qquad u^*(\cdot, 0) = u_0. \tag{3.4}$$

We can show that under some additional conditions on u_0 and for small $T \leq T(u_0)$ the pair (u^*, Ω^*) is the minimal element of $\mathcal{B}$ and indeed a classical solution of P. The conditions on u_0 are as follows. First

$$u_0 \in C^{0,1}(\overline{\Omega}_0) \quad \text{and} \quad \lim_{\Omega_0 \ni x \to \partial\Omega_0} |\nabla u_0(x)| = 1; \tag{C2}$$

and, next, there exists a constant $M > 0$ such that

$$\Delta u_0(x) + M(u_0(x) - \nabla u_0(x) \cdot x) \geq 0 \quad \text{for every } x \in \Omega_0. \tag{C3}$$

Let us point out that if (C2) holds, then (C3) can be replaced by

$$\Delta u_0(x) \geq -M' \quad \text{for every } x \in \Omega_0. \tag{C3'}$$

for certain $M' > 0$. Indeed, (C3') readily follows from (C3) in view of the boundedness of $w(x) = u_0(x) - \nabla u_0(x) \cdot x$ in Ω_0. Suppose now that (C3') holds. We have $w(x) \geq u_0(x) > 0$ in Ω_0 and

$$\liminf_{\Omega_0 \ni x \to \partial\Omega_0} w(x) \geq \liminf_{\Omega_0 \ni x \to \partial\Omega_0} -\nabla u_0(x) \cdot x \geq \rho,$$

where $\rho > 0$ is the diameter of the circle, centered at $x_0 = 0$ and inscribed in Ω_0. Hence, there is an $\varepsilon > 0$ such that $w(x) \geq \varepsilon$ in Ω_0 and (C3) follows with $M = M'/\varepsilon$.

Remark 3.2. Before we proceed, we check that the class $\mathcal{B}$ is not empty and that so is Ω^*. For the latter it suffices to show the existence of a subsolution of P, in view of the comparison principle (Lemma 2.4.)

Indeed, let u_0 satisfy (C1)–(C3). Consider the functions

$$u(x,t) = u_0(x) \tag{3.5}$$

and

$$v(x,t) = \sqrt{1 - 2Mt}\, u_0\left(\frac{x}{\sqrt{1-2Mt}}\right) \tag{3.6}$$

in Q_T. We claim that the first is a super- and the second is a subsolution of P in Q_T for $T \leq 1/(2M)$, but in a sense a little bit different from Definition 2.1, which will not affect on the comparison principle. The matter is that (3.5) and (3.6) are not caloric in their positivity sets but super- and subcaloric respectively. Indeed, for (3.5) this follows from superharmonicity of u_0 in Ω_0 and for (3.6) from the direct computation

$$\Delta v(x,t) - v_t(x,t) = (1 - 2Mt)^{-1/2}(\Delta u_0(\xi) + M(u_0(\xi) - \nabla u_0(\xi) \cdot \xi)) \geq 0,$$

for $\xi = x/\sqrt{1-2Mt} \in \Omega_0$ and $t \in [0, 1/(2M)]$. The gradient condition for both functions follows from (C2).

4. Lipschitz regularity in time

The following lemma plays one of the fundamental roles in our study.

Lemma 4.1. *Let u_0 satisfy (C1)–(C3) and let Ω^* be given by* (3.3)*. Then $\partial\Omega^* \cap Q_{1/(2M)}$ is Lipschitz regular in time.*

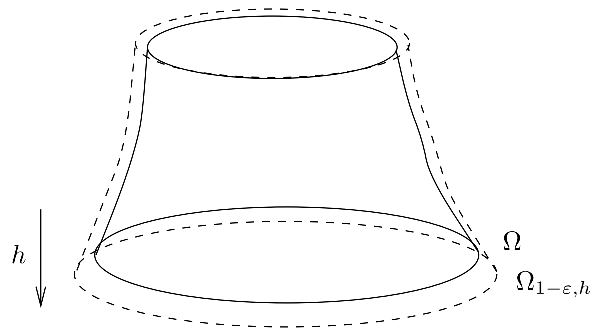

FIGURE 2. Construction of $\Omega_{1-\varepsilon,h}$

Proof. Let $(u,\Omega)\in\mathcal{B}$. For small ε, $h>0$ let us define

$$w(x,t)=\frac{1}{1-\varepsilon}u((1-\varepsilon)x,(1-\varepsilon)^2(t+h))$$

in $Q_{(1-\varepsilon)^{-2}T-h}$, that is we first stretch Ω a little and then drop it down and thus obtain

$$\Omega_{1-\varepsilon,h}=\{(x,t):((1-\varepsilon)x,(1-\varepsilon)^2(t+h)\in\Omega\}.$$

Clearly, a pair $(w,\Omega_{1-\varepsilon,h})$ again will lie in $\mathcal{B}$ if the condition

$$w(\cdot,0)\geq u_0 \tag{4.1}$$

is verified. In view of Remark 3.2 and Lemma 2.4

$$\begin{aligned}w(x,0)&=\frac{1}{1-\varepsilon}u((1-\varepsilon)x,(1-\varepsilon)^2h)\geq\frac{1}{1-\varepsilon}v((1-\varepsilon)x,(1-\varepsilon)^2h)\\&=\frac{\sqrt{1-2M(1-\varepsilon)^2h}}{1-\varepsilon}u_0\left(\frac{1-\varepsilon}{\sqrt{1-2M(1-\varepsilon)^2h}}x\right)=u_0(x)\end{aligned}$$

if we choose

$$h=\frac{2\varepsilon-\varepsilon^2}{2M(1-\varepsilon)^2}. \tag{4.2}$$

Therefore $(w,\Omega_{1-\varepsilon,h})\in\mathcal{B}$ if h is given by (4.2). Note now that the time levels of $\Omega_{1-\varepsilon,h}$ are given by the identity

$$\frac{1}{1-\varepsilon}\Omega(t)=\Omega_{1-\varepsilon,h}\left(\frac{t}{(1-\varepsilon)^2}-h\right).$$

Running over all $(u,\Omega)\in\mathcal{B}$, we may conclude therefore that

$$\frac{1}{1-\varepsilon}\Omega^*(t)\supset\Omega^*\left(\frac{t}{(1-\varepsilon)^2}-h\right). \tag{4.3}$$

Since $\Omega^*(t)$ shrinks in time, the inclusion (4.3) is not trivial if

$$\frac{t}{(1-\varepsilon)^2}-h<t,$$

for certain small ε, $h>0$, given by (4.2). The latter is equivalent to to the inequality $t<1/(2M)$. Besides, for these t, (4.3) implies also the Lipschitz regularity of $\partial\Omega^*$ in time variable. □

Remark 4.2. Let u^* be a solution of (3.4) for $T < 1/(2M)$ and continue the line of reasonings from the previous lemma. For small ε, $h > 0$, given by (4.2), the following inequality will be satisfied

$$\frac{1}{1-\varepsilon}u^*((1-\varepsilon)x,(1-\varepsilon)^2(t+h)) - u^*(x,t) \geq 0.$$

Let now ε go to 0. We will obtain

$$u^*(x,t) - \nabla u^*(x,t)\cdot x + (1/M - 2t)u_t^*(x,t) \geq 0$$

in $\Omega^* \cap Q_{1/(2M)}$. The reader can see the similarity of this condition with (C3). In particular, we have that u_t^* is bounded from below in every $\Omega^* \cap Q_T$ with $T < 1/(2M)$.

5. Some technical lemmas

For supersolutions $(u,\Omega) \in \mathcal{B}$ of P one can replace the gradient condition (ii) in Definition 2.1 with

$$\limsup_{\Omega\ni(x,t)\to(x_0,t_0)} \frac{u(x,t)}{d_\Omega(x,t)} \leq 1$$

for every $(x_0,t_0) \in \partial\Omega \cap Q_T$, where

$$d_\Omega(x,t) = \mathrm{distance}(x, \partial\Omega(t)).$$

This is taken care of in the next lemma.

Lemma 5.1. *Let Ω be a bounded domain in Q_T such that $\Omega(t)$ is convex for $t \in (0,T)$ and $\partial\Omega \cap Q_T$ is Lipschitz in time. Let also u be a nonnegative function, continuously vanishing on $\partial\Omega \cap Q_T$, and such that $\Delta u - u_t = 0$ in Ω. Then*

$$\limsup_{\Omega\ni(x,t)\to(x_0,t_0)} |\nabla u(x,t)| = \limsup_{\Omega\ni(x,t)\to(x_0,t_0)} \frac{u(x,t)}{d_\Omega(x,t)}, \tag{5.1}$$

for every $(x_0,t_0) \in \partial\Omega \cap Q_T$.

Proof. Denote by α the left $\limsup$ in (5.1) and by β the right one. The inequality $\alpha \geq \beta$ follows immediately from the finite-increment formula, and therefore we focus on the inequality $\alpha \leq \beta$. If $\beta = \infty$ there is nothing to prove. Therefore assume β is finite. Let a sequence $\{(x_k,t_k)\}$ be such that $\alpha = \lim_k |\nabla u(x_k,t_k)|$. Let also $z_k \in \partial\Omega(t_k)$ be chosen such that $d_k := d_\Omega(x_k,t_k) = |x_k - z_k|$. Set

$$u_k(y,s) = \frac{1}{d_k}u(z_k + d_k y, t_k + d_k^2 s).$$

Let $e_k = (x_k - z_k)/d_k$ and assume that $\{e_k\}$ converges to $e = (0,\dots,0,1)$. In view of the Lipschitz regularity of $\partial\Omega \cap Q_T$ in t and the convexity of $\Omega(t)$'s, the positivity sets $\Omega_k = \{u_k > 0\}$ will converge (over a subsequence) to a subset of $\Pi^+ = \{(x,t) : x\cdot e > 0\}$ that has a cylindrical form $D \times \mathbf{R}$, where D is an unbounded convex subset of the halfspace $\{x \cdot e > 0\}$, containing a spatial ball $B(e,1)$. Since u_k satisfy the heat equation in Ω_k and are locally uniformly bounded (since we assume β is finite), a subsequence of $\{u_k\}$ will converge in C^1-norm on compact subsets of $D^\circ \times \mathbf{R}$ to a nonnegative caloric function v, which enjoys the following properties

$$|\nabla v(e,0)| = \alpha \tag{5.2}$$

$$0 \leq v(x,t) \leq \beta\,\mathrm{distance}(x,\partial D) \quad \text{for every } (x,t) \in D \times \mathbf{R}. \tag{5.3}$$

Suppose first that D is not a halfspace. Then by a theorem of Phragmén-Lindelöf type, the condition (5.3) will imply that $v \equiv 0$ in $D \times \mathbf{R}$; in particular we will have $\alpha = 0$ and hence $\alpha \le \beta$.

If D is a halfspace, that is $D \times \mathbf{R} = \Pi^+$, condition (5.3) is rewritten as

$$0 \le v(x,t) \le \beta(x \cdot e) \quad \text{for every } (x,t) \in \Pi^+ \tag{5.4}$$

and we observe that the only caloric functions in Π^+ with property (5.4) are functions of the form $v(x,t) = \gamma(x \cdot e)$ for nonnegative $\gamma \le \beta$. We give a short proof of this statement. Extend v into a caloric function in the whole $\mathbf{R}^n \times \mathbf{R}$ by the odd reflection $v(x', x_n, t) = -v(x', -x_n, t)$, where $x' = (x_1, \dots, x_{n-1})$ and $x_n = x \cdot e$, and consider the derivative $\partial v/\partial x_n$. It is a caloric function in $\mathbf{R}^n \times \mathbf{R}$ and from (5.4), by the interior gradient estimate, $|\partial v/\partial x_n| \le C\beta$, where C is an absolute constant. From the Liouville theorem $\partial v/\partial x_n$ is identically constant, say γ, and therefore $v(x) = \gamma x_n$, since it vanishes on $\Pi = \partial\Pi^+$. That $0 \le \gamma \le \beta$ follows from (5.4).

Return to our v. In this case (5.2) implies $\gamma = \alpha$ and therefore $\alpha \le \beta$. The proof is complete. □

The following lemma is an elliptic counterpart of the lemma above and is proved in a similar way; therefore, the proof is omitted.

Lemma 5.2. *Let D be a bounded spatial convex domain and u a nonnegative function in D, continuously vanishing on ∂D and such that $\Delta u = f$ in D with f bounded. Then*

$$\limsup_{D \ni x \to x_0} |\nabla u(x)| = \limsup_{D \ni x \to x_0} \frac{u(x)}{d(x)}, \tag{5.5}$$

where $d(x) = \operatorname{distance}(x, \partial D)$. □

6. The minimal element of $\mathcal{B}$

From Lemma 4.1 we know that if (C1)–(C3) are satisfied then Ω^* given by (3.3) will have a Lipschitz in time lateral boundary in $Q_{1/(2M)}$. Then the Dirichlet problem (3.4) is solvable in the classical sense. In this section we show that (u^*, Ω^*) is a supersolution of P and hence the minimal element of $\mathcal{B}$.

Lemma 6.1. *Let u_0 satisfy (C1)–(C3) and $T \le 1/(2M)$. Then the pair (u^*, Ω^*) is the minimal element of $\mathcal{B}$.*

Proof. The only thing we have to verify is that (u^*, Ω^*) is a supersolution of P. Let $(u_k, \Omega_k) \in \mathcal{B}$ be such that

(i) $\Omega^* = \bigcap_k \Omega_k$;
(ii) the sequence $\{\Omega_k\}$ is decreasing;
(iii) $\Omega_k(0) = \Omega_0$ and $u_k(\cdot, 0) = u_0$.

We can construct such a sequence as follows. Let $(u_k, \Omega_k) \in \mathcal{B}$ satisfy (i). Next, in order to have (ii) we observe the following. Denote by $u_{k,m}$ the solution of the Dirichlet problem in $\Omega_{k,m} = \Omega_k \cap \Omega_m$ with the initial function $u_{k,m}(\cdot, 0) = \min\{u_k(\cdot, 0), u_m(\cdot, 0)\}$ and vanishing on the lateral boundary, then

$$u_{k,m}(x,t) \le \min\{u_k(x,t), u_m(x,t)\} \quad \text{for every } (x,t) \in \Omega_{k,m}.$$

Besides, for the distance functions we will have

$$d_{\Omega_{k,m}}(x,t) = \min\{d_{\Omega_k}(x,t), d_{\Omega_m}(x,t)\} \quad \text{for every } (x,t) \in \Omega_{k,m}.$$

Therefore, using Lemma 5.1, we can conclude that $(u_{k,m}, \Omega_{k,m}) \in \mathcal{B}$. If now (iii) is not satisfied, we can replace Ω_k by the intersection of all Ω_m with $m \le k$ and thus to make $\{\Omega_k\}$ decreasing.

In order to have (iii), let us take as Ω_1 in the original sequence the cylindrical domain $\Omega_0 \times (0,T)$ (see Remark 3.2). Now, since we can assume Ω_k are decreasing, we will have $\Omega_k(0) = \Omega_0$. To satisfy the second condition in (iii) just replace u_k with the solution of the corresponding Dirichlet problem in Ω_k.

Denote now by ω_k the caloric measure of Ω_0 with respect to Ω_k; that is

$$\Delta\omega_k - (\omega_k)_t = 0 \quad \text{in } \Omega_k; \quad \omega_k = 0 \quad \text{on } \partial\Omega_k \cap Q_T; \quad \omega(\cdot,0) = 1 \quad \text{in } \Omega_0. \tag{6.1}$$

Let

$$C = \sup_{x\in\Omega_0} |\nabla u_0(x)|. \tag{6.2}$$

Then we can control the growth of $|\nabla u_k|$ in Ω_k. Namely,

$$|\nabla u_k(x,t)| \le 1 + (C-1)\omega_k(x,t) \tag{6.3}$$

for every (x,t) in Ω_k. This follows from the maximum principle for subcaloric functions, since $v(x,t) = |\nabla u_k|$ satisfies $\Delta v - v_t \ge 0$ in Ω_k.

For the next step, observe that since u_k are caloric in Ω_k and uniformly bounded, a subsequence of $\{u_k\}$ will converge in C^1 norm on compact subsets of $\Omega^* = \bigcap_k \Omega_k$ to a function u^*. We may assume also that over this subsequence, the corresponding caloric measures ω_k converge to a caloric measure ω^* of Ω_0 with respect to Ω^*. Then in the limit we will obtain from (6.3)

$$|\nabla u^*(x,t)| \le 1 + (C-1)\omega^*(x,t) \quad \text{for every } (x,t) \in \Omega^*.$$

As a consequence, (u^*, Ω^*) is in $\mathcal{B}$ and therefore is its minimal element. □

7. Further properties of the minimal element

The method used in this and the next section is due to A. Henrot and H. Shahgholian [10, 11]. Originally it was applied to the Bernoulli type free boundary problem for p-Laplace operator, an elliptic problem, whose free boundary condition is analogous to that of P.

Definition 7.1. A point $(x,t) \in \partial\Omega \cap Q_T$, where Ω satisfies (3.1), is said to be *extreme* if $x \in \partial\Omega(t)$ is extreme for $\Omega(t)$. The latter means that x is not a convex combination of points on $\partial\Omega(t)$, other than x.

Lemma 7.2. *Let u_0 satisfy (C1)–(C3) and $T \le 1/(2M)$. Then the pair (u^*, Ω^*) satisfies*

$$\limsup_{\Omega^*\ni(x,t)\to(x_0,t_0)} |\nabla u^*(x,t)| = 1 \tag{7.1}$$

for every extreme point $(x_0,t_0) \in \partial\Omega^* \cap Q_T$.

Proof. Let us point out that it is enough to prove the lemma in the case when x_0 is an *extremal* point of $\partial\Omega^*(t_0)$, which means that there is a spatial supporting hyperplane to $\Omega^*(t_0)$, touching $\partial\Omega^*(t_0)$ at x_0 only. This follows from the fact that the extremal points are dense among the extreme points.

Suppose now $x_0 \in \partial\Omega^*(t_0)$ is extremal and that (7.1) is not true. Then, in view of Lemmas 5.1 and 4.1, there exists a (space-time) neighborhood V of (x_0, t_0) and $\alpha > 0$ such that

$$u^*(x,t) \leq (1-\alpha) d_{\Omega^*}(x,t) \tag{7.2}$$

for every $(x,t) \in V \cap \Omega^*$. We may assume additionally that the intersection $V \cap \Omega^*$ is given by (3.2).

Let now Π be a spatial supporting hyperplane to $\Omega^*(t_0)$, such that $\Pi \cap \partial\Omega^*(t_0) = \{x_0\}$. By translation and rotation, we may assume that $x_0 = 0$ and that $\Pi = \{x_n = 0\}$. Moreover let $\Omega^*(t_0) \subset \{x_n > 0\}$. Using the extremality of (x_0, t_0), it is easy to see that there are $\delta_0 > 0$ and $\eta_0 > 0$ such that

$$\{(x,t) \in \Omega^* : x_n \leq \eta_0 \text{ and } t \in [t_0 - \delta_0, t_0]\} \subset V.$$

Let us consider the function

$$h(t) = -\min_{x \in \Omega^*(t)} x_n$$

for $t \in [t_0 - \delta_0, t_0]$. In view of Lipschitz regularity of $\partial\Omega^*$ in time,

$$h(t) \leq L(t_0 - t)$$

for $t \in [t_0 - \delta_0, t_0]$, where L is the Lipschitz constant of f in t. Let now $\eta_1 \in (0, \eta_0)$ be very small and a constant $C \geq L$ be chosen such that

$$h(t_0 - \delta_0) \leq C\delta_0 - \eta_1.$$

Further, we can find $\delta_1 \in (0, \delta_0]$ such that

$$h(t_0 - \delta_1) = C\delta_1 - \eta_1$$

and

$$h(t) \geq C(t_0 - t) - \eta_1$$

for every $t \in [t_0 - \delta_1, t_0]$.

Define now a domain $\Omega \subset Q_T$ by giving its time sections as follows

$$\Omega(t) = \begin{cases} \Omega^*(t), & t \in (0, t_0 - \delta_1) \\ \Omega^*(t) \cap \{x_n > \eta_1 - C(t_0 - t)\}, & t \in [t_0 - \delta_1, t_0] \\ \Omega(t_0), & t \in (t_0, T). \end{cases} \tag{7.3}$$

Let also u be a solution to the Dirichlet problem in Ω for the heat equation, zero on the lateral boundary and equal u_0 in Ω_0. We claim that if η_1 is small enough, then (u, Ω) is in $\mathcal{B}$. This will lead to a contradiction since $\Omega^* \not\subset \Omega$ and the lemma will follow. Since Ω satisfies (3.1), we need to verify only that (u, Ω) is a supersolution of P

By Lemma 5 in [14], there exists $\varepsilon = \varepsilon(n, L) > 0$ such that in a neighborhood of (x_0, t_0), the function

$$w^*(x) = w^*(x;t) = u^*(x,t) + u^*(x,t)^{1+\varepsilon}$$

is subharmonic in x. Moreover the size of the neighborhood depends only on n and L. We may suppose V has this property. Next, note that we can take $C \leq L + 1$ if η_1 is sufficiently small and δ_0 is fixed. This will make the boundary of new constructed Ω $(L+1)$-Lipschitz in time. Therefore we may assume also that

$$w(x) = w(x;t) = u(x,t) + u(x,t)^{1+\varepsilon}$$

is subharmonic in the neighborhood V.

We are ready to prove now that (u, Ω) is indeed a supersolution.

Case 1. $t \in (0, t_0 - \delta_1)$. This is the simplest case, since $u = u^*$ in $\Omega \cap \{0 < t < t_0 - \delta_1\}$: all the properties of u there follows from those of u^*.

Case 2. $t \in [t_0 - \delta_1, t_0]$, the most interesting case. First of all, since $\Omega(t) \subset \Omega^*(t)$ for these t, we have also $u \le u^*$ there. Let now consider a part $D(t)$ of $\Omega^*(t)$ between to planes: $\Pi_1 = \{x_n = \eta_1 - C(t_0 - t)\}$ and $\Pi_0 = \{x_n = \eta_0 - C(t_0 - t)\}$. Compare there two functions $w(x) = w(x; t)$ and $\ell(x) = x_n - \eta_1 + C(t_0 - t)$. On Π_1 both functions are 0. Next

$$\ell(x) = \eta_0 - \eta_1 \quad \text{on } \Pi_0.$$

To estimate w on Π_0, let us first estimate u on Π_0. Thus

$$u(x,t) \le u^*(x,t) \le (1-\alpha) d_{\Omega^*}(x,t) \le (1-\alpha)(x_n + L(t_0 - t)) \le (1-\alpha)\eta_0$$

and therefore, if η_0 is small enough, we will obtain

$$w(x) \le (1 - \alpha/2)\eta_0 \quad \text{on } \Pi_0.$$

Choose now η_1 so small that $(1 - \alpha/2)\eta_0 \le \eta_0 - \eta_1$. Then $w \le \ell$ on $\partial D(t)$ and, since w is subharmonic and ℓ is harmonic (linear), we conclude that $w \le \ell$ in $D(t)$. Along with $u \le u^*$ this gives

$$\limsup_{\Omega \ni (x,t) \to \partial\Omega} \frac{u(x,t)}{d_\Omega(x,t)} \le 1, \tag{7.4}$$

where t is free to vary within $[t_0 - \delta_1, t_0]$.

Case 3. $t \in (t_0, T)$. Since $\Omega(t)$ shrink in time and u_0 is superharmonic in Ω_0, considering the time derivative u_t in Ω, we can infer from the maximum principle for the heat equation that $u_t \le 0$ in Ω. In particular, we will have in the cylindrical portion of Ω with $t > t_0$ that $u(x,t) \le u(x,t_0)$ in $\Omega(t) = \Omega(t_0)$ and applying Lemma 5.1 we conclude that (7.4) is valid also in this case.

Summing up, we see that (7.4) holds for all $t \in (0,T)$, and by Lemma (5.1) this implies $(u, \Omega) \in \mathcal{B}$, which is a contradiction. □

8. The classical solution of P for short time

In this section we prove

Theorem 8.1. *Let u_0 satisfy (C1)–(C3). Then the minimal element (u^*, Ω^*) of $\mathcal{B}$ is a classical solution of P in $Q_{1/(2M)}$. Moreover this classical solution is unique,*

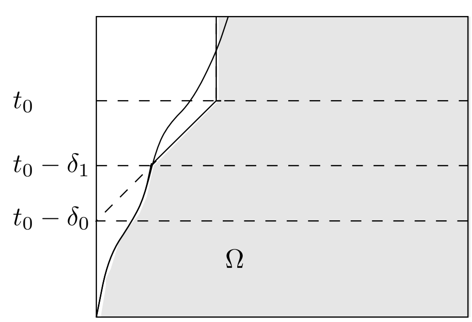

Figure 3. Construction of Ω from Ω^* in profile

the time sections $\Omega^*(t)$ are convex and shrinking in time and $u^*(\cdot,t)$ are log-*concave in $\Omega^*(t)$, $0<t<1/(2M)$.*

Up to now we have not used fully the log-concavity condition on u_0. The following lemma will exploit this property.

Lemma 8.2. *Let u be a solution of a Dirichlet problem for the heat equation in a bounded domain Ω in Q_T, zero on $\partial\Omega\cap Q_T$ and $u(\cdot,0)=u_0$. If the time sections $\Omega(t)$ are convex for every $t\in[0,T]$ and u_0 is* log-*concave in Ω_0 then so is $u(\cdot,t)$ in $\Omega(t)$ for every $t\in(0,T)$.*

Proof. The proof, using Korevaar's Concavity maximum principle [15] can be found in [9]. We point out that an alternative proof can be given based on Brascamp and Leib's paper [16]. □

Remark 8.3. However, we can replace the condition of log-concavity of u_0 by another one that guarantees the convexity of level sets $\{u(\cdot,t)>s\}$ for every u as in Lemma 8.2. In fact only this property will be used.

The convexity of level sets is used in the following lemma, which is mainly due to [10], [11].

Lemma 8.4. *Let D be a bounded spatial convex domain with C^1 regular boundary, V a neighborhood of ∂D and w a smooth positive subharmonic function in $D\cap V$, continuously vanishing on ∂D. If the level lines $\{w=s\}$ are strictly convex surfaces for $0<s<s_0$ then the condition*

$$\limsup_{D\cap V\ni x\to x_0}|\nabla w(x)|\geq 1$$

for every extreme point $x_0\in\partial D$ implies that

$$|\nabla w(x)|\geq 1$$

for every x with $0<w(x)<s_0$.

Proof. Let $y_0\in D\cap V$ be such that $w(y_0)=s\in(0,s_0)$, so $\ell_s=\{w(x)=s\}$ is a strictly convex surface. Denote by Π a tangent hyperplane to ℓ_s at y_0. By translation and rotation we may assume that $y_0=0$ and $\Pi=\{x_n=0\}$ and that ℓ_s lies in the lower halfspace $\{x_n\leq 0\}$. Choose now an extreme point $x_0\in\partial D\cap\{x_n\geq 0\}$ such that it has the maximal x_n-coordinate among the points of ∂D. Although this point can be not uniquely defined, we will denote it $x_0(y_0)$ to indicate the way it was constructed from y_0.

Suppose now, that along with C^1 regularity of ∂D, w is C^1 regular up to ∂D. The core inequality is then

$$|\nabla w(y_0)|\geq|\nabla w(x_0(y_0))|. \tag{8.1}$$

To prove, let $\alpha=|\nabla w(x_0(y_0))|$ and $\beta\in(0,\alpha)$. Consider a function $f(x)=w(x)+\beta x_n$ in $D^+=D\cap\{x_n>0\}$. The function f is subharmonic in D^+ and therefore admits its maximum on ∂D^+. A simple analysis shows that the maximum of f is admitted either at the origin y_0 or at x_0. Let us exclude the latter possibility. We have that $\partial_{x_n}w(x_0)=-|\nabla w(x_0)|=-\alpha$ and hence $\partial_{x_n}f(x_0)=\beta-\alpha<0$, which is impossible if x_0 is a maximum point. Therefore the maximum of f is admitted at $y_0=0$ and as a consequence we have

$$|\nabla w(0)|=-\partial_{x_n}w(0)=-\lim_{h\to 0+}\frac{w(he_n)-w(0)}{h}\geq\lim_{h\to 0+}\frac{\beta h}{h}=\beta.$$

Letting $\beta \to \alpha-$, we obtain (8.1). This, of course, proves the lemma in the case considered.

Consider now the general case. Choose a sequence of points $\{x_j \in D \cap V\}$ converging to x_0 so that

$$\lim_{j\to\infty} |\nabla w(x_j)| = \limsup_{x\to x_0} |\nabla w(x)| \geq 1.$$

Let $s_j = w(x_j)$ and a domain D_j be bounded by the level surface ℓ_{s_j}. Construct points $y_j \in \ell_s$ on the same level surface as y_0 so that $x_j = x_0(y_j)$ for the domain D_j. It can be done as follows. Take the tangent hyperplane Π_j to ℓ_{s_j} at x_j and move it down towards ℓ_s until the plane touches ℓ_s and define y_j to be the touching point. Now, the function w is C^1 regular up to the boundary of D_j and therefore we may apply (8.1) to obtain

$$|\nabla w(y_j)| \geq |\nabla w(x_j)|.$$

It is clear that the proof will be completed as soon as we show that $y_j \to y_0$. Due to strict convexity of ℓ_s, this is indeed so, if the outer normals ν_j of the tangent planes Π_j to ℓ_{s_j} at x_j converge to the unit vector $e_n = (0, 0, \cdots, 1)$. In its turn, the latter statement is a consequence of the C^1 regularity of ∂D and the proof of the lemma is complete. □

Lemma 8.5. *Let D be a bounded spatial convex domain and x_0 a singular point on ∂D, such that there are more than one supporting hyperplane to D at x_0. Let also V be a neighborhood of x_0 and w a nonnegative subharmonic function in $D \cap V$, continuously vanishing on $\partial D \cap V$. Then*

$$\lim_{D\cap V \ni x \to x_0} \frac{w(x)}{d(x)} = 0, \tag{8.2}$$

where $d(x) = \mathrm{distance}(x, \partial D)$.

Proof. As a first step, we note, that placing D between two supporting hyperplanes at x_0 which form an angle of opening $\alpha < \pi$, one can easily construct a barrier function and prove that for small r

$$S(r) := \max_{B_r(x_0)\cap D} w(x) \leq C_1 r^{\pi/\alpha} \tag{8.3}$$

where C_1 is some constant, and $B_r(x_0) = \{x : |x - x_0| < r\}$.

Next, we claim that

$$w(x) \leq C_2 d(x) S(4r)/r \qquad \text{for } x \in \partial B_r(x_0) \cap D \tag{8.4}$$

with a constant $C_2 > 0$.

Indeed, let x_1 be a point on ∂D with $|x - x_1| = d(x)$. Note that $d(x) \leq r$, hence $|x_0 - x_1| \leq 2r$ and in particular $B_{2r}(x_1) \subset B_{4r}(x_0)$. Let Π be a supporting hyperplane to D at x_1 and Π^+ be the halfspace, which contains D. Let ω be a harmonic measure of $\partial B_{2r}(x_1) \cap \Pi^+$ with respect to $B_{2r}(x_1) \cap \Pi^+$ and $v := S(4r)\omega$. Then $\Delta v = 0 \leq \Delta w$ in $B_{2r}(x_1) \cap D$ and $v \geq w$ on $\partial(B_{2r}(x_1) \cap D)$ and therefore $v(y) \geq w(y)$ in $B_{2r}(x_1) \cap D$ by the maximum principle. In particular, at x we have

$$w(x) \leq v(x) \leq S(4r)\omega(x) \leq S(4r)\frac{C_3}{r} d(x)$$

with C_3 a universal constant, such that

$$\omega(y) \le \frac{C_3}{r}|y - x_1|$$

for $y \in B_r(x_1) \cap \Pi^+$, and (8.4) follows.

Now, the estimates (8.3) and (8.4) complete the proof. $\square$

Proof of Theorem 8.1. First observe that we need only to show that (u^*, Ω^*) is a subsolution. The properties of (u^*, Ω^*) in the second part of the theorem follow from inclusion $(u^*, \Omega^*) \in \mathcal{B}$ and Lemma 8.2. The uniqueness follows from Lemma 2.4.

Recall that from Lemma 7.2 we know that

$$\limsup_{\Omega^* \ni (x,t) \to (x_0,t_0)} |\nabla u^*(x,t)| = 1$$

for every extreme point $(x_0, t_0) \in \partial\Omega^* \cap Q_{1/(2M)}$. Denote by $\mathcal{R}$ the set of all $t_0 \in (0, 1/(2M))$ such that

$$\limsup_{\Omega^*(t_0) \ni x \to x_0} |\nabla u^*(x, t_0)| = 1$$

for every extreme point $x_0 \in \partial\Omega^*(t_0)$.

The reader can easily see the difference between these two properties: if in the former case (x, t) goes to (x_0, t_0) by varying in space and time, in the latter case the time t_0 is fixed.

Let us prove that the complement $\mathcal{J} = (0, 1/(2M)) \setminus \mathcal{R}$ is a union of a countable family of nowhere dense subsets of $(0, 1/(2M))$. This follows from the continuity of $|\nabla u^*|$ in Ω^*. Indeed, let $\{U_k\}$ be an open countable basis for topology in $\mathbf{R}^n$. If $t_0 \in \mathcal{J}$ then there exist an extreme point $x_0 \in \partial\Omega^*(t_0)$, and natural numbers k and m such that $x_0 \in U_k$ and $|\nabla u^*(x, t_0)| \le 1 - (1/m)$ for every $x \in U_k \cap \Omega^*(t_0)$. Denote now the set of all $t_0 \in \mathcal{J}$ with such k and m by $\mathcal{J}_{k,m}$. Then $\mathcal{J} = \bigcup_{k,m} \mathcal{J}_{k,m}$. Besides, as easy to see, $\mathcal{J}_{k,m}$ are nowhere dense in $(0, 1/(2M))$ and our assertion follows.

Consider now, as in the proof of Lemma 7.2, the function

$$w^*(x) = w^*(x; t) = u^*(x, t) + u^*(x, t)^{1+\varepsilon}.$$

Let $t_0 \in (0, 1/(2M))$. By Lemma 5 in [14], there exist $\varepsilon > 0$, $\delta > 0$ and $s_0 > 0$ such that $w^* = w^*(\cdot; t)$ is subharmonic in a convex ring $D(t) = \{0 < w^*(x, t) < s_0\}$ whenever $t \in (t_0 - \delta, t_0 + \delta)$. Besides, from Lemma 8.2, the level surfaces of u^* and therefore those of w^* are convex. Moreover, they are strictly convex due to real analyticity of $u^*(\cdot, t)$ in $\Omega^*(t)$.

Now, we point out that if $t \in \mathcal{R}$, then $\partial\Omega^*(t)$ is C^1 regular. Otherwise there would exist a singular extreme point $x_0 \in \partial\Omega^*(t)$ with

$$\lim_{\Omega^*(t) \ni x \to x_0} |\nabla u^*(x, t)| = 0, \tag{8.5}$$

which contradicts to the definition of $\mathcal{R}$. Indeed, if $x_0 \in \partial\Omega^*(t)$ is singular then by Lemma 8.5 $w^*(x; t)/d_\Omega(x, t) \to 0$, or, equivalently, $u^*(x, t)/d_\Omega(x, t) \to 0$ as $x \to x_0$, and (8.5) will follow from Lemma 5.2. Note that Lemma 5.2 is applicable here since $f(x) := u_t^*(x, t) = \Delta u^*(x, t)$ is bounded in $\Omega^*(t)$ (see Remark 4.2).

Let now $t \in (t_0 - \delta, t_0 + \delta) \cap \mathcal{R}$. Then Lemma 8.4 implies that

$$|\nabla w^*(x; t)| \ge 1,$$

if $0 < w^*(x;t) < s_0$ and t is as above. This inequality is extended for all $t \in (t_0 - \delta, t_0 + \delta)$ because of everywhere density of $\mathcal{R}$ and continuity of $|\nabla w^*(x;t)|$ in Ω^*. Since $|\nabla w^*|$ and $|\nabla u^*|$ are asymptotically equivalent when $u^* \to 0$, we obtain immediately that

$$\liminf_{\Omega^* \ni (x,t) \to (x_0,t_0)} |\nabla u^*(x,t)| \geq 1$$

whenever $x_0 \in \partial\Omega^*(t_0)$. Since $t_0 \in (0, 1/(2M))$ was arbitrary, we conclude that (u^*, Ω^*) is indeed a classical solution of P in $Q_{1/(2M)}$. The theorem is proved. □

9. Regularity and convexity up to the extinction time

As we have seen in Section 2, problem P has a unique limit solution for $t > 0$ and Theorem 8.1 tells us that this solution will be classical for a short time interval, if u_0 satisfies convexity assumptions (C1)–(C3). However the natural hypothesis is that the solution should stay classical as long as $\Omega(t)$ is nonempty, i.e. until the *extinction time* t^*.

In this section we prove that this is indeed so. Let $T > 0$ denote the maximal time for which the solution u of P is classical in Q_T and that the time sections $\Omega(t)$ are convex and nonempty for $t \in [0, T)$. We are going to show that if $\Omega(T)$ is nonempty, the solution u is of parabolic Hölder class $H_{2+\alpha}(\Omega \cap \overline{Q}_T)$ (see G. Lieberman's book [17] for notations.) This will ensure that the function $u(\cdot, T)$ satisfies conditions analogous to (C1)–(C3) and hence the solution will stay classical also for some time after T. Hence we will arrive at the contradiction.

The first step in this direction is the following lemma.

Lemma 9.1. *If $\Omega(T)$ is nonempty, then*

$$\liminf_{(x,t) \to \partial\Omega \cap \overline{Q}_T} |\nabla u(x,t)| \geq \delta > 0, \qquad (x,t) \in \Omega \cap \overline{Q}_T$$

Proof. Since $\Omega(t)$ are convex for $t \in [0, T]$, the spatial level sets $\{u(\cdot, t) > s\}$ are also convex, and for sufficiently small $0 < s < s_0$ contain a certain ball, which we assume is $B_r = B_r(0)$. Since also Ω_0 is bounded, all these level sets are contained in a bigger ball B_R. This implies immediately that if $x \in \{u(\cdot, t) = s\}$ for $0 < s < s_0$ then

$$\frac{\nabla u(x,t)}{|\nabla u(x,t)|} \cdot \frac{x}{|x|} \geq \gamma = \gamma(R/r) > 0. \tag{9.1}$$

Let us now take a point $x^* \in \partial\Omega(T)$. Without loss of generality we may assume that $x^*/|x^*| = e_n$ and consider partial derivative $w(x,t) = \partial_{x_n} u(x,t)$ in a small neighborhood $U_\epsilon = \{(x,t) : |x_n - x_n^*| < \epsilon, |x' - x^{*\prime}| < c\epsilon, T - \epsilon < t < T\}$ of (x^*, T) for some $\epsilon > 0$ and a constant $c = c(R/r) > 0$. The constant c is chosen such that $U_\epsilon \cap \partial\Omega$ can be represented as a graph $x_n = f(x')$ of a certain function f. (In fact one can take $c = r/R$.)

Next, the function w satisfies the heat equation in $U_\epsilon \cap \Omega$ and is nonnegative there for sufficiently small ϵ, as it follows from (9.1). Besides, since u is a classical solution of P for $t < T$, at any point $(\tilde{x}, \tilde{t}) \in U_\epsilon \cap \partial\Omega$, we have

$$\lim_{(x,t) \to (\tilde{x}, \tilde{t})} |\nabla u(x,t)| = 1, \qquad (x,t) \in \Omega$$

which together with (9.1) implies

$$\lim_{(x,t) \to (\tilde{x}, \tilde{t})} w(x,t) > \gamma/2 > 0, \qquad (x,t) \in U_\epsilon \cap \Omega.$$

This is enough to conclude that $w \geq \delta > 0$ in $U_{\epsilon/2} \cap \Omega$. Indeed, consider the solution h of the Dirichlet problem in U_ϵ for the heat equation such that

$$h = \gamma/2 \text{ on } \partial_p U_\epsilon \cap \{x_n = \epsilon\} \quad \text{and} \quad h = 0 \text{ on } \partial_p U_\epsilon \setminus \{x_n = \epsilon\},$$

where $\partial_p U_\epsilon = \partial U_\epsilon \setminus \{t = T\}$ is the parabolic boundary of U_ϵ. Then $h \geq \delta$ in $U_{\epsilon/2}$ for some constant $\delta > 0$. Now applying comparison principle in $U_\epsilon \cap \Omega$, we conclude that $w \geq h$ there and hence $w \geq \delta > 0$ in $U_{\epsilon/2} \cap \Omega$. It is not difficult to understand that this proves the lemma. □

In the next step we make the change of variables, which reduces the free boundary problem P to a problem with fixed (and flat) boundary.

Lemma 9.2. *If $\Omega(T)$ is nonempty, $u(\cdot, t)$ are uniformly $C^{2,\theta}$ for $T - \eta \leq t \leq T$, for some $\theta, \eta > 0$.*

Proof. Take $x^* \in \partial\Omega(T)$. Then from the proof of the previous lemma it follows that there is a direction ν such that $u_\nu(x, t) \geq \delta > 0$ for $(x, t) \in U_\epsilon \cap \Omega$ for some $\epsilon > 0$ and we assume $\nu = e_n$. Consider the mapping $(x, t) \mapsto (y, t) = (x', u(x, t), t)$ of $U_\epsilon \cap \Omega$ to an open subset $V_\epsilon$of $\{(y, t) : y_n > 0\}$. The image of the free boundary $U_\epsilon \cap \partial\Omega$ under this mapping lies on the hyperplane $y_n = 0$. Since the Jacobian of this mapping equals $u_n \geq \delta > 0$, from the implicit function theorem we may assume that there is an inverse mapping $(y, t) \mapsto (y', v(y, t), t)$, where the function v satisfies

$$x_n = v(x', u(x, t), t). \tag{9.2}$$

We refer to the paper by Kinderlehrer and Nirenberg [18] for more details on this and related transformations with applications to free boundary problems. Differentiating (9.2), we find

$$u_n = \frac{1}{v_n}, \quad u_i = -\frac{v_i}{v_n}, \ i = 1, \ldots, n-1, \quad u_t = -\frac{v_t}{v_n} \tag{9.3}$$

where subscripts denote the partial derivatives in the respective direction. Further differentiations reveal that the function v satisfies

$$\sum_{i,j=1}^{n} a_{ij}(\nabla v) v_{ij} - v_t = 0 \qquad \text{in } V_\epsilon, \tag{9.4}$$

where $a_{ij} = 0$ if $i, j < n$, $i \neq j$ and

$$a_{ii} = 1, \ a_{in} = \frac{v_i}{v_n}, \ i = 1, \ldots, n-1 \quad \text{and} \quad a_{nn} = \frac{1 + \sum_{i=1}^{n-1} v_i^2}{v_n^2}. \tag{9.5}$$

Next, the free boundary condition $|\nabla u| = 1$ on $U_\epsilon \cap \partial\Omega$ will read now

$$v_n = \left(1 + \sum_{i=1}^{n-1} v_i^2\right)^{1/2} \qquad \text{on } \partial V_\epsilon \cap \{y_n = 0\}. \tag{9.6}$$

In this form, the problem (9.4)–(9.6) becomes an oblique derivative problem for a quasilinear uniformly parabolic equation, which was studied by Gary M. Lieberman in [17]. Thus, applying Lemma 13.21 from [17], we find first that v is of Hölder class $H_{1+\alpha}$ in $\overline{V}_{\epsilon/2}$ and then conclude from Theorem 14.22 that v is of class $H_{2+\theta}$ in $\overline{V}_{\epsilon/4}$ for some $\theta > 0$.

Now we go back to function u in $U_{\epsilon/4}$. Since $-u_i/u_n = v_i$ are uniformly $C^{1,\theta}$, this implies immediately that the spatial level lines of $u(\cdot,t)$ are uniformly $C^{2,\theta}$ near the free boundary for $T-\epsilon/4 \le t \le T$. Also, it follows that the functions u itself is $C^{2,\theta}$ in space variable on $\overline{U_{\epsilon/4}\cap\Omega}$. Since we can cover by such sets a ring-shaped neighborhood of the free boundary for $T-\eta \le t \le T$, for some $\eta > 0$, we conclude that $u(\cdot,t) \in C^{2,\theta}(\overline{\Omega(t)})$ for $T-\eta \le t \le T$. □

As a consequence we obtain the following theorem.

Theorem 9.3. *Let u_0 satisfy (C1)–(C3). Then the limit solution of P is a classical solution up to the extinction time t^*. Moreover, the time sections $\Omega(t)$ are convex and $C^{2,\alpha}$ regular for every $t \in (0,t^*)$.* □

References

[1] Caffarelli, L.A.; Vazquez, J.L. A free-boundary problem for the heat equation arising in flame propagation. Trans. Amer. Math. Soc. **1995**, *347*(2), 411–441.

[2] Vazquez, J.L. The free boundary problem for the heat equation with fixed gradient condition. In *Free boundary problems, theory and applications (Zakopane, 1995)*; Pitman Res. Notes Math. Ser. 363; Longman: Harlow, 1996; 277–302.

[3] Caffarelli, L.A; Lederman, C.; Wolanski, N. Uniform estimates and limits for a two phase parabolic singular perturbation problem. Indiana Univ. Math. J. **1997** *46*(2), 453–489.

[4] Caffarelli, L.A.; Lederman C.; Wolanski, N. Pointwise and viscosity solutions for the limit of a two phase parabolic singular perturbation problem. Indiana Univ. Math. J. **1997**, *46*(3), 719–740.

[5] Lederman, C.; Vazquez, J.L.; Wolanski, N. Uniqueness of solution to a free boundary problem from combustion, Trans. Amer. Math. Soc. **2001**, *353*(2), 655–692.

[6] Galaktionov, V.A.; Hulshof, J.; Vazquez, J.L. Extinction and focusing behaviour of spherical and annular flames described by a free boundary problem. J. Math. Pures Appl. (9) **1997**, *76*(7), 563–608.

[7] Meĭrmanov, A.M. On a problem with a free boundary for parabolic equations.(In Russian.) Mat. Sb. (N.S.) **1981**, *115(157)*(4), 532–543.

[8] Andreucci, D.; Gianni, R. Classical solutions to a multidimensional free boundary problem arising in combustion theory. Comm. Partial Differential Equations **1994**, *19*(5–6), 803–826.

[9] Petrosyan, A. Convexity and uniqueness in a free boundary problem arising in combustion theory. To appear in Rev. Mat. Iberoamericana.

[10] Henrot, A.; Shahgholian, H. Existence of classical solutions to a free boundary problem for the p-Laplace operator: (I) the exterior convex case. To appear in J. Reine Angew. Math.

[11] Henrot, A.; Shahgholian, H. Existence of classical solutions to a free boundary problem for the p-Laplace operator: (II) the interior convex case. To appear in Indiana Univ. Math. J.

[12] Daskalopoulos, P.; Lee, K. All the time existence of a smooth solution for the one-phase free boundary problem of the flame type. Preprint.

[13] Beurling, A. On free-boundary problems for the Laplace equation. Sem. analytic functions **1958**, *1*, 248–263.

[14] Athanasopoulos, I.; Caffarelli, L.; Salsa, S. Caloric functions in Lipschitz domains and the regularity of solutions to phase transition problems. Annals of Math. **1996**, *143*, 413–434.

[15] Korevaar, N. Convex solutions to nonlinear elliptic and parabolic boundary value problems. Indiana Univ. Math. J. **1983**, *32*(4), 603–614.

[16] Brascamp, H.J.; Lieb, E.H. On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. J. Functional Analysis **1976**, *22*(4), 366–389.

[17] Lieberman, G.M. *Second order parabolic differential equations*; World Scientific Publishing Co., Inc.: River Edge, NJ, 1996.

[18] Kinderlehrer, D.; Nirenberg, L. Regularity in free boundary problems. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **1977**, *4*(2), 373–391.

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