# PARABOLIC BOUNDARY HARNACK PRINCIPLES IN DOMAINS WITH THIN LIPSCHITZ COMPLEMENT

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Abstract. We prove forward and backward parabolic boundary Harnack principles for nonnegative solutions of the heat equation in the complements of thin parabolic Lipschitz sets given as subgraphs

$$E = \{(x,t) : x_{n-1} \leq f(x'',t), x_n = 0\} \subset \mathbb{R}^{n-1} \times \mathbb{R}$$

for parabolically Lipschitz functions $f$ on $\mathbb{R}^{n-2} \times \mathbb{R}$.

We are motivated by applications to parabolic free boundary problems with thin (i.e co-dimension two) free boundaries. In particular, at the end of the paper we show how to prove the spatial $C^{1,\alpha}$ regularity of the free boundary in the parabolic Signorini problem.

## 1. Introduction

The purpose of this paper is to study forward and backward boundary Harnack principles for nonnegative solutions of the heat equation in a certain type of domains in $\mathbb{R}^n \times \mathbb{R}$, which are, roughly speaking, complements of thin parabolically Lipschitz sets $E$. By the latter we understand closed sets, lying in the vertical hyperplane $\{x_n = 0\}$, and which are locally given as subgraphs of parabolically Lipschitz functions (see Fig. 1).

This kind of sets appear naturally in free boundary problems governed by parabolic equations, where the free boundary lies in a given hypersurface and thus has co-dimension two. Such free boundaries are also known as thin free boundaries. In particular, our study was motivated by the parabolic Signorini problem, recently studied in [DGPT13]. The boundary Harnack principles that we prove in this paper provide important technical tools in problems with thin free boundaries. For instance, they open up the possibility for proving that the thin Lipschitz free boundaries have Hölder continuous spatial normals, following the original idea in [AC85]. In particular, we show that this argument indeed can be successfully carried out in the parabolic Signorini problem.

We have to point out that the elliptic counterparts of the results in this paper are very well known, see e.g. [AC85, CSS08, ALM03]. However, there are significant differences between the elliptic and parabolic boundary Harnack principles, mostly because of the time-lag in the parabolic Harnack inequality. This results in two types of the boundary Harnack principles for the parabolic equations: the forward one (also known as the Carleson estimate) and the backward one. Besides, those

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results are known only for a much smaller class of domains than in the elliptic case. Thus, to put our results in a better perspective, we start with a discussion of the known results both in the elliptic and parabolic cases.

**Elliptic boundary Harnack principle.** By now classical boundary Harnack principle for harmonic functions [Kem72a, Dah77, Wu78] says that if $D$ is a bounded Lipschitz domain in $\mathbb{R}^n$, $x_0 \in \partial D$, and $u$ and $v$ are positive harmonic functions on $D$ vanishing on $B_r(x_0) \cap \partial D$ for a small $r > 0$, then there exist positive constants $M$ and $C$, depending only on the dimension $n$ and the Lipschitz constant of $D$, such that

$$\frac{u(x)}{v(x)} \leq C \frac{u(y)}{v(y)} \quad \text{for } x, y \in B_{r/M}(x_0) \cap D.$$

Note that this result is scale-invariant, hence by a standard iterative argument, one then immediately obtains that the ratio $u/v$ extends to $\overline{D} \cap B_{r/M}(x_0)$ as a Hölder continuous function. Roughly speaking, this theorem says that two positive harmonic functions vanishing continuously on a certain part of the boundary will decay at the same rate near that part of the boundary.

The above boundary Harnack principle depends heavily on the geometric structure of the domains. The scale invariant boundary Harnack principle (among other classical theorems of real analysis) was extended by [JK82] from Lipschitz domains to the so-called NTA (non-tangentially accessible) domains. Moreover, if the Euclidean metric is replaced by the internal metric, then similar results hold for so-called uniform John domains [ALM03, Aik05].

In particular, the boundary Harnack principle is known for the domains of the following type

$$D = B_1 \setminus E_f, \quad E_f = \{x \in \mathbb{R}^n : x_{n-1} \leq f(x''), x_n = 0\},$$

where $f$ is a Lipschitz function on $\mathbb{R}^{n-2}$, with $f(0) = 0$, where it is used for instance in the thin obstacle problem [AC85, ACS08, CSS08]. In fact, there is a relatively simple proof of the boundary Harnack principle for the domains as above, already indicated in [AC85]: there exists a bi-Lipschitz transformation from $D$ to a halfball $B_1^+$, which is a Lipschitz domain. The harmonic functions in $D$ transform to solutions of a uniformly elliptic equation in divergence form with bounded measurable coefficients in $B_1^+$, for which the boundary Harnack principle is known [CFMS81].

**Parabolic boundary Harnack principle.** The parabolic version of the boundary Harnack principle is much more challenging than the elliptic one, mainly because of the time-lag issue in the parabolic Harnack inequality. The latter is called sometimes the forward Harnack inequality, to emphasize the way it works: for non-negative caloric functions (solutions of the heat equation), if the earlier value is positive at some spatial point, after a necessary waiting time, one can expect that the value will become positive everywhere in a compact set containing that point. Under the condition that the caloric function vanishes on the lateral boundary of the domain, one may overcome the time-lag issue and get a backward type Harnack principle (so combining together one gets an elliptic-type Harnack inequality)

The forward and backward boundary Harnack principle are known for parabolic Lipschitz domains, not necessarily cylindrical, see [Kem72b, FGS84, Sal81]. Moreover, they were shown more recently in [HLN04] to hold for unbounded parabolically

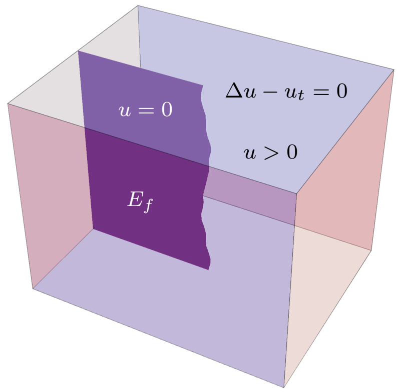


FIGURE 1. Domain with a thin Lipschitz complement

Reifenberg flat domains. In this paper, we will generalize the parabolic boundary Harnack principle to the domains of the following type (see Figure 1):

$$D = \Psi_1 \setminus E_f,$$

where

$$\Psi_1 = \{(x,t) : |x_i| < 1, i = 1, \dots, n-2, |x_{n-1}| < 4nL, |x_n| < 1, |t| < 1\};$$
$$E_f = \{(x,t) : x_{n-1} \le f(x'',t), x_n = 0\}$$

and $f(x'',t)$ is a parabolically Lipschitz function satisfying

$$|f(x'',t) - f(y'',s)| \le L(|x''-y''|^2 + |t-s|)^{1/2}; \quad f(0,0) = 0.$$

Note that $D$ is not cylindrical ($E_f$ is not time invariant), and it does not fall into any category of the domains on which the forward or backward Harnack principle is known. Inspired by the elliptic inner NTA domains (see e.g. [ACS08]), it seems natural to equip the domain $D$ with the intrinsic geodesic distance $\rho_D((x,t),(y,s))$, where $\rho_D((x,t),(y,s))$ is defined as the infimum of the Euclidean length of rectifiable curves $\gamma$ joining $(x,t)$ and $(y,s)$ in $D$, and consider the abstract completion $D^*$ of $D$ with respect to this inner metric $\rho_D$. We will not be working directly with the inner metric in this paper, since it seems easier to work with the Euclidean parabolic cylinders due to the time-lag issues and different scales in space and time variables. However, we do use the fact that the interior points of $E_f$ (in relative topology) correspond to two different boundary points in the completion $D^*$.

Even though we assume in this paper that $E_f$ lies on the hyperplane $\{x_n = 0\}$ in $\mathbb{R}^n \times \mathbb{R}$, our proofs, except those on the doubling of the caloric measure and the backward boundary Harnack principle, are easily generalized to the case when $E_f$ is a hypersurface which is Lipschitz in space variable and independent of time variable.

**Structure of the paper.** The paper is organized as follows.

In Section 2 we give basic definitions and introduce the notations used in this paper.

In Section 3 we consider the Perron-Wiener-Brelot (PWB) solution to the Dirichlet problem of the heat equation for $D$. We show that $D$ is regular and has a Hölder continuous barrier function at each parabolic boundary point.

In Section 4 we establish a forward boundary Harnack inequality for nonnegative caloric functions vanishing continuously on a part of the lateral boundary following the lines of Kemper's paper ([Kem72b]).

In Section 5 we study the kernel functions for the heat operator. We show that each boundary point $(y, s)$ in the interior of $E_f$ (as a subset of the hyperplane $\{x_n = 0\}$) corresponds to two independent kernel functions. Hence the parabolic Euclidean boundary for $D$ is not homeomorphic to the parabolic Martin boundary.

In Section 6 we show the doubling property of the caloric measure with respect to $D$, which will imply a backward Harnack inequality for caloric functions vanishing on the whole lateral boundary.

Section 7 is dedicated to various forms of the boundary Harnack principle from Sections 4 and 6, including a version for solutions of the heat equation with a nonzero right-hand side. We conclude the section and the paper with an application to the parabolic Signorini problem.

## 2. Notation and Preliminaries

### 2.1. Basic Notation.

| | |
|---|---|
| $\mathbb{R}^n$ | the $n$-dimensional Euclidean space |
| $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ | for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ |
| $x'' = (x_1, \dots, x_{n-2}) \in \mathbb{R}^{n-2}$ | for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ |

Sometimes it will be convenient to identify $x'$, $x''$ with $(x', 0)$ and $(x'', 0, 0)$, respectively.

| | |
|---|---|
| $x \cdot y = \sum_{i=1}^{n} x_i y_i,$ | the inner product for $x, y \in \mathbb{R}^n$ |
| $\lvert x\rvert = (x \cdot x)^{1/2}$ | the Euclidean norm of $x \in \mathbb{R}^n$ |
| $\lVert (x, t)\rVert = (\lvert x\rvert^2 + \lvert t\rvert)^{1/2}$ | the parabolic norm of $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ |
| $\overline{E}, E^\circ, \partial E$ | the closure, the interior, the boundary of $E$ |
| $\partial_p E$ | the parabolic boundary of $E$ in $\mathbb{R}^n \times \mathbb{R}$ |
| $B_r(x) := \{y \in \mathbb{R}^n : \lvert x - y\rvert < r\}$ | open ball in $\mathbb{R}^n$ |
| $B'_r(x'), B''_r(x'')$ | (thin) open balls in $\mathbb{R}^{n-1}$, $\mathbb{R}^{n-2}$ |
| $Q_r(x, t) := B_r(x) \times (t - r^2, t)$ | lower parabolic cylinders in $\mathbb{R}^n \times \mathbb{R}$ |
| $\operatorname{dist}_p(E, F) = \inf_{\substack{(x,t)\in E\\ (y,s)\in F}} \lVert (x - y, t - s)\rVert$ | the parabolic distance between sets $E$, $F$ |

We will also need the notion of *parabolic Harnack chain* in a domain $D \subset \mathbb{R}^n \times \mathbb{R}$. For two points $(z_1, h_1)$ and $(z_2, h_2)$ in $D$ with $h_2 - h_1 \geq \mu^2 \lvert z_2 - z_1\rvert^2$, $0 < \mu < 1$, we say that a sequence of parabolic cylinders $Q_{r_i}(x_i, t_i) \subset D$, $i = 1, \dots, N$ is a Harnack chain from $(z_1, h_1)$ to $(z_2, h_2)$ with a constant $\mu$ if

$$(z_1,h_1)\in Q_{r_1}(x_1,t_1),\quad (z_2,h_2)\in Q_{r_N}(x_N,t_N)$$
$$\mu\, r_i\le \operatorname{dist}_p(Q_{r_i}(x_i,t_i),\partial_p D)\le \frac{1}{\mu}r_i,\quad i=1,\dots,N,$$
$$Q_{r_{i+1}}(x_{i+1},t_{i+1})\cap Q_{r_i}(x_i,t_i)\neq\emptyset,\quad i=1,\dots,N-1,$$
$$t_{i+1}-t_i\ge \mu^2 r_i^2,\quad i=1,\dots,N-1.$$

The number $N$ is called the length of the Harnack chain. By the parabolic Harnack inequality, if $u$ is a nonnegative caloric function in $D$ and there is a Harnack chain of length $N$ and constant $\mu$ from $(z_1,h_1)$ to $(z_2,h_2)$, then

$$u(z_1,h_1)\le C(\mu,n,N)\,u(z_2,h_2).$$

Further, for given $L\ge 1$ and $r>0$ we also introduce the (elongated) parabolic boxes, specifically adjusted to our purposes

$$\Psi''_r=\{(x'',t)\in\mathbb{R}^{n-2}\times\mathbb{R}:|x_i|<r, i=1,\dots,n-2,|t|<r^2\}$$
$$\Psi'_r=\{(x',t)\in\mathbb{R}^{n-1}\times\mathbb{R}:(x'',t)\in\Psi''_r,|x_{n-1}|<4nLr\}$$
$$\Psi_r=\{(x,t)\in\mathbb{R}^{n}\times\mathbb{R}:(x',t)\in\Psi'_r,|x_n|<r\}$$
$$\Psi_r(y,s)=(y,s)+\Psi_r.$$

We also define the following neighborhoods

$$\mathcal{N}_r(E):=\bigcup_{(y,s)\in E}\Psi_r(y,s),\quad \text{for any set } E\subset\mathbb{R}^n\times\mathbb{R}.$$

2.2. **Domains with thin Lipschitz complement.** Let $f:\mathbb{R}^{n-2}\times\mathbb{R}\to\mathbb{R}$ be a parabolically Lipschitz function with a Lipschitz constant $L\ge 1$ in a sense that

$$|f(x'',t)-f(y'',s)|\le L(|x''-y''|^2+|t-s|)^{1/2},\quad (x'',t),(y'',s)\in\mathbb{R}^{n-2}\times\mathbb{R}$$

Then consider the following two sets:

$$G_f=\{(x,t):x_{n-1}=f(x'',t),x_n=0\}$$
$$E_f=\{(x,t):x_{n-1}\le f(x'',t),x_n=0\}$$

We will call them *thin Lipschitz graph* and *subgraph* respectively (with "thin" indicating their lower dimension). We are interested in a behavior of caloric functions in domains of the type $\Omega\setminus E_f$, where $\Omega$ is open in $\mathbb{R}^n\times\mathbb{R}$. We will say that $\Omega\setminus E_f$ is a domain with a *thin Lipschitz complement.*

We are interested mostly in local behavior of caloric functions near the points on $G_f$ and therefore we concentrate our study on the case

$$D=D_f:=\Psi_1\setminus E_f$$

with a normalization condition

$$f(0,0)=0\iff (0,0)\in G_f.$$

We will state most of our results for $D$ defined as above, however, the results will still hold, if we replace $\Psi_1$ in the construction above with a rectangular box

$$\tilde\Psi=\Big(\prod_{i=1}^n(a_i,b_i)\Big)\times(\alpha,\beta)$$

such that for some constants $c_0, C_0 > 0$ depending on $L$ and $n$, we have

$$\tilde\Psi \subset \Psi_{C_0},\quad \Psi_{c_0}(y,s)\subset\tilde\Psi,\quad \text{for all } (y,s)\in G_f,\ s\in[\alpha+c_0^2,\beta-c_0^2]$$

and consider the complement

$$\tilde D = \tilde D_f := \tilde\Psi\setminus E_f.$$

Even more generally, one may take $\tilde\Psi$ to be a cylindrical domain of the type $\tilde\Psi = \mathcal{O}\times(\alpha,\beta)$ where $\mathcal{O}\subset\mathbb{R}^n$ has the property that $\mathcal{O}_\pm = \mathcal{O}\cap\{\pm x_n > 0\}$ are Lipschitz domains. For instance, we can take $\mathcal{O} = B_1$. Again, most of the results that we state will be valid also in this case, with a possible change in constants that appear in estimates.

2.3. **Corkscrew points.** Since will be working in $D = \Psi_1\setminus E_f$ as above, it will be convenient to redefine sets $E_f$ and $G_f$ as follows:

$$G_f = \{(x,t)\in\overline\Psi_1 : x_{n-1} = f(x'',t), x_n = 0\},$$
$$E_f = \{(x,t)\in\overline\Psi_1 : x_{n-1} \le f(x'',t), x_n = 0\},$$

so that they are subsets of $\overline\Psi_1$. It is easy to see from the definition of $D$ that it is connected and its parabolic boundary is given by

$$\partial_p D = \partial_p\Psi_1\cup E_f.$$

As we will see, the domain $D$ has a parabolic NTA-like structure, with the catch that at points on $E_f$ (and close to it) we need to define two pairs of future and past corkscrew points, pointing into $D_+$ and $D_-$ respectively, where

$$D_+ = D\cap\{x_n>0\} = (\Psi_1)_+,\quad D_- = D\cap\{x_n<0\} = (\Psi_1)_-.$$

More specifically, fix $0<r<1/4$ and $(y,s)\in\mathcal{N}_r(E_f)\cap\partial_p D$, define

$$\overline{A}_r^\pm(y,s) = (y'', y_{n-1}+r/2, \pm r/2, s+2r^2),\quad \text{if } s\in[-1,1-4r^2),$$
$$\underline{A}_r^\pm(y,s) = (y'', y_{n-1}+r/2, \pm r/2, s-2r^2),\quad \text{if } s\in(-1+4r^2,1].$$

Note that by definition, we always have $\overline{A}_r^+(y,s)$, $\underline{A}_r^+(y,s)\in D_+$ and $\overline{A}_r^-(y,s)$, $\underline{A}_r^-(y,s)\in D_-$. We also have that

$$\overline{A}_r^\pm(y,s), \underline{A}_r^\pm(y,s)\in\Psi_{2r}(y,s),$$
$$\Psi_{r/2}(\overline{A}_r^\pm(y,s)), \Psi_{r/2}(\underline{A}_r^\pm(y,s))\cap\partial D = \emptyset.$$

Moreover, the corkscrew points have the following property.

**Lemma 2.1** (Harnack chain property I). *Let $0<r<1/4$, $(y,s)\in\partial_p D\cap\mathcal{N}_r(E_f)$, and $(x,t)\in D$ be such that*

$$(x,t)\in\Psi_r(y,s)\quad\text{and}\quad \Psi_{\gamma r}(x,t)\cap\partial_p D = \emptyset.$$

*Then there exists a Harnack chain in $D$ with a constant $\mu$ and length $N$, depending only on $\gamma$, $L$, and $n$, from $(x,t)$ to either $\overline{A}_r^+(y,s)$ or $\overline{A}_r^-(y,s)$, provided $s\le 1-4r^2$, and from either $\underline{A}_r^+(y,s)$ or $\underline{A}_r^-(y,s)$ to $(x,t)$, provided $s\ge -1+4r^2$.*

*In particular, there exists a constant $C = C(\gamma, L, n) > 0$ such that for any nonnegative caloric function $u$ in $D$*

$$u(x,t)\le C\max\{u(\overline{A}_r^+(y,s)), u(\overline{A}_r^-(y,s))\},\quad \text{if } s\le 1-4r^2,$$
$$u(x,t)\ge C^{-1}\min\{u(\underline{A}_r^+(y,s)), (\underline{A}_r^-(y,s))\},\quad \text{if } s\ge -1+4r^2.$$

*Proof.* This is easily seen when $(y,s) \notin \mathcal{N}_r(G_f)$ (in this case the chain length $N$ does not depend on $L$). When $(y,s) \in \mathcal{N}_r(G_f)$, one needs to use the parabolic Lipschitz continuity of $f$. $\square$

Next, we want to define the corkscrew points when $(y,s)$ is further away for $E_f$. Namely, if $(y,s) \in \partial_p D \setminus (\mathcal{N}_r(E_f))$, we define a single pair of future and past corkscrew points by

$$\overline{A}_r(y,s) = (y(1-r), s+2r^2), \quad \text{if } s \in [-1, 1-4r^2)$$
$$\underline{A}_r(y,s) = (y(1-r), s-2r^2), \quad \text{if } s \in (-1+4r^2, 1].$$

Note that the points $\overline{A}_r(y,s)$ and $\underline{A}_r(y,s)$ will have properties similar to those of $\overline{A}_r^{\pm}(y,s)$ and $\underline{A}_r^{\pm}(y,s)$. That is,

$$\overline{A}_r(y,s), \underline{A}_r(y,s) \in \Psi_{2r}(y,s),$$
$$\Psi_{r/2}(\overline{A}_r(y,s)), \Psi_{r/2}(\underline{A}_r(y,s)) \cap \partial D = \emptyset,$$

and we have the following version of Lemma 2.1 above

**Lemma 2.2** (Harnack chain property II)**.** *Let $r \in (0,1/4)$, $(y,s) \in \partial_p D \setminus \mathcal{N}_r(E_f)$ and $(x,t) \in D$ be such that*

$$(x,t) \in \Psi_r(y,s) \quad \text{and} \quad \Psi_{\gamma r}(x,t) \cap \partial_p D = \emptyset.$$

*Then there exists a Harnack chain in $D$ with a constant $\mu$ and length $N$, depending only on $\gamma$, $L$, and $n$, from $(x,t)$ to $\overline{A}_r(y,s)$, provided $s \le 1-4r^2$, and from $\underline{A}_r(y,s)$ to $(x,t)$, provided $s \ge -1+4r^2$.*

*In particular, there exists a constant $C = C(\gamma, L, n) > 0$ such that for any nonnegative caloric function $u$ in $D$*

$$u(x,t) \le C\, u(\overline{A}_r(y,s)) \qquad \text{if } s \le 1-4r^2,$$
$$u(x,t) \ge C^{-1} u(\underline{A}_r(y,s)) \quad \text{if } s \ge -1+4r^2.$$

$\square$

To state our next lemma, we need to use parabolic scaling operator on $\mathbb{R}^n \times \mathbb{R}$. For any $(y,s) \in \mathbb{R}^n \times \mathbb{R}$ and $r > 0$ we define

$$T^r_{(y,s)} : (x,t) \mapsto \left( \frac{x-y}{r}, \frac{t-s}{r^2} \right).$$

**Lemma 2.3** (Localization property)**.** *For $r \in (0,1/4)$ and $(y,s) \in \partial_p D$ and there exists a point $(\tilde{y},\tilde{s}) \in \partial_p D \cap \Psi_{2r}(y,s)$ and $\tilde{r} \in [r, 4r]$ such that*

$$\Psi_r(y,s) \cap D \subset \Psi_{\tilde{r}}(\tilde{y},\tilde{s}) \cap D \subset \Psi_{8r}(y,s) \cap D$$

*and the parabolic scaling $T^{\tilde{r}}_{(\tilde{y},\tilde{s})}(\Psi_{\tilde{r}}(\tilde{y},\tilde{s}) \cap D)$ is either*

1. *a rectangular box $\tilde{\Psi}$ such that $\Psi_{c_0} \subset \tilde{\Psi} \subset \Psi_{C_0}$ for some positive constants $c_0$ and $C_0$ depending on $L$ and $n$.*
2. *union of two rectangular boxes as in (1) with a common vertical side;*
3. *domain $\tilde{D}_{\tilde{f}} = \tilde{\Psi} \setminus E_f$ with a thin Lipschitz complement at the end of Section 2.2.*

*Proof.* Consider the following cases:

1) $\Psi_r(y,s)\cap E_f=\emptyset$. In this case we take $(\tilde y,\tilde s)=(y,s)$ and $\rho=r$. Then $\Psi_r(y,s)\cap\Psi_1$ falls into category (1).

2) $\Psi_r(y,s)\cap E_f\neq\emptyset$, but $\Psi_{2r}(y,s)\cap G_f=\emptyset$. In this case we take $(\tilde y,\tilde s)=(y,s)$ and $\rho=2r$. In this case $\Psi_{2r}(y,s)\cap D$ splits into the disjoint union of $\Psi_{2r}(y,s)\cap(\Psi_1)_\pm$ that falls into category (2).

3) $\Psi_{2r}(y,s)\cap G_f\neq\emptyset$. In this case choose $(\tilde y,\tilde s)\in\Psi_{3r}(y,s)\cap G_f$ with an additional property $-1+r^2/4\leq\tilde s\leq 1-r^2/4$ and let $\rho=4r$. Then $\Psi_\rho(\tilde y,\tilde s)\cap D=(\Psi_\rho(\tilde y,\tilde s)\setminus E_f)\cap\Psi_1$ falls into category (3). $\square$

## 3. Regularity of $D$ for the heat equation

In this section we show that the domains $D$ with thin Lipschitz complement $E_f$ are regular for the heat equation by using the existence of an exterior thin cone at points on $E_f$ and applying Wiener-type criterion for the heat equation [EG82]. Furthermore, we show the existence of Hölder continuous local barriers at the points on $E_f$, which we will use in the next section to prove the Hölder continuity regularity of the solutions up to the parabolic boundary.

3.1. **PWB solutions.** ([Doo01, Lie96]) Given an open subset $\Omega\subset\mathbb{R}^n\times\mathbb{R}$, let $\partial\Omega$ be its Euclidean boundary. Define the parabolic boundary $\partial_p\Omega$ of $\Omega$ to be the set of all points $(x,t)\in\partial\Omega$ such that for any $\varepsilon>0$ the lower parabolic cylinder $Q_\varepsilon(x,t)$ contains points not in $\Omega$.

We say that a function $u:\Omega\to(-\infty,+\infty]$ is supercaloric if $u$ is lower semicontinuous, finite on dense subsets of $\Omega$, and satisfies the comparison principle in each parabolic cylinder $Q\Subset\Omega$: if $v\in C(\overline{Q})$ solves $\Delta v-\partial_t v=0$ in $Q$ and $v=u$ on $\partial_pQ$, then $v\leq u$ in $Q$.

A subcaloric function is defined as the negative of a supercaloric function. A function is caloric if it is supercaloric and subcaloric.

Given $g$, any real-valued function defined on $\partial_p\Omega$, we define the upper solution

$$\begin{aligned}\overline{H}_g=\inf\{u:\ &u\text{ is supercaloric or identically }+\infty\text{ on each component of }\Omega,\\ &\liminf_{(y,s)\to(x,t)}u(y,s)\geq g(x,t)\text{ for all }(x,t)\in\partial_p\Omega, u\text{ bounded below on }\Omega\},\end{aligned}$$

and the lower solution

$$\begin{aligned}\underline{H}_g=\sup\{u:\ &u\text{ is subcaloric or identically }-\infty\text{ on each component of }\Omega,\\ &\limsup_{(y,s)\to(x,t)}u(y,s)\leq g(x,t)\text{ for all }(x,t)\in\partial_p\Omega, u\text{ bounded above on }\Omega\}.\end{aligned}$$

If $\overline{H}_g=\underline{H}_g$, then $H_g=\overline{H}_g=\underline{H}_g$ is the Perron-Wiener-Brelot (PWB) solution to the Dirichlet problem for $g$. It is shown in 1.VIII.4 and 1.XVIII.1 in [Doo01] that if $g$ is a bounded continuous function, then the PWB solution $H_g$ exists and is unique for any bounded domain $\Omega$ in $\mathbb{R}^n\times\mathbb{R}$.

Continuity of the PWB solution at points of $\partial_p\Omega$ is not automatically guaranteed. A point $(x,t)\in\partial_p\Omega$ is a regular boundary point if $\lim_{(y,s)\to(x,t)}H_g(y,s)=g(x,t)$ for every bounded continuous function $g$ on $\partial_pD$. A necessary and sufficient condition for a parabolic boundary point to be regular is the existence of a local barrier for earlier time at that point (Theorem 3.26 in [Lie96]). By a local barrier at $(x,t)\in\partial_p\Omega$ we mean here a nonnegative continuous function $w$ in $\overline{Q_r(x,t)\cap\Omega}$ for some $r > 0$, which has the following properties: (i) $w$ is supercaloric in $Q_r(x,t)\cap\Omega$; (ii) $w$ vanishes only at $(x,t)$.

3.2. **Regularity of $D$ and barrier functions.** For the domain $D$ defined in the introduction we have $\partial_p D = \partial_p \Psi_1 \cup E_f$. The regularity of $(x,t) \in \partial_p \Psi_1$ follows immediately from the exterior cone condition for the Lipschitz domain. For $(x,t) \in E_f$, instead of the full exterior cone we only know the existence of a flat exterior cone centered at $(x,t)$ by the Lipschitz nature of the thin graph. This will still be enough for the regularity, by the Wiener-type criterion for the heat equation. We give the details below.

For $(x,t) \in E_f$, with parabolically Lipschitz $f$, there exist $c_1, c_2 > 1$, depending on $n$ and $L$, such that the exterior of $D$ contains a flat parabolic cone $\mathcal{C}(x,t)$ defined by

$$\mathcal{C}(x,t) = (x,t) + \mathcal{C}$$
$$\mathcal{C} = \{(y,s) \in \mathbb{R}^n \times \mathbb{R} : s \le 0, y_{n-1} \le -c_1|y''| - c_2\sqrt{-s}, y_n = 0\}.$$

Then by the Wiener-type criterion for the heat equation, established in [EG82], the regularity of $(x,t) \in E_f$ will follow once we show that

$$\sum_{k=1}^{\infty} 2^{kn/2} \operatorname{cap}(\mathcal{A}(2^{-k}) \cap \mathcal{C}) = +\infty,$$

where

$$\mathcal{A}(c) = \{(y,s) : (4\pi c)^{-n/2} \le \Gamma(y,-s) \le (2\pi c)^{-n/2}\},$$

$\Gamma$ is the heat kernel

$$\Gamma(y,s) = \begin{cases} (4\pi s)^{-n/2} e^{-|y|^2/4s}, & s > 0, \\ 0, & s \le 0, \end{cases}$$

and $\operatorname{cap}(K)$ is the thermal capacity for compact set $K$ defined by

$$\operatorname{cap}(K) = \sup\{\mu(K) : \mu \text{ is a nonnegative Radon measure supported in } K, \text{ s.t. } \mu * \Gamma \le 1 \text{ on } \mathbb{R}^n \times \mathbb{R}\}.$$

Because of the self-similarity of $\mathcal{C}$, it is enough to verify that

$$\operatorname{cap}(\mathcal{A}(1) \cap \mathcal{C}) > 0.$$

The latter is easy to see, since we can take as $\mu$ the restriction of $H^n$ Hausdorff measure to $\mathcal{A}(1) \cap \mathcal{C}$ and note that

$$\begin{aligned}(\mu * \Gamma)(x,t) &= \int_{\mathcal{A}(1)\cap\mathcal{C}} \Gamma(x-y, t-s)\, dy'\, ds \\ &\le \int_{-1}^{0} \frac{1}{\sqrt{4\pi(t-s)^+}} ds \le \int_{-1}^{0} \frac{1}{\sqrt{4\pi(-s)}} ds < \infty\end{aligned}$$

for any $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. Since $H^n(\mathcal{A}(1) \cap \mathcal{C}) > 0$, we therefore conclude that $\operatorname{cap}(\mathcal{A}(1) \cap \mathcal{C}) > 0$. We therefore established the following fact.

**Proposition 3.1.** *The domain $D = D_f$ is regular for the heat equation.* □

We next show that we can use the self-similarity of $\mathcal{C}$ to construct a Hölder continuous barrier function at every $(x,t) \in E_f$.

**Lemma 3.2.** *There exist a nonnegative continuous function $U$ on $\overline{\Psi_1}$ with the following properties:*

(i) $U > 0$ *in* $\overline{\Psi_1} \setminus \{(0,0)\}$ *and* $U(0,0) = 0$
(ii) $\Delta U - \partial_t U = 0$ *in* $\Psi_1 \setminus \mathcal{C}$.
(iii) $U(x,t) \le C(|x|^2 + |t|)^{\alpha/2}$ *for* $(x,t) \in \Psi_1$ *and some* $C > 0$ *and* $0 < \alpha < 1$ *depending only on* $n$ *and* $L$.

*Proof.* Let $U$ be a solution of the Dirichlet problem in $\Psi_1 \setminus \mathcal{C}$ with boundary values $U(x,t) = |x|^2 + |t|$ on $\partial_p(\Psi_1 \setminus \mathcal{C})$. Then $U$ will be continuous on $\overline{\Psi}_1$ and will satisfy the following properties:

(i) $U > 0$ in $\overline{\Psi_1} \setminus \{(0,0)\}$ and $U(0,0) = 0$;
(ii) $\Delta U - \partial_t U = 0$ in $\Psi_1 \setminus \mathcal{C}$.

In particular, there exists $c_0 > 0$ and $\lambda > 0$ such that

$$U \ge c_0 \quad \text{on } \partial_p \Psi_1, \quad U \le c_0/2 \quad \text{on } \Psi_\lambda.$$

We then can compare $U$ with its own parabolic scaling. Indeed, let $M_U(r) = \sup_{\Psi_r} U$, for $0 < r < 1$. Then by the comparison principle for the heat equation we will have

$$U(x,t) \le \frac{M_U(r)}{c_0} U(x/r, t/r^2), \quad \text{for } (x,t) \in \Psi_r.$$

(Carefully note that this inequality is satisfied on $\mathcal{C}$ by the homogeneity of the boundary data on $\mathcal{C}$). Hence, we will obtain that

$$M_U(\lambda r) \le \frac{M_U(r)}{2}, \quad \text{for any } 0 < r < 1,$$

which will imply the Hölder continuity of $U$ at the origin by the standard iteration. The proof is complete □

## 4. Forward Boundary Harnack Inequalities

In this section, we show the boundary Hölder regularity of the solutions to the Dirichlet problem and follow the lines of [Kem72b] to show the forward boundary Harnack inequality (Carleson estimate).

We also need the notion of the caloric measure. Given a domain $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ and $(x,t) \in \Omega$, the caloric measure on $\partial_p\Omega$ is denoted by $\omega_\Omega^{(x,t)}$. The following facts about caloric measures can be found in [Doo01]. For $B$ a Borel subset of $\partial_p\Omega$, we have $\omega_\Omega^{(x,t)}(B) = H_{\chi_B}(x,t)$, which is the PWB solution to the Dirichlet problem

$$\Delta u - u_t = 0 \quad \text{in } \Omega; \quad u = \chi_B \text{ on } \partial_p\Omega,$$

where $\chi_B$ is the characteristic function of $B$. Given $g$ a bounded and continuous function on $\partial_p\Omega$, the PWB solution to the Dirichlet problem

$$\Delta u - u_t = 0 \text{ in } \Omega; \quad u = g \text{ on } \partial_p\Omega.$$

is given by $u(x,t) = \int_{\partial_p\Omega} g(y,s) d\omega_\Omega^{(x,t)}(y,s)$. For a regular domain $\Omega$, one has the following useful property of caloric measures ([Doo01]):

**Proposition 4.1.** *If $E$ is a fixed Borel subset of $\partial_p\Omega$, then the function $(x,t) \mapsto \omega_\Omega^{(x,t)}(E)$ extends to $(y,s) \in \partial_p\Omega$ continuously provided $\chi_E$ is continuous at $(y,s)$.*

4.1. **Forward boundary Harnack principle.** From now on, we will write the caloric measure with respect to $D = \Psi_1 \setminus E_f$ as $\omega^{(x,t)}$ for simplicity. Before we prove the forward boundary Harnack inequality, we first show the Hölder continuity of the caloric functions up to the boundary, which follows from the estimates on the barrier function constructed in Section 3.

In what follows, for $0 < r < 1/4$ and $(y, s) \in \partial_p D$ we will denote

$$\Delta_r(y, s) = \Psi_r(y, s) \cap \partial_p D,$$

and call it the *parabolic surface ball* at $(y, s)$ of radius $r$.

**Lemma 4.2.** *Let $0 < r < 1/4$ and $(y, s) \in \partial_p D$. Then there exist $C = C(n, L) > 0$ and $\alpha = \alpha(n, L) \in (0, 1)$ such that if $u$ is positive and caloric in $\Psi_r(y, s) \cap D$ and vanishes continuously on $\Delta_r(y, s)$, then*

$$u(x,t) \leq C \left( \frac{|x-y|^2 + |t-s|}{r^2} \right)^{\alpha/2} M_u(r) \tag{4.1}$$

*for all $(x, t) \in \Psi_r(y, s) \cap D$, where $M_u(r) = \sup_{\Psi_r(y,s)\cap D} u$.*

*Proof.* Let $U$ be the barrier function at $(0, 0)$ in Lemma 3.2 and $c_0 = \inf_{\partial_p \Psi_1} U > 0$. We then use the parabolic scaling $T^r_{(y,s)}$ to construct a barrier function at $(y, s)$. If $(y, s) \in \mathcal{N}_r(E_f)$, then there is an exterior cone $\mathcal{C}(y, s)$ at $(y, s)$ with a universal opening, depending only on $n, L$, and

$$U^r_{(y,s)} := U \circ T^r_{(y,s)}$$

will be a local barrier function at $(y, s)$ and will satisfy

$$0 \leq U^r_{(y,s)}(x,t) \leq C \left( \frac{|x-y|^2 + |t-s|}{r^2} \right)^{\alpha/2}, \text{ for } (x,t) \in \Psi_r(y,s). \tag{4.2}$$

This construction can be made also at $(y, s) \in \partial_p D \setminus \mathcal{N}_r(E_f)$ as these points also have the exterior cone property and we may still use the same formula for $U^r_{(y,s)}$, but after a possible rotation of the coordinate axes in $\mathbb{R}^n$.

Then, by the maximum principle in $\Psi_r(y, s) \cap D$, we easily obtain that

$$u(x,t) \leq \frac{M_u(r)}{c_0} U^r_{(y,s)}(x,t), \quad \text{for } (x,t) \in \Psi_r(y,s) \cap D. \tag{4.3}$$

Combining (4.2) and (4.3) we obtain (4.1). □

The main result in this section is the following forward boundary Harnack principle, also known as the Carleson estimate.

**Theorem 4.3** (Forward boundary Harnack principle or Carleson estimate)**.** *Let $r \in (0, 1/4)$, $(y, s) \in \partial_p D$ with $s \leq 1 - 4r^2$, and $u$ be a nonnegative caloric function in $D$, continuously vanishing on $\Delta_{3r}(y, s)$. Then there exists $C = C(n, L) > 0$ such that for $(x, t) \in \Psi_{r/2}(y, s) \cap D$*

$$u(x,t) \leq C \begin{cases} \max\{u(\overline{A}^+_r(y,s)), u(\overline{A}^-_r(y,s))\}, & \textit{if } (y,s) \in \partial_p D \cap \mathcal{N}_r(E_f) \\ u(\overline{A}_r(y,s)), & \textit{if } (y,s) \in \partial_p D \setminus \mathcal{N}_r(E_f) \end{cases} \tag{4.4}$$

To prove the Carleson estimate above, we need the following two lemmas on the properties of the caloric measure in $D$, which correspond to Lemmas 1.1 and 1.2 in [Kem72b], respectively.

**Lemma 4.4.** *For $0 < r < 1/4$, $(y,s) \in \partial_p D$ with $s \le 1 - 4r^2$, and $\gamma \in (0,1)$, there exists $C = C(\gamma, L) > 0$ such that*

$$\omega^{(x,t)}(\Delta_r(y,s)) \ge C, \quad \text{for } (x,t) \in \Psi_{\gamma r}(y,s) \cap D.$$

*Proof.* Suppose first $(y,s) \in \mathcal{N}_r(E_f)$. Consider the caloric function

$$v(x,t) := \omega^{(x,t)}_{\Psi_r(y,s)\setminus \mathcal{C}(y,s)}(\mathcal{C}(y,s)),$$

where $\mathcal{C}(y,s)$ is the flat exterior cone defined in Section 3. The domain $\Psi_r(y,s) \setminus \mathcal{C}(y,s)$ is regular, hence by Proposition 4.1, $v(x,t)$ is continuous on $\overline{\Psi_{\gamma r}(y,s)}$. We next claim that there exists $C = C(\gamma, n, L) > 0$ such that

$$v(x,t) \ge C \quad \text{in } \Psi_{\gamma r}(y,s).$$

Indeed, consider the normalized version of $v$

$$v_0(x,t) := \omega^{(x,t)}_{\Psi_1 \setminus \mathcal{C}}(\mathcal{C}),$$

which is related to $v$ through the identity $v = v_0 \circ T^r_{(y,s)}$. Then, from the continuity of $v_0$ in $\overline{\Psi_\gamma}$, equality $v_0 = 1$ on $\mathcal{C}$, and the strong maximum principle we obtain that $v_0 \ge C = C(\gamma, n, L) > 0$ on $\overline{\Psi_\gamma}$. Using the parabolic scaling, we obtain the claimed inequality for $v$. Moreover, applying comparison principle to $v(x,t)$ and $\omega^{(x,t)}(\Delta_r(y,s))$ in $D \cap \Psi_r(y,s)$, we have

$$\omega^{(x,t)}(\Delta_r(y,s)) \ge v(x,t) \ge C, \quad \text{for } (x,t) \in D \cap \Psi_{\gamma r}(y,s).$$

In the case when $(y,s) \in \partial_p D \setminus \mathcal{N}_r(E_f)$, we may modify the proof by changing the flat cone $\mathcal{C}(y,s)$ with the full cone contained in the complement of $D$, or directly applying Kemper's Lemma 1.1 in [Kem72b]. □

**Lemma 4.5.** *For $0 < r < 1/4$, $(y,s) \in \partial_p D$ with $s \le 1 - 4r^2$, there exists a constant $C = C(n, L) > 0$, such that for any $r' \in (0, r)$ and $(x,t) \in D \setminus \Psi_r(y,s)$, we have*

$$\omega^{(x,t)}(\Delta_{r'}(y,s)) \le C \begin{cases} \omega^{\overline{A}_r(y,s)}(\Delta_{r'}(y,s)), & \text{if } (y,s) \notin \mathcal{N}_r(E_f); \\ \max\{\omega^{\overline{A}^+_r(y,s)}(\Delta_{r'}(y,s)), \omega^{\overline{A}^-_r(y,s)}(\Delta_{r'}(y,s))\}, & \\ & \text{if } (y,s) \in \mathcal{N}_r(E_f). \end{cases} \tag{4.5}$$

*Proof.* For notational simplicity, we define

$$\begin{aligned} &\Delta' := \Delta_{r'}(y,s), \quad \Delta := \Delta_r(y,s), \\ &\Psi^k := \Psi_{2^{k-1}r'}(y,s), \\ &\overline{A}^\pm_k := \overline{A}^\pm_{2^{k-1}r'}(y,s), \quad \text{if } \Psi^k \cap E_f \ne \emptyset \\ &\overline{A}_k := \overline{A}_{2^{k-1}r'}(y,s), \quad \text{if } \Psi^k \cap E_f = \emptyset \\ &\qquad \text{for } k = 0, 1, \dots, \ell \quad \text{with} \quad 2^{\ell-1}r' < 3r/4 < 2^\ell r'. \end{aligned}$$

We want to clarify here that for $(y,s) \notin E_f$ and small $r'$ and $k$, it may happen that $\Psi^k$ does not intersect $E_f$. To be more specific, let $\ell_0$ be the smallest nonnegative integer such that $\Psi^{\ell_0} \cap E_f \ne \emptyset$. Then we define $\overline{A}_k$ for $0 \le k \le \min\{\ell_0 - 1, \ell\}$ and the pair $\overline{A}^\pm_k$ for $\ell_0 \le k \le \ell$.

To prove the lemma, we want to show that there exists a universal constant $C$, in particular independent of $k$, such that for $(x,t) \in D \setminus \Psi^k$

$$(S_k) \qquad \omega^{(x,t)}(\Delta') \le C \begin{cases} \omega^{\overline{A}_k}(\Delta'), & \text{if } 1 \le k \le \min\{\ell_0 - 1, \ell\}, \\ \max\{\omega^{\overline{A}_k^+}(\Delta'), \omega^{\overline{A}_k^-}(\Delta')\}, & \text{if } \ell_0 \le k \le \ell. \end{cases}$$

Once this is established, (4.5) will follow from $(S_l)$ and the Harnack inequality.

The proof of $(S_k)$ is going to be by induction in $k$. We start with an observation that by the Harnack inequality, there is $C_1 > 0$ independent of $k$, $r'$ such that (4.6)

$$\begin{aligned} \omega^{\overline{A}_k}(\Delta') &\le C_1 \omega^{\overline{A}_{k+1}}(\Delta') && \text{for } 0 \le k \le \min\{\ell_0 - 2, \ell - 1\} \\ \omega^{\overline{A}_{\ell_0 - 1}}(\Delta') &\le C_1 \max\{\omega^{\overline{A}_{\ell_0}^+}(\Delta'), \omega^{\overline{A}_{\ell_0}^-}(\Delta')\}, && \text{if } \ell_0 \le \ell \\ \omega^{\overline{A}_k^\pm}(\Delta') &\le C_1 \omega^{\overline{A}_{k+1}^\pm}(\Delta'), && \text{for } \ell_0 \le k \le \ell - 1. \end{aligned}$$

*Proof of* $(S_1)$: Without loss of generality assume $(y,s) \in \partial_p D \cap \overline{D}_+$.

Case 1) Suppose first that $\Psi^1 \cap E_f = \emptyset$, i.e., $\ell_0 > 1$. In this case $\overline{A}_0 = \overline{A}_{r'/2}(y,s) \in \Psi_{(3/4)r'}(y,s)$ and by Lemma 4.4 there exists a universal $C_0 > 0$, such that $\omega^{\overline{A}_0}(\Delta') \ge C_0$. By (4.6) we have $\omega^{\overline{A}_0}(\Delta') \le C_1 \omega^{\overline{A}_1}(\Delta')$. Letting $C_2 = C_1/C_0$, we then have

$$\omega^{(x,t)}(\Delta') \le 1 \le C_2 \omega^{\overline{A}_1}(\Delta'). \tag{4.7}$$

Case 2) Suppose now, $\Psi^1 \cap E_f \neq \emptyset$, but $\Psi^0 \cap E_f = \emptyset$, i.e., $\ell_0 = 1$. In this case we start as in Case 1) and finish by applying the second inequality in (4.6), which yields

$$\omega^{(x,t)}(\Delta') \le 1 \le C_2 \max\{\omega^{\overline{A}_1^+}(\Delta'), \omega^{\overline{A}_1^-}(\Delta')\}. \tag{4.8}$$

Case 3) Finally, assume that $\Psi^0 \cap E_f \neq \emptyset$, i.e., $\ell_0 = 0$. Without loss of generality assume also that $(y,s) \in \partial_p D \cap \overline{D}_+$. In this case $\overline{A}_0^+ \in \Psi_{(3/4)r'}(y,s)$ and therefore $\omega^{\overline{A}_0^+}(\Delta') \ge C_0$. Besides, by (4.6), we have that $\omega^{\overline{A}_0^+}(\Delta') \le C_1 \omega^{\overline{A}_1^+}(\Delta')$, which yields

$$\omega^{(x,t)}(\Delta') \le 1 \le C_2 \omega^{\overline{A}_1^+}(\Delta'). \tag{4.9}$$

This proves $(S_1)$ with the constant $C = C_2$.

We now turn to the proof of the induction step.

*Proof of* $(S_k) \Rightarrow (S_{k+1})$: More precisely, we will show that if $(S_k)$ holds with some universal constant $C$ (to be specified) then $(S_{k+1})$ also holds with the same constant.

By the maximum principle, we need to verify $(S_{k+1})$ for $(x,t) \in \partial_p(D \setminus \Psi^{k+1})$. Since $\omega^{(x,t)}(\Delta')$ vanishes on $(\partial_p D) \setminus \Psi^{k+1}$, we may assume that $(x,t) \in (\partial \Psi^{k+1}) \cap D$. We will need to consider three cases, as in the proof of $(S_1)$:

1) $\Psi^{k+1} \cap E_f = \emptyset$, i.e., $\ell_0 > k+1$;
2) $\Psi^{k+1} \cap E_f \neq \emptyset$, but $\Psi^k \cap E_f = \emptyset$, i.e., $\ell_0 = k+1$;
3) $\Psi^k \cap E_f \neq \emptyset$, i.e., $\ell_0 \le k$.

Since the proof is similar in all three cases, we will treat only Case 2) in detail.

Case 2) So suppose $\Psi^{k+1} \cap E_f \neq \emptyset$ but $\Psi^k \cap E_f = \emptyset$. We consider two subcases, depending weather $(x,t) \in \partial\Psi^{k+1}$ is close to $\partial_p D$ or not.

Case 2a) First assume that $(x,t) \in \mathcal{N}_{\mu 2^k r'}(\partial_p D)$ for some small positive $\mu = \mu(L,n) < 1/2$ (to be specified). Take $(z,h) \in \Psi_{\mu 2^k r'}(x,t) \cap \partial_p D$ and observe that $\omega^{(x,t)}(\Delta')$ is caloric in $\Psi_{2^{k-1}r'}(z,h) \cap D$ and vanishes continuously on $\Delta_{2^{k-1}r'}(z,h)$ (by Proposition 4.1). Besides, by the induction assumption that $(S_k)$ holds, we have

$$\omega^{(x,t)}(\Delta') \leq C\omega^{\overline{A}_k}(\Delta'), \quad \text{for } (x,t) \in \Psi_{2^{k-1}r'}(z,h) \cap D \subset D \setminus \Psi^k.$$

Hence, by Lemma 4.2, if $\mu = \mu(n,L) > 0$ is small enough, we obtain that

$$\omega^{(x,t)}(\Delta') \leq \frac{1}{C_1} C\omega^{\overline{A}_k}(\Delta'), \quad \text{for } (x,t) \in \Psi_{\mu 2^k r'}(z,h).$$

Here $C_1$ is the constant in (4.6). This, combined with (4.6), gives

$$\begin{aligned}
\omega^{(x,t)}(\Delta') &\leq \frac{C}{C_1}\omega^{\overline{A}_k}(\Delta') \\
&\leq \frac{C}{C_1} \cdot C_1 \max\{\omega^{\overline{A}^+_{k+1}}(\Delta'), \omega^{\overline{A}^-_{k+1}}(\Delta')\} \\
&= C\max\{\omega^{\overline{A}^+_{k+1}}(\Delta'), \omega^{\overline{A}^-_{k+1}}(\Delta')\}.
\end{aligned}$$

This proves $(S_{k+1})$ for $(x,t) \in \mathcal{N}_{\mu 2^k r'}(\partial_p D) \cap \partial\Psi^{k+1}$.

Case 2b) Assume now $\Psi_{\mu 2^k r'}(x,t) \cap \partial_p D = \emptyset$. In this case, it is easy to see that we can construct a parabolic Harnack chain in $D$ of universal length from $(x,t)$ to either $\overline{A}^+_{k+1}$ or $\overline{A}^-_{k+1}$, which implies that for some universal constant $C_3 > 0$

$$\omega^{(x,t)}(\Delta') \leq C_3 \max\{\omega^{\overline{A}^+_{k+1}}(\Delta'), \omega^{\overline{A}^-_{k+1}}(\Delta')\}.$$

Thus, combing Cases 2a) and 2b), we obtain that $(S_{k+1})$ holds with provided $C = \max\{C_2, C_3\}$. This completes the proof of our induction step in Case 2). As we mentioned earlier, Cases 1) and 3) are obtained by a small modification from Case 1) as in the proof of $(S_1)$. This completes the proof of the lemma. $\square$

Now we prove the Carleson estimate. With Lemma 4.4 and Lemma 4.5 at hand, we use ideas similar to those in [Sal81].

*Proof of Theorem 4.3.* We start with a remark that if $(y,s) \notin \mathcal{N}_{r/4}(E_f)$ then we can restrict $u$ to $D_+$ or $D_-$ and obtain the second estimate in (4.4) from the known result for parabolic Lipschitz domains. We thus consider only the case $(y,s) \in \mathcal{N}_{r/4}(E_f)$. Besides, replacing $(y,s)$ with $(y',s') \in \Psi_{r/4}(y,s) \cap E_f$ we may further assume that $(y,s) \in E_f$, but then we will need to change the assumption that $u$ vanishes on $\Delta_{2r}(y,s)$ and prove the estimate (4.4) for $(x,t) \in \Psi_r(y,s) \cap D$.

With the above assumptions in mind, let $0 < r < 1/4$ and $R = 8r$. Let $\tilde{D}_R(y,s) := \Psi_{\tilde{R}}(\tilde{y},\tilde{s}) \cap D$ be given by the localization property Lemma 2.3. Note that we will be either in case (2) or (3) of that lemma, moreover, we can choose $(\tilde{y},\tilde{s}) = (y,s)$.

For the notational brevity, let $\omega_R^{(x,t)} := \omega_{\tilde{D}_R(y,s)}^{(x,t)}$ be the caloric measure with respect to $\tilde{D}_R(y,s)$. We will also skip the center $(y,s)$ in the notations $\tilde{D}_R(y,s)$ for $\Psi_\rho(y,s)$ and $\Delta_\rho(y,s)$.

Since $u$ is caloric in $\tilde{D}_R$ and continuously vanishes up to $\Delta_{2r}$, we have

$$u(x,t) = \int_{(\partial_p \tilde{D}_R)\setminus\Delta_{2r}} u(z,h)\, d\omega_R^{(x,t)}(z,h), \quad (x,t)\in\tilde{D}_R. \tag{4.10}$$

Note that for $(x,t)\in\Psi_r\cap D$, we have $(x,t)\notin\Psi_{r/2}(z,h)$ for any $(z,h)\in(\partial_p\tilde{D}_R)\setminus\Delta_{2r}$. Hence, applying Lemma 4.5[1] to $\omega_R^{(x,t)}$ in $\tilde{D}_R$, we will have that for $(x,t)\in\Psi_r\cap D$ and sufficiently small $r'$

$$\omega_R^{(x,t)}(\Delta_{r'}(z,h)) \le C\max\left\{\omega_R^{\overline{A}^+_{r/2,R}(z,h)}(\Delta_{r'}(z,h)), \omega_R^{\overline{A}^-_{r/2,R}(z,h)}(\Delta_{r'}(z,h))\right\}$$
$$\text{for } (z,h)\in\mathcal{N}_{r/2}(E_f)\cap(\partial_p\tilde{D}_R)\setminus\Delta_{2r}$$

and

$$\omega_R^{(x,t)}(\Delta_{r'}(z,h)) \le C\,\omega_R^{\overline{A}_{r/2,R}(z,h)}(\Delta_{r'}(z,h)),$$
$$\text{for } (z,h)\in\partial_p\tilde{D}_R\setminus(\mathcal{N}_{r/2}(E_f)\cup\Delta_{2r}),$$

where $C=C(L,n)$ and by $\overline{A}^\pm_{r/2,R}$ and $\overline{A}_{r/2,R}$ we denote the corkscrew points with respect to the domain $\tilde{D}_R$. To proceed, we note that for $(z,h)\in\partial_p\tilde{D}_R$ with $h>s+r^2$, by the maximum principle

$$\omega_R^{(x,t)}(\Delta_{r'}(z,h)) = 0$$

for any $(x,t)\in\Psi_r\cap D$ provided $r'$ is small enough. For $(z,h)\in(\partial_p\tilde{D}_R)\setminus\Delta_{2r}$ with $h\le s+r^2$, we note that with the help of Lemmas 2.1 and 2.2 we can construct a Harnack chain of controllable length in $D$ from $\overline{A}^\pm_{r/2,R}(z,h)$ or $\overline{A}_{r/2,R}(z,h)$ to $\overline{A}^+_r(y,s)$ or $\overline{A}^-_r(y,s)$ (corkscrew points with respect to the original $D$). This will imply that for $(x,t)\in\Psi_r\cap D$ and $(z,h)\in\partial_p\tilde{D}_R\setminus\Delta_{2r}$

$$\omega_R^{(x,t)}(\Delta_{r'}(z,h)) \quad\le\quad C\max\{\omega_R^{\overline{A}^+_r(y,s)}(\Delta_{r'}(z,h)), \omega_R^{\overline{A}^-_r(y,s)}(\Delta_{r'}(z,h))\}. \tag{4.11}$$

We now want to apply Besicovitch's theorem on the differentiation of Radon measures. However, since $\partial_p\tilde{D}_R$ locally is not topologically equivalent to a Euclidean space, we make the following symmetrization argument. For $x\in\mathbb{R}^n$ let $\hat{x}$ be its mirror image with respect to the hyperplane $\{x_n=0\}$. We then can write

$$\begin{aligned}
u(x,t)+u(\hat{x},t) &= \int_{\partial_p\tilde{D}_R\setminus\Delta_{2r}} [u(z,h)+u(\hat{z},h)]\,d\omega_R^{(x,t)}(z,h)\\
&= \frac{1}{2}\int_{\partial_p\tilde{D}_R\setminus\Delta_{2r}} [u(z,h)+u(\hat{z},h)]\left(d\omega_R^{(x,t)}(z,h)+d\omega_R^{(\hat{x},t)}(z,h)\right)\\
&= \int_{\partial_p((\tilde{D}_R)_+)\setminus\Delta_{2r}} [u(z,h)+u(\hat{z},h)]\chi\left(d\omega_R^{(x,t)}(z,h)+d\omega_R^{(\hat{x},t)}(z,h)\right),
\end{aligned}$$

where $\chi=1/2$ on $\partial_p((\tilde{D}_R)_+)\cap\{x_n=0\}$ and $\chi=1$ on the remaining part of $\partial_p((\tilde{D}_R)_+)$ and the measures $d\omega_R^{(x,t)}$ and $d\omega_R^{(\hat{x},t)}$ are extended as zero on the thin space outside $E_f$, i.e., on $\partial_p((\tilde{D}_R)_+)\setminus\partial_p\tilde{D}_R$. We then use the estimate (4.11) for $(x,t)$ and $(\hat{x},t)$ in $\Psi_r\cap D$. Now note that in this situation we can apply Besicovitch's

[1] We have to scale the domain $\tilde{D}_R$ with $T^{\tilde{R}}_{(\tilde{y},\tilde{s})}$ first and apply Lemma 4.5 to $r/2\tilde{R}<1/8$ if we are in case (3) of the localization property Lemma 2.3; in the case (2) we apply the known results for parabolic Lipschitz domains.

theorem on differentiation, since we can locally project $\partial_p((\tilde{D}_R)_+)$ to hyperplanes, similarly to [Hun70]. This will yield

$$\frac{d\omega_R^{(x,t)}(z,h) + d\omega_R^{(\hat{x},t)}(z,h)}{d\omega_R^{\overline{A}_r^+(y,s)}(z,h) + d\omega_R^{\overline{A}_r^-(y,s)}(z,h)} \le C \tag{4.12}$$

for $(z,h) \in \partial_p((\tilde{D}_R)_+) \setminus \Delta_{2r}$ and $(x,t) \in \Psi_r \cap D$. Hence, we obtain

$$\begin{aligned} &u(x,t) + u(\hat{x},t) \\ &\quad \le C \int_{\partial_p((\tilde{D}_R)_+)\setminus\Delta_{2r}} [u(z,h) + u(\hat{z},h)] \left( d\omega_R^{\overline{A}_r^+(y,s)}(z,h) + d\omega_R^{\overline{A}_r^-(y,s)}(z,h) \right) \\ &\quad \le C \left( u(\overline{A}_r^+(y,s)) + u(\overline{A}_r^-(y,s)) \right), \\ &\quad \le C \max\{u(\overline{A}_r^+(y,s)), u(\overline{A}_r^-(y,s))\}, \quad (x,t) \in \Psi_r \cap D. \end{aligned}$$

This completes the proof of the theorem. $\square$

The following theorem is a useful consequence of Theorem 4.3, whose proof is similar to that of Theorem 1.1 in [FGS86] with Theorem 4.3 above in hand. Hence here we only state the theorem without giving a proof.

**Theorem 4.6.** *For $0 < r < 1/4$, $(y,s) \in \partial_p D$ with $s \le 1 - 4r^2$, let $u$ be caloric in $D$ and continuously vanishes on $\partial_p D \setminus \Delta_{r/2}(y,s)$. Then there exists $C = C(n,L)$ such that for $(x,t) \in D \setminus \Psi_r(y,s)$ we have*

$$u(x,t) \le C \begin{cases} \max\{u(\overline{A}_r^+(y,s)), u(\overline{A}_r^-(y,s))\}, & \text{if } (y,s) \in \mathcal{N}_r(E_f) \\ u(\overline{A}_r(y,s)), & \text{if } (y,s) \notin \mathcal{N}_r(E_f). \end{cases} \tag{4.13}$$

*Moreover, applying Lemma 4.4 and the maximum principle we have: for $(x,t) \in D \setminus \Psi_r(y,s)$,*

$$\begin{aligned} u(x,t) \le C\omega^{(x,t)}(\Delta_{2r}(y,s)) \times \\ \times \begin{cases} \max\{u(\overline{A}_r^+(y,s)), u(\overline{A}_r^-(y,s))\}, & \text{if } (y,s) \in \mathcal{N}_r(E_f) \\ u(\overline{A}_r(y,s)), & \text{if } (y,s) \notin \mathcal{N}_r(E_f). \end{cases} \end{aligned} \tag{4.14}$$

## 5. Kernel functions

Before proceeding to the backward boundary Harnack principle, we need the notion of kernel functions associated to the heat operator and the domain $D$. In [FGS86], the backward Harnack principle is a consequence of the global comparison principle (Theorem 6.4) by a simple time-shifting argument. In our case, since $D$ is not cylindrical, the above simple argument does not work. So we will first prove some properties of the kernel functions which can be used to show the doubling property of the caloric measures as in [Wu79]. Then, using arguments as in [FGS86], we obtain the the backward Harnack principle.

5.1. **Existence of kernel functions.** Let $(X,T) \in D$ be fixed. Given $(y,s) \in \partial_p D$ with $s < T$, a function $K(x,t;y,s)$ defined in $D$ is called a kernel function at $(y,s)$ for the heat equation with respect to $(X,T)$ if,

(i) $K(\cdot,\cdot;y,s) \ge 0$ in $D$,
(ii) $(\Delta - \partial_t)K(\cdot,\cdot;y,s) = 0$ in $D$,

(iii) $\lim\limits_{\substack{(x,t)\to(z,h)\\(x,t)\in D}} K(x,t;y,s)=0$ for $(z,h)\in\partial_p D\setminus\{(y,s)\}$,

(iv) $K(X,T;y,s)=1$.

If $s\geq T$, $K(x,t;y,s)$ will be taken identically equal to zero. We note that by maximum principle $K(x,t;y,s)=0$ when $t<s$.

The existence of the kernel functions (for the heat operator on domain $D$) follows directly from Theorem 4.3. Let $(y,s)\in\partial_p D$ with $s<T-\delta^2$ for some $\delta>0$, consider

$$v_n(x,t)=\frac{\omega^{(x,t)}(\Delta_{\frac{1}{n}}(y,s))}{\omega^{(X,T)}(\Delta_{\frac{1}{n}}(y,s))},\quad (x,t)\in D,\quad \frac{1}{n}<\delta. \tag{5.1}$$

We clearly have $v_n(x,t)\geq 0$, $(\Delta-\partial_t)v_n(x,t)=0$ in $D$ and $v_n(X,T)=1$. Given $\varepsilon\in(0,1/4)$ small, by Theorem 4.6 and the Harnack inequality $\{v_n\}$ is uniformly bounded on $\overline{D\setminus\Psi_\varepsilon(y,s)}$ if $n\geq 2/\varepsilon$. Moreover, by the up to the boundary regularity (see Proposition 4.1 and Lemma 4.2), the family $\{v_n\}$ is uniformly Hölder in $\overline{D\setminus\Psi_\varepsilon(y,s)}$. Hence, up to a subsequence, $\{v_n\}$ converges uniformly on $\overline{D\setminus\Psi_\varepsilon(y,s)}$ to some nonnegative caloric function $v$ satisfying $v(X,T)=1$. Since $\varepsilon$ can be taken arbitrarily small, $v$ vanishes on $\partial_p D\setminus\{(y,s)\}$. Therefore, $v(x,t)$ is a kernel function at $(y,s)$.

**Convention 5.1.** From now on, to avoid cumbersome details we will make a time extension of domain $D$ for $1\leq t<2$ by looking at

$$\tilde{D}=\tilde{\Psi}\setminus E_f,\quad \tilde{\Psi}=(-1,1)^n\times(-1,2)$$

as in Section 2.2. We then fix $(X,T)$ with $T=3/2$ and $X\in\{x_n=0\}$, $X_{n-1}>3nL$ and normalize all kernels $K(\cdot,\cdot;\cdot,\cdot)$ at this point $(X,T)$. In this way we will be able to state the results in this section for our original domain $D$. Alternatively, we could fix $(X,T)\in D$, and then state the results in the part of the domain $D\cap\{(x,t):-1<t<T-\delta^2\}$ with some $\delta>0$, with the additional dependence of constants on $\delta$.

5.2. **Nonuniqueness of kernel functions at $E_f\setminus G_f$.** The idea is, if we consider the completion $D^*$ of domain $D$ with respect to the inner metric $\rho_D$ and let $\partial^*D=D^*\setminus D$, then it is clear that each Euclidean boundary point $(y,s)\in G_f$ and $(y,s)\in\partial_p\Psi_1$ will correspond to only one $(y,s)^*\in\partial^*D$, and each $(y,s)\in E_f\setminus G_f$ will correspond to exactly two points $(y,s)^*_+$, $(y,s)^*_-\in\partial^*D$. It is not hard to imagine that the kernel functions corresponding to $(y,s)^*_+$ and $(y,s)^*_-$ are linearly independent and they are the two linearly independent kernel functions at $(y,s)$. In this section we will make this idea precise by considering two-sided caloric measures $\vartheta_+$ and $\vartheta_-$. We will study the properties of $\vartheta_+$ and $\vartheta_-$ and their relationship with the caloric measure $\omega_D$.

First we introduce some more notations. Given $(y,s)\in\partial_p D\setminus G_f$, let

$$r_0=\sup\{r\in(0,1/4):\Delta_{2r}(y,s)\cap G_f=\emptyset\}. \tag{5.2}$$

Note that $r_0$ is a constant depending on $(y,s)$ and is such that for any $0<r<r_0$, $\Psi_{2r}(y,s)\cap D$ is either separated by $E_f$ into two disjoint sets $\Psi^+_{2r}$ and $\Psi^-_{2r}$ or $\Psi_{2r}(y,s)\cap D\subset D_+$ (or $D_-$). We define for $0<r<r_0$ the following shifting

operators $F_r^+$ and $F_r^-$:

$$F_r^+(x,t) = (x'', x_{n-1} + 4nLr, x_n + r, t + 4r^2), \tag{5.3}$$

$$F_r^-(x,t) = (x'', x_{n-1} + 4nLr, x_n - r, t + 4r^2). \tag{5.4}$$

For any $0 < r < r_0$, define

$$D_r^+ = D \setminus (E_{r,1}^+ \cup E_{r,2}^+ \cup E_{r,3}^+ \cup E_{r,4}^+), \tag{5.5}$$

where

$$\begin{aligned}
E_{r,1}^+ &:= \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : x_{n-1} \le f(x'',t), -r \le x_n \le 0\},\\
E_{r,2}^+ &:= \{(x,t) : 1 - r \le x_n \le 1\},\\
E_{r,3}^+ &:= \{(x,t) : 4nL(1-r) \le x_{n-1} \le 4nL\},\\
E_{r,4}^+ &:= \{(x,t) : 1 - 4r^2 \le t \le 1\}.
\end{aligned}$$

It is easy to see that $D_r^+ \subset D$ and $F_r^+(D_r^+) \subset D$. Similarly we can define $D_r^- \subset D$ satisfying $F_r^-(D_r^-) \subset D$. Notice that $D_r^+ \nearrow D$, $D_r^- \nearrow D$ as $r \searrow 0$. Moreover, it is clear that for each $r \in (0, 1/4)$

$$\mathcal{N}_{1/4}(E_f) \cap \partial_p D \subset (\partial_p D_r^+ \cup \partial_p D_r^-) \cap \partial_p D, \tag{5.6}$$

$$E_f \subset \partial_p D_r^+ \cap \partial_p D_r^-. \tag{5.7}$$

Let $\omega_r^+$ and $\omega_r^-$ denote the caloric measures with respect to $D_r^+$ and $D_r^-$ respectively. Given $(x,t) \in D$ and $r > 0$ small enough such that $(x,t) \in D_r^+ \cap D_r^-$, $\omega_r^{\pm^{(x,t)}}$ are Radon measures on $\partial_p(D_r^\pm) \cap \partial_p(D_\pm)$ (recall $D_+(D_-) = D \cap \{x_n > 0 (< 0)\}$). Moreover, let $K$ be a relatively compact Borel subset of $\partial_p(D_r^\pm) \cap \partial_p(D_\pm)$, by the comparison principle $\omega_r^{\pm^{(x,t)}}(K) \le \omega_{r'}^{\pm^{(x,t)}}(K)$ for $0 < r' < r$. Hence there exist Radon measures $\vartheta_\pm^{(x,t)}$ on $\partial_p(D_r^\pm) \cap \partial_p(D_\pm)$, such that

$$\omega_r^{\pm^{(x,t)}}|_{\partial_p(D_r^\pm) \cap \partial_p(D_\pm)} \overset{*}{\rightharpoonup} \vartheta_\pm^{(x,t)}, \quad r \to 0.$$

For $(y,s) \in (\mathcal{N}_{1/4}(E_f) \cap \partial_p D) \setminus G_f$ and $0 < r < r_0$, denote

$$\Delta_r^\pm(y,s) := \Delta_r(y,s) \cap \partial_p D_\pm, \quad \text{if } \Delta_r(y,s) \cap \partial_p(D_\pm) \ne \emptyset.$$

Note that if $\Delta_r(y,s) \subset E_f$, then $\Delta_r^\pm(y,s) = \Delta_r(y,s)$. It is easy to see that $(x,t) \mapsto \vartheta_\pm^{(x,t)}(\Delta_r^\pm(y,s))$ are caloric in $D$.

To simplify the notations we will write $\Delta_r$, $\Delta_r^\pm$ instead of $\Delta_r(y,s)$, $\Delta_r^\pm(y,s)$. If $\Delta_r(y,s) \cap \partial_p(D_+)$ (or $\Delta_r(y,s) \cap \partial_p(D_-)$) is empty, we set $\vartheta_+^{(x,t)}(\Delta_r^+(y,s)) = 0$ (or $\vartheta_-^{(x,t)}(\Delta_r^-(y,s)) = 0$).

We also note that with Convention 5.1 in mind, the future corkscrew points $\overline{A}_r^\pm(y,s)$ or $\overline{A}_r(y,s)$, $0 < r < r_0$ and are defined for all $s \in [-1, 1]$.

**Proposition 5.2.** *Given $(y,s) \in (\mathcal{N}_{1/4}(E_f) \cap \partial_p D) \setminus G_f$, for $0 < r < r_0$ we have,*

(i)

$$\sup_{(x,t) \in \partial_p D_{r'}^+ \cap D} \vartheta_+^{(x,t)}(\Delta_r^+), \qquad \sup_{(x,t) \in \partial_p D_{r'}^- \cap D} \vartheta_-^{(x,t)}(\Delta_r^-) \to 0, \ \text{as } r' \to 0.$$

(ii) $\vartheta_+^{(x,t)}(\Delta_r^+) + \vartheta_-^{(x,t)}(\Delta_r^-) = \omega^{(x,t)}(\Delta_r)$, *for $(x,t) \in D$.*

(iii) *There exists a constant $C = C(n, L)$ such that for any $0 < r' < r$*

$$\vartheta_+^{(x,t)}(\Delta_{r'}^+) \leq C\vartheta_+^{\overline{A}_r^+(y,s)}(\Delta_{r'}^+)\vartheta_+^{(x,t)}(\Delta_{2r}^+), \quad \textit{for } (x,t) \in D \setminus \Psi_r^+(y,s),$$
$$\vartheta_-^{(x,t)}(\Delta_{r'}^-) \leq C\vartheta_-^{\overline{A}_r^-(y,s)}(\Delta_{r'}^-)\vartheta_-^{(x,t)}(\Delta_{2r}^-), \quad \textit{for } (x,t) \in D \setminus \Psi_r^-(y,s).$$

(iv) *For $(X, T)$ as defined above and $(y, s) \in E_f \setminus G_f$, there exists a positive constant $C = C(n, L, r_0)$ such that*

$$C^{-1}\vartheta_+^{(X,T)}(\Delta_r^+) \leq \vartheta_-^{(X,T)}(\Delta_r^-) \leq C\vartheta_+^{(X,T)}(\Delta_r^+).$$

*Proof. Proof of (i)*: We assume that $\Delta_r^\pm \neq \emptyset$. If either of them is empty, the conclusion holds obviously.

For $0 < r < r_0$ we have

$$\partial_p D_r^+ \cap D = \{(x,t) \in D : x_{n-1} = 4nL(1-r) \text{ or } x_n = 1 - r\} \cup$$
$$\{(x,t) \in D : x_{n-1} \leq f(x'', t), x_n = -r \text{ or } x_{n-1} = f(x'', t), -r \leq x_n < 0\}.$$

Given $(y, s) \in (\mathcal{N}_{1/4}(E_f) \cap \partial_p D) \setminus G_f$, let $0 < r'' < r' < r_0$, then $\omega_{r''}^{+(x,t)}(\Delta_r^+(y, s))$ is caloric in $D_{r''}^+$ and from the way $r_0$ is chosen vanishes continuously on $\Delta_{r_0}(z, h)$ for each $(z, h) \in \partial_p D_{r''}^+ \cap D$. Notice that

$$\partial_p D_{r'}^+ \cap D \subset \bigcup_{(z,h) \in \partial_p D_{r''}^+ \cap D} \Psi_{r_0}(z, h),$$

hence applying Lemma 4.2 in each $\Psi_{r_0}(z, h) \cap D_{r''}^+$ we obtain constants $C = C(n, L)$ and $\gamma = \gamma(n, L)$, $\gamma \in (0, 1)$ such that

$$\omega_{r''}^{+(x,t)}(\Delta_r^+) \leq C\left(\frac{|x - z| + |t - h|^{\frac{1}{2}}}{r_0}\right)^\gamma \leq C\left(\frac{r'}{r_0}\right)^\gamma, \quad \forall (x,t) \in \partial_p D_{r'}^+ \cap D. \tag{5.8}$$

The constant $C$ and $\gamma$ above do not depend on $(z, h) \in \partial_p D_{r''}^+ \cap D$, $r$ or $r''$ because of the existence of the exterior flat parabolic cones centered at each $(z, h)$ with an uniform opening depending only on $n$ and $L$ .

Let $r'' \to 0$ in (5.8), then we get

$$\vartheta_+^{(x,t)}(\Delta_r^+) \leq C\left(\frac{r'}{r_0}\right)^\gamma, \quad \text{uniformly for } (x,t) \in \partial_p D_{r'}^+ \cap D.$$

Therefore,

$$\lim_{r' \to 0} \sup_{(x,t) \in \partial_p D_{r'}^+ \cap D} \vartheta_+^{(x,t)}(\Delta_r^+) = 0,$$

which finishes the proof.

*Proof of (ii)*: Let $\chi_{\Delta_r}$ be the characteristic function of $\Delta_r$ on $\partial_p D$. Let $g_n$ be a sequence of nonnegative continuous functions on $\partial_p D$ such that $g_n \nearrow \chi_{\Delta_r}$. Let $u_n$ be the solution to the heat equation in $D$ with boundary values $g_n$. Then by the maximum principle, $u_n(x, t) \nearrow \omega^{(x,t)}(\Delta_r)$ for $(x, t) \in D$.

Now we estimate $\vartheta_+^{(x,t)}(\Delta_r^+) + \vartheta_-^{(x,t)}(\Delta_r^-)$. Let $u_{n,r'}^+(x, t)$ be the solution to the heat equation in $D_{r'}^+$ with boundary value equal to $g_n$ on $\partial_p D_{r'}^+ \cap \partial_p D$ and equal to $\vartheta_+^{(x,t)}(\Delta_r^+)$ otherwise. Since $\vartheta_+^{(x,t)}(\Delta_r^+) = \lim_{r'' \to 0} \omega_{r''}^{+(x,t)}(\Delta_r^+)$ takes the boundary value $\chi_{\Delta_r^+}$ on $\partial_p D_{r'}^+ \cap \partial_p D$, then by the maximum principle we have $u_{n,r'}^+(x, t) \leq \vartheta_+^{(x,t)}(\Delta_r^+)$ for $(x, t) \in D_{r'}^+$. Similarly, $u_{n,r'}^-(x, t) \leq \vartheta_-^{(x,t)}(\Delta_r^-)$ for

$(x,t) \in D^-_{r'}$. Therefore, for $(x,t) \in D^+_{r'} \cap D^-_{r'}$ and $0 < r' < r$ sufficiently small we have

$$u^+_{n,r'}(x,t) + u^-_{n,r'}(x,t) \le \vartheta_+^{(x,t)}(\Delta^+_r) + \vartheta_-^{(x,t)}(\Delta^-_r). \tag{5.9}$$

Let $r' \searrow 0$, then $D^+_{r'} \cap D^-_{r'} \nearrow D$. By the comparison principle there is a nonnegative function $\tilde u_n$ in $\Psi_1$ and caloric in $D$ such that

$$u^+_{n,r'}(x,t) + u^-_{n,r'}(x,t) \nearrow \tilde u_n(x,t) \text{ as } r' \searrow 0, \quad (x,t) \in D. \tag{5.10}$$

By (i) just shown above and (5.9),

$$\begin{aligned} &\sup_{\partial_p D^+_{r'} \cap D} u^+_{n,r'}(x,t) + \sup_{\partial_p D^-_{r'} \cap D} u^-_{n,r'}(x,t) \\ &\le \sup_{\partial_p D^+_{r'} \cap D} \vartheta_+^{(x,t)}(\Delta^+_r) + \sup_{\partial_p D^-_{r'} \cap D} \vartheta_-^{(x,t)}(\Delta^-_r) \to 0 \quad \text{ as } r' \to 0, \end{aligned}$$

hence it is not hard to see that $\tilde u_n$ takes the boundary value $g_n$ continuously on $\partial_p D$. Hence by the maximum principle $\tilde u_n = u_n$ in $D$. This combined with (5.9) and (5.10) gives

$$u_n(x,t) \le \vartheta_+^{(x,t)}(\Delta^+_r) + \vartheta_-^{(x,t)}(\Delta^-_r). \tag{5.11}$$

Letting $n \to \infty$ in (5.11), we obtain

$$\omega^{(x,t)}(\Delta_r) \le \vartheta_+^{(x,t)}(\Delta^+_r) + \vartheta_-^{(x,t)}(\Delta^+_r).$$

By taking the approximation $g_n \searrow \chi_{\Delta_r}$, $0 \le g_n \le 2$ and $\operatorname{supp} g_n \subset \mathcal{N}_{2r}(E_f) \cap \partial_p D$ we obtain the reverse inequality and hence the equality.

*Proof of (iii)*: We only show it for $\vartheta_+$ and assume additionally $\Delta^\pm_{r'} \neq \emptyset$.

First for $0 < r'' < r' < r_0$, by Lemma 1.1 in [Kem72b] there exists $C = C(n) \ge 0$ such that

$$\omega^{\overline{A}^+_{r'}(y,s)}_{\Psi_{2r'}(y,s) \cap D_+}(\Delta^+_{r'}) \ge C.$$

Applying the comparison principle in $\Psi_{2r'}(y,s) \cap D_+$ we have

$$\vartheta_+^{\overline{A}^+_{r'}(y,s)}(\Delta^+_{r'}) \ge C. \tag{5.12}$$

Next for $0 < r'' < r' < r_0$, applying the same induction arguments as in Lemma 4.5 we have

$$\omega^{+\,(x,t)}_{r''}(\Delta^+_{r'}) \le C \omega^{+\,\overline{A}^+_r(y,s)}_{r''}(\Delta^+_{r'}), \quad \text{for } (x,t) \in D^+_{r''} \setminus (\Psi_r(y,s))_+, \tag{5.13}$$

where $C = C(n,L)$ is independent of $r'$ and $r''$. The reason that $C$ is uniform in $r''$ is as follows. By the maximum principle it is enough to show (5.13) for $(x,t) \in \partial(\Psi_r(y,s))_+ \cap D^+_{r''}$, which is contained in $D_+$. Hence the same iteration procedure as in Lemma 4.5 but only on the $D_+$ side gives (5.13), and the proof is uniform in $r''$. Therefore, letting $r'' \to 0$ in (5.13), we obtain

$$\vartheta_+^{(x,t)}(\Delta^+_{r'}) \le C \vartheta_+^{\overline{A}^+_r(y,s)}(\Delta^+_{r'}).$$

Applying Lemma 4.4 and the maximum principle, we deduce (iii).

*Proof of (iv)*: Applying (iii), (ii), Harnack inequality and Lemma 4.4 we have that for given $(y,s)\in E_f\setminus G_f$ and $0<r<r_0$,

$$\begin{aligned}\vartheta_-^{(X,T)}(\Delta_r^-) &\le C\vartheta_-^{\overline{A}^-_{r_0}(y,s)}(\Delta_r^-)\le C\omega^{\overline{A}^-_{r_0}(y,s)}(\Delta_r)\\ &\le C\omega^{\overline{A}^+_{2r_0}(y,s)}(\Delta_r)\le C\vartheta_+^{\overline{A}^+_{2r_0}(y,s)}(\Delta_r^+)\\ &\le C\vartheta_+^{(X,T)}(\Delta_r^+),\end{aligned}$$

for $C=C(n,L,r_0)$. The second last inequality holds because

$$\vartheta_+^{\overline{A}^+_{2r_0}(y,s)}(\Delta_r^+)\ge\vartheta_-^{\overline{A}^+_{2r_0}(y,s)}(\Delta_r^-), \tag{5.14}$$

which follows from the $x_n$ symmetry of $D$ and the comparison principle. (5.14) together with (ii) just shown above yield the result.

□

Now we use $\vartheta_+$ and $\vartheta_-$ to construct two linear independent kernel functions at $(y,s)\in E_f\setminus G_f$.

**Theorem 5.3.** *Given $(y,s)\in E_f\setminus G_f$, there exist at least two linearly independent kernel functions at $(y,s)$.*

*Proof.* Given $(y,s)\in E_f\setminus G_f$, let $r_0$ be as in (5.2). For $m>1/r_0$ we consider the sequence

$$v_m^+(x,t)=\frac{\vartheta_+^{(x,t)}(\Delta^+_{\frac{1}{m}}(y,s))}{\vartheta_+^{(X,T)}(\Delta^+_{\frac{1}{m}}(y,s))},\quad (x,t)\in D. \tag{5.15}$$

By Proposition 5.2(iii) and the same arguments as in Section 5.1, we have, up to a subsequence, that $v_m(x,t)$ converges to a kernel function at $(y,s)$ normalized at $(X,T)$. We denote it by $K^+(x,t;y,s)$.

If we consider instead

$$v_m^-(x,t)=\frac{\vartheta_-^{(x,t)}(\Delta^-_{\frac{1}{m}}(y,s))}{\vartheta_-^{(X,T)}(\Delta^-_{\frac{1}{m}}(y,s))},\quad (x,t)\in D, \tag{5.16}$$

we will obtain another kernel function at $(y,s)$, which we will denote $K^-(x,t;y,s)$.

We now show that for $(y,s)$ fixed, $K^+(\cdot,\cdot;y,s)$ and $K^-(\cdot,\cdot;y,s)$ are linearly independent. In fact, by Proposition 5.2(i), (5.15) and (5.16) we have $K^+(x,t;y,s)\to 0$ as $(x,t)\to(y,s)$ from $D_-$ and $K^-(x,t;y,s)\to 0$ as $(x,t)\to(y,s)$ from $D_+$. If $K^+(\cdot,\cdot;y,s)=K^-(\cdot,\cdot;y,s)$, then we also have $K^+(x,t;y,s)\to 0$ as $(x,t)\to(y,s)$ from $D_+$, which will mean that $K^+(x,t;y,s)$ is a caloric function continuously vanishing on the whole $\partial_pD$. By the maximum principle $K^+$ will vanish in the entire $D$, which contradicts the normalization condition $K^+(X,T;y,s)=1$. Moreover, since $K^+(X,T;y,s)=K^-(X,T;y,s)=1$, it is impossible that $K^+(\cdot,\cdot;y,s)=\lambda K^-(\cdot,\cdot;y,s)$ for a constant $\lambda\ne 1$. Hence $K^+$ and $K^-$ are linearly independent. □

*Remark* 5.4. The non-uniqueness of the kernel functions at $(y,s)$ shows that the parabolic Martin boundary of $D$ is not homeomorphic to Euclidean parabolic boundary $\partial_pD$.

Next we show $K^+$ and $K^-$ in fact span the space of all the kernel functions at $(y,s)$. We use an argument similar to the one in [Kem72b].

**Lemma 5.5.** *Let $(y,s) \in E_f \setminus G_f$. There exists a positive constant $C = C(n, L, r_0)$ such that if $u$ is a kernel function at $(y,s)$ in $D$, we have either*

$$u \geq CK^+, \tag{5.17}$$

*or*

$$u \geq CK^-. \tag{5.18}$$

*Here $K^+$, $K^-$ are the kernel functions at $(y,s)$ constructed from* (5.15) *and* (5.16).

*Proof.* For $0 < r < r_0$ we consider $u_r^\pm : D_r^\pm \to \mathbb{R}$, where $u_r^\pm(x,t) = u(F_r^\pm(x,t))$. $u_r^\pm$ are caloric in $D_r^\pm$ and continuous up to the boundary. Then for $(x,t) \in D_r^\pm$,

$$\begin{aligned} u_r^\pm(x,t) &= \int_{\partial_p D_r^\pm} u_r^\pm(z,h) d\omega_r^{\pm^{(x,t)}}(z,h) \geq \int_{\Delta_r^\pm(y,s)} u_r^\pm(z,h) d\omega_r^{\pm^{(x,t)}}(z,h) \\ &\geq \inf_{(z,h)\in\Delta_r^\pm(y,s)} u_r^\pm(z,h) \omega_r^{\pm^{(x,t)}}(\Delta_r^\pm(y,s)). \end{aligned}$$

Note that the parabolic distance between $F_r^\pm(\Delta_r^\pm(y,s))$ and $\partial_p D$ is equivalent to $r$ and the time lag between it and $\overline{A}_r^\pm(y,s)$ is equivalent to $r^2$, hence by the Harnack inequality there exists $C = C(n,L)$ such that

$$\inf_{(z,h)\in\Delta_r^\pm(y,s)} u_r^\pm(z,h) \geq Cu(\overline{A}_r^\pm(y,s)).$$

Hence,

$$u_r^\pm(x,t) \geq Cu(\overline{A}_r^\pm(y,s)) \omega_r^{\pm^{(x,t)}}(\Delta_r^\pm(y,s)), \quad \text{for } (x,t) \in D_r^\pm. \tag{5.19}$$

On the other hand, $u$ is a kernel function at $(y,s)$ and vanishes on $\partial_p D \setminus \Delta_{r/4}(y,s)$ for any $0 < r < 1$. Applying Theorem 4.6 we obtain

$$\begin{aligned} u(x,t) \leq C \max\{u(\overline{A}^+_{r/2}(y,s)), u(\overline{A}^-_{r/2}(y,s))\} \omega^{(x,t)}(\Delta_r(y,s)), \\ \text{for } (x,t) \in D \setminus \Psi_{r/2}(y,s). \end{aligned} \tag{5.20}$$

*Case 1.* $u(\overline{A}^+_{r/2}(y,s)) \geq u(\overline{A}^-_{r/2}(y,s))$ in (5.20).

By Proposition 5.2(ii) and the Harnack inequality,

$$u(x,t) \leq Cu(\overline{A}_r^+(y,s))(\vartheta_+^{(x,t)}(\Delta_r^+) + \vartheta_-^{(x,t)}(\Delta_r^-)), \quad (x,t) \in D \setminus \Psi_{r/2}(y,s)$$

In particular,

$$1 = u(X,T) \leq Cu(\overline{A}_r^+(y,s))(\vartheta_+^{(X,T)}(\Delta_r^+) + \vartheta_-^{(X,T)}(\Delta_r^-)). \tag{5.21}$$

Now (5.19) for $u_r^+$, (5.21) and Proposition 5.2(iv) yield the existence of $C_1 = C_1(n, L, r_0)$ such that for any $0 < r < r_0$,

$$u_r^+(x,t) \geq C \frac{\omega_r^{+^{(x,t)}}(\Delta_r^+)}{\vartheta_+^{(X,T)}(\Delta_r^+) + \vartheta_-^{(X,T)}(\Delta_r^-)} \geq C_1 \frac{\omega_r^{+^{(x,t)}}(\Delta_r^+)}{\vartheta_+^{(X,T)}(\Delta_r^+)}, \quad (x,t) \in D_r^+. \tag{5.22}$$

Since by the maximum principle in $D_r^+$

$$\omega_r^{+^{(x,t)}}(\Delta_r^+) \geq \vartheta_+^{(x,t)}(\Delta_r^+) - \sup_{(z,h)\in\partial_p D_r^+ \cap D} \vartheta_+^{(z,h)}(\Delta_r^+), \tag{5.23}$$

then (5.22) can be written as

$$u_r^+(x,t) \geq C_1 \left( \frac{\vartheta_+^{(x,t)}(\Delta_r^+)}{\vartheta_+^{(X,T)}(\Delta_r^+)} - \sup_{(z,h)\in\partial_p D_r^+\cap D} \frac{\vartheta_+^{(z,h)}(\Delta_r^+)}{\vartheta_+^{(X,T)}(\Delta_r^+)} \right), \quad (x,t)\in D_r^+. \tag{5.24}$$

By Proposition 5.2(iii) and the Harnack inequality, there exists $C_2 = C_2(n, L, r_0)$ such that for $(z,h) \in \partial_p D_r^+ \cap D$,

$$\frac{\vartheta_+^{(z,h)}(\Delta_r^+)}{\vartheta_+^{(X,T)}(\Delta_r^+)} \leq C \frac{\vartheta_+^{\overline{A}^+_{r_0}}(\Delta_r^+)}{\vartheta_+^{(X,T)}(\Delta_r^+)} \cdot \vartheta_+^{(z,h)}(\Delta_{r_0}^+) \leq C_2 \vartheta_+^{(z,h)}(\Delta_{r_0}^+), \tag{5.25}$$

Hence (5.24) and (5.25) imply

$$u_r^+(x,t) \geq C_1 \left( \frac{\vartheta_+^{(x,t)}(\Delta_r^+)}{\vartheta_+^{(X,T)}(\Delta_r^+)} - C_2 \sup_{(z,h)\in\partial_p D_r^+\cap D} \vartheta_+^{(z,h)}(\Delta_{r_0}^+) \right), \quad (x,t)\in D_r^+.$$

*Case 2.* $u(\overline{A}^+_{r/2}(y,s)) \leq u(\overline{A}^-_{r/2}(y,s))$ in (5.20). Similarly,

$$u_r^-(x,t) \geq C_1 \left( \frac{\vartheta_-^{(x,t)}(\Delta_r^-)}{\vartheta_-^{(X,T)}(\Delta_r^-)} - C_2 \sup_{(z,h)\in\partial_p D_r^-\cap D} \vartheta_-^{(z,h)}(\Delta_{r_0}^-) \right), \quad (x,t)\in D_r^-.$$

Note that as $r \searrow 0$, $D_r^\pm \nearrow D$, and $u_r^\pm \to u$. Let $r_j \to 0$ be such that either Case 1 applies for all $r_j$ or Case 2 applies. Hence, over a subsequence, it follows by Proposition 5.2(i) and (5.15) that either

$$\begin{aligned} u(x,t) &\geq C_1 \lim_{r_j\to 0} \left( \frac{\vartheta_+^{(x,t)}(\Delta_{r_j}^+)}{\vartheta_+^{(X,T)}(\Delta_{r_j}^+)} - C_2 \sup_{(z,h)\in\partial_p D_{r_j}^+\cap D} \vartheta_+^{(z,h)}(\Delta_{r_0}^+) \right) \\ &= C_1 K^+(x,t), \quad \text{for all } (x,t)\in D, \end{aligned}$$

or

$$u(x,t) \geq C_1 K^-(x,t), \quad \text{for all } (x,t) \in D.$$

□

The next theorem says that $K^+(\cdot,\cdot;y,s)$ and $K^-(\cdot,\cdot;y,s)$ span the space of kernel functions at $(y,s)$.

**Theorem 5.6.** *If $u$ is a kernel function at $(y,s) \in E_f \setminus G_f$ normalized at $(X,T)$, then there exists a constant $\lambda \in [0,1]$ which may depend on $(y,s)$, such that $u(\cdot,\cdot) = \lambda K^+(\cdot,\cdot;y,s) + (1-\lambda)K^-(\cdot,\cdot;y,s)$ in $D$, where $K^+$ and $K^-$ are kernel function obtained from* (5.15) *and* (5.16)*.*

*Proof.* By Lemma 5.5 if $u$ is a kernel function at $(y,s)$, then either (i) $u \geq CK^+$ or (ii) $u \geq CK^-$ with $C = C(r_0,n,L)$.

If (i) takes place, let

$$\lambda = \sup\{C : u(x,t) \geq CK^+(x,t), \forall (x,t) \in D\},$$

then we must have $\lambda \leq 1$, because $u(X,T) = K^+(X,T) = 1$. If $\lambda = 1$, then $u(x,t) = K^+(x,t)$ for all $(x,t)\in D$ by the strong maximum principle and we are done. If $\lambda < 1$, consider

$$u_1(x,t) := \frac{u(x,t) - \lambda K^+(x,t)}{1-\lambda},$$

which is another kernel function at $(y,s)$ satisfying either (i) or (ii). If (i) holds for $u_1$ for some $C>0$, then $u(x,t) \geq (C(1-\lambda)+\lambda)K^+(x,t)$, with $C(1-\lambda)+\lambda > \lambda$ which contradicts to the supreme of $\lambda$. Hence (ii) must be true for $u_1$. Let

$$\tilde{\lambda} = \sup\{C : u_1(x,t) \geq CK^-(x,t), \forall (x,t) \in D\}.$$

The same reason as above gives $\tilde{\lambda} \leq 1$. We claim $\tilde{\lambda} = 1$.

Proof of the claim: If not, then $\tilde{\lambda} < 1$. We get

$$u_2(x,t) := \frac{u_1(x,t) - \tilde{\lambda}K^-(x,t)}{1-\tilde{\lambda}}$$

is again a kernel function at $(y,s)$. If $u_2$ satisfies (i) for some $C>0$, then

$$u_1(x,t) \geq u_1(x,t) - \tilde{\lambda}K^-(x,t) \geq C(1-\tilde{\lambda})K^+(x,t),$$

which implies

$$u(x,t) \geq (\lambda + C(1-\tilde{\lambda}))K^+(x,t)$$

is again a contradiction to the supreme of $\lambda$. Hence $u_2$ has to satisfy (ii) for some $C>0$, then we have

$$u_2(x,t) \geq (C(1-\tilde{\lambda})+\tilde{\lambda})K^-(x,t),$$

but this contradicts to the supreme of $\tilde{\lambda}$. Hence we proved the claim.

The fact that $\tilde{\lambda} = 1$ implies that $u_1(x,t) = K^-(x,t)$ in $D$ by the strong maximum principle. Hence if (i) applies to $u$ we have $u(x,t) = \lambda K^+(x,t) + (1-\lambda)K^-(x,t)$ with $\lambda \in (0,1]$. If (ii) applies to $u$ we get the equality with $\lambda \in [0,1)$. □

5.3. **Radon-Nikodym derivative as a kernel function.** We first show that the kernel function at $(y,s) \in G_f$ or $(y,s) \in \partial_p D \setminus E_f$ is unique. The proof for the uniqueness is similar as Lemma 1.6 and Theorem 1.7 in [Kem72b]. More precisely, we will need the following direction shift operator $F_r^0$:

$$\begin{aligned} F_r^0(x,t) &= (x'', x_{n-1} + 4nLr, x_n, t+8r^2), \quad 0<r<1/4 \\ D_r^0 &= \{(x,t) \in D : F_r^0(x,t) \in D\}. \end{aligned} \tag{5.26}$$

Let $\omega_r^0$ denote the caloric measure for $D_r^0$. Note that $D_r^0$ is also a cylindrical domain with a thin Lipschitz complement.

**Theorem 5.7.** *For all $(y,s) \in \partial_p D$ the limit of* (5.1) *exists. If we denote the limit by $K_0(\cdot,\cdot;y,s)$, i.e.*

$$K_0(x,t;y,s) = \lim_{n\to\infty} \omega^{(x,t)}(\Delta_{\frac{1}{n}}(y,s))/\omega^{(X,T)}(\Delta_{\frac{1}{n}}(y,s)).$$

*then*

(i) *For $(y,s) \in G_f$ or $(y,s) \in \partial_p D \setminus E_f$, $K_0$ is the unique kernel function at $(y,s)$.*

(ii) *If $(y,s) \in E_f \setminus G_f$, then $K_0$ is a kernel function at $(y,s)$ and*

$$K_0(x,t;y,s) = \frac{1}{2}K^+(x,t;y,s) + \frac{1}{2}K^-(x,t;y,s), \tag{5.27}$$

*where $K^+$ and $K^-$ are kernel functions at $(y,s)$ given by the limit of* (5.15) *and* (5.16).

*Proof.* For $(y,s) \in G_f$ and $r$ small enough, we denote $\overline{\overline{A}}_r(y,s) = (y'', y_{n-1} + 4nrL, 0, s+4r^2)$, which is on $\{x_n = 0\}$ and have a time-lag $2r^2$ above $\overline{A}_r^{\pm}$. Then by the Harnack inequality,

$$\omega^{\overline{A}_r^{\pm}(y,s)}(\Delta_{r'}(y,s)) \le C(n,L)\omega^{\overline{\overline{A}}_r(y,s)}(\Delta_{r'}(y,s)), \quad \forall 0 < r' < r.$$

Then one can proceed as in Lemma 1.6 of [Kem72b] by using $F_r^0$, $D_r^0$, $\omega^0$ to show that any kernel function (at $(y,s)$) $u$ satisfies $u \ge CK_0$ for some $C > 0$. Then the uniqueness follows from Theorem 1.7, Remark 1.8 of [Kem72b].

For $(y,s) \in \partial_p D \setminus E_f$, for $r$ sufficiently small one has either $\Psi_r(y,s) \cap D \subset D_+$ or $\Psi_r(y,s) \cap D \subset D_-$. In either case one can proceed as in Lemma 1.6, Theorem 1.7 and Remark 1.8 of [Kem72b].

For $(y,s) \in E_f \setminus G_f$, by Theorem 5.6, $K_0(x,t;y,s) = \lambda K^+(x,t;y,s) + (1-\lambda)K^-(x,t;y,s)$ for some $\lambda \in [0,1]$. By Proposition 5.2(ii), the symmetry of the domain about $x_{n-1}$ and the definition of $K^{\pm}$, one has $\lambda = 1/2$. □

*Remark* 5.8. From Theorem 5.7 we can conclude that the Radon-Nikodym derivative $d\omega^{(x,t)}/d\omega^{(X,T)}$ exists at every $(y,s) \in \partial_p D$ and it is the kernel function $K_0(x,t;y,s)$ with respect to $(X,T)$.

The following corollary is an easy consequence of Theorems 5.6 and 5.7.

**Corollary 5.9.** *For fixed $(x,t) \in D$, the function $(y,s) \mapsto K_0(x,t;y,s)$ is continuous on $\partial_p D$, where $K_0$ is given by the limit of* (5.1).

*Proof.* Given $(y,s) \in \partial_p D$, let $(y_m, s_m) \in \partial_p D$ with $(y_m,s_m) \to (y,s)$ as $m \to \infty$.

If $(y,s) \in G_f$ or $\partial_p D \setminus E_f$, continuity follows from the uniqueness of the kernel function.

If $(y,s) \in E_f \setminus G_f$, by Theorem 5.7(ii) for each $m$ we have

$$K_0(x,t;y_m,s_m) = \frac{1}{2}K^+(x,t;y_m,s_m) + \frac{1}{2}K^-(x,t;y_m,s_m). \tag{5.28}$$

Given $\varepsilon > 0$, $K^+(\cdot,\cdot;y_m,s_m)$ is uniformly bounded and equicontinuous on $D \setminus \Psi_\varepsilon(y,s)$ for $m$ large enough, hence by a similar argument as in Section 5.1, up to a subsequence, $K^+(\cdot,\cdot;y_m,s_m) \to v^+(\cdot,\cdot;y,s)$ uniformly on compact subsets, where $v^+(\cdot,\cdot;y,s)$ is some kernel function at $(y,s)$. Moreover, by Theorem 5.6 we have

$$v^+(\cdot,\cdot;y,s) = \lambda K^+(\cdot,\cdot;y,s) + (1-\lambda)K^-(\cdot,\cdot;y,s), \quad \text{for some } \lambda \in [0,1]. \tag{5.29}$$

By Proposition 5.2(i),

$$\sup_{(x,t)\in\partial_p D_r^+ \cap D} K^+(x,t;y_m,s_m) \to 0, \quad r \to 0$$

which is uniform in $m$ from the proof of the proposition. Hence after $m \to \infty$, $v^+$ satisfies

$$\sup_{(x,t)\in\partial_p D_r^+ \cap D} v^+(x,t) \to 0, \quad r \to 0,$$

which combined with

$$K^-(x,t;y,s) \not\to 0, \text{ as } (x,t) \to (y,s), \quad \text{for } (x,t) \in D_-$$

gives $\lambda = 1$ in (5.29).

Similarly, up to a subsequence $K^-(x,t;y_m,s_m) \to K^-(x,t;y,s)$.

Thus along a subsequence $K(\cdot,\cdot;y_m,s_m)\to K_0(\cdot,\cdot;y,s)$ by (5.27). Since this holds for all the converging subsequences, then $K_0(x,t;y,s)$ is continuous on $\partial_p D$ for fixed $(x,t)$. □

By using Corollary 5.9, Remark 5.8 and Theorem 4.6 we can prove some uniform behavior of $K_0$ on $\partial_p D$ as in Lemmas 2.2 and 2.3 of [Kem72b]. We state the results in the following two lemmas and omit the proof.

**Lemma 5.10.** *Let $(y,s)\in\partial_p D$. Then for $0<r<1/4$,*

$$\sup_{(y',s')\in\partial_p D\setminus\Delta_r(y,s)} K_0(x,t;y',s')\to 0,\ \text{ as } (x,t)\to(y,s) \text{ in } D.$$

The following lemma says that if $D'$ is a domain obtained by a perturbation of a portion of $\partial_p D$ where $\omega^{(x,t)}$ vanishes, then the caloric measure $\omega_{D'}$ is equivalent to $\omega_D$ on the common boundary of $D'$ and $D$. We recall here $\omega_r^0$ is the caloric measure with respect to the domain $D_r^0$ defined in (5.26) and $\omega_r^\pm$ is the caloric measure with respect to $D_r^\pm$ defined in (5.5).

**Lemma 5.11.**

(i) *Let $r\in(0,1/4)$ and $(y,s)\in G_f\cup(\partial_p D\setminus E_f)$ with $s>-1+4r^2$. Then there exists $\rho_0=\rho_0(n,L)>0$, $C=C(n,L)>0$ such that for $0<\rho<\rho_0$ we have*

$$\omega_\rho^{0^{(X',T')}}(\Delta_r(y,s))\geq C\omega^{(X',T')}(\Delta_r(y,s)),\quad (X',T')\in\Psi_{1/4}(X,T), \tag{5.30}$$

*provided also $r<|y_n|$ for $(y,s)\in\partial_p D\setminus E_f$.*

(ii) *Let $(y,s)\in(\mathcal{N}_r(E_f)\cap\partial_p D)\setminus G_f$. Then there exists $\delta_0=\delta_0(n,L)>0$, such that for $0<r'<\delta_0$ we have*

$$\omega_{r'}^{+^{(X',T')}}(\Delta_r^+(y,s))+\omega_{r'}^{-^{(X',T')}}(\Delta_r^-(y,s))\geq\frac{1}{2}\omega^{(X',T')}(\Delta_r(y,s)) \tag{5.31}$$

*for $(X',T')\in\Psi_{1/4}(X,T)$ and $0<r<r_0$, where $r_0$ is the constant defined in (5.2).*

*Proof.* To show (5.31) we first argue similarly as in [Kem72b] to show there exists $\delta_0=\delta_0(n,L)>0$ such that for any $0<r'<\delta_0$

$$\omega_{r'}^{\pm^{(X',T')}}(\Delta_r^\pm(y,s))\geq\frac{1}{2}\vartheta_\pm^{(X',T')}(\Delta_r^\pm(y,s)) \tag{5.32}$$

for each $\Delta_r^\pm(y,s)$ with $0<r<r_0$. Then using Proposition 5.2(ii) we get the conclusion.

□

## 6. Backward Boundary Harnack Principle

In this section, we follow the lines of [FGS84] to build up a backward Harnack inequality for nonnegative caloric functions in $D$. To prove this kind of inequalities, we have to ask the nonnegative caloric functions to vanish on the *lateral boundary*

$$S:=\partial_p D\cap\{s>-1\},$$

or at least a portion of it. This will allow to control the time-lag issue in the parabolic Harnack inequality.

Some of the proofs in this section follow the lines of the corresponding proofs in [FGS84]. For that reason, we will omit the parts that don't require modifications or additional arguments.

For $(x,t)$ and $(y,s) \in D$, denote by $G(x,t;y,s)$ the Green's function for the heat equation in the domain $D$. Since $D$ is a regular domain, Green's function can be written in the form

$$G(x,t;y,s) = \Gamma(x,t;y,s) - V(x,t;y,s),$$

where $\Gamma(\cdot,\cdot;y,s)$ is the fundamental solution of the heat equation with pole at $(y,s)$ and $V(\cdot,\cdot;y,s)$ is a caloric function in $D$ that equals $\Gamma(\cdot,\cdot;y,s)$ on $\partial_p D$. We note that by the maximum principle we have $G(x,t;y,s) = 0$ whenever $(x,t) \in D$ with $t \le s$.

In this section, similarly to Section 5, we will work under Convention 5.1. In particular, in Green's function we will allow pole $(y,s)$ to be in $\tilde{D}$ with $s \ge 1$. But in that case we simply have $G(x,t;y,s) = 0$ for all $(x,t) \in D$.

**Lemma 6.1.** *Let $0 < r < 1/4$ and $(y,s) \in S$ with $s \ge -1 + 8r^2$. Then there exists a constant $C = C(n,L) > 0$ such that for $(x,t) \in D \cap \{t \ge s + 4r^2\}$, we have*

$$\begin{aligned} C^{-1} r^n \max\{G(x,t;\overline{A}_r^{\pm}(y,s))\} &\le \omega^{(x,t)}(\Delta_r(y,s)) \\ &\le C r^n \max\{G(x,t;\underline{A}_r^{\pm}(y,s))\}, \quad \textit{if } (y,s) \in \mathcal{N}_r(E_f), \end{aligned} \tag{6.1}$$

$$\begin{aligned} C^{-1} r^n G(x,t;\overline{A}_r(y,s)) &\le \omega^{(x,t)}(\Delta_r(y,s)) \\ &\le C r^n G(x,t;\underline{A}_r(y,s)), \quad \textit{if } (y,s) \notin \mathcal{N}_r(E_f). \end{aligned} \tag{6.2}$$

*Proof.* The proof uses Lemma 4.4 and Theorem 4.3 and is similar to that of Lemma 1 in [FGS84]. □

**Theorem 6.2** (Interior backward Harnack inequality)**.** *Let $u$ be a positive caloric function in $D$ vanishing continuously on $S$. Then for any compact $K \Subset D$ there exists a constant $C = C(n, L, \operatorname{dist}(K, \partial_p D))$ such that*

$$\max_K u \le C \min_K u$$

*Proof.* The proof is similar to that of Theorem 1 in [FGS84] and uses Theorem 4.3 and the Harnack inequality. □

**Theorem 6.3** (Local comparison theorem)**.** *Let $0 < r < 1/4$ and $(y,s) \in S$ with $s \ge -1 + 18r^2$, and $u, v$ be two positive caloric functions in $\Psi_{3r}(y,s) \cap D$ vanishing continuously on $\Delta_{3r}(y,s)$. Then there exists $C = C(n,L) > 0$ such that for $(x,t) \in \Psi_{r/8}(y,s) \cap D$ we have:*

$$\frac{u(x,t)}{v(x,t)} \le C \frac{\max\{u(\overline{A}_r^{+}(y,s)), u(\overline{A}_r^{-}(y,s))\}}{\min\{v(\underline{A}_r^{+}(y,s)), v(\underline{A}_r^{-}(y,s))\}}, \quad \textit{if } (y,s) \in \mathcal{N}_r(E_f) \tag{6.3}$$

*and*

$$\frac{u(x,t)}{v(x,t)} \le C \frac{u(\overline{A}_r(y,s))}{v(\underline{A}_r(y,s))}, \qquad \textit{if } (y,s) \notin \mathcal{N}(E_f). \tag{6.4}$$

*Proof.* The proof is similar to that of Theorem 3 in [FGS84]. First, note that if $\Psi_{r/8}(y,s) \cap E_f = \emptyset$, we can consider restriction of $u$ and $v$ to $D_+$ or $D_-$ (which are Lipschitz cylinders) and apply the arguments from [FGS84] directly there. Thus, we may assume that $\Psi_{r/8}(y,s) \cap E_f \ne \emptyset$. If we now argue as in the proof of the localization property (Lemma 2.3) by replacing $(y,s)$ and $r$ with $(\tilde{y},\tilde{s}) \in \Psi_{(3/8)r}(y,s) \cap E_f$ we may further assume that $(y,s) \in E_f$, and that $\Psi_r(y,s) \cap D$ falls either into category (2) or (3) in the localization property. For definiteness, we will assume category (3). To account for the possible change in $(y,s)$ we then change the hypothesis to $u=0$ on $\Delta_{2r}(y,s)$ and prove (6.3) for $(x,t)\in\Psi_{r/2}(y,s)\cap D$.

With the above simplification in mind, we proceed as in the proof of Theorem 3 in [FGS84]. By using Lemma 6.1 and Theorem 4.6 we first show

$$\omega_r^{(x,t)}(\alpha_r)\leq C\omega_r^{(x,t)}(\beta_r),\quad (x,t)\in\Psi_{r/2}(y,s)\cap D \tag{6.5}$$

where $\alpha_r=\partial_p(\Psi_r(y,s)\cap D)\setminus S$, $\beta_r=\partial_p(\Psi_r(y,s)\cap D)\setminus\mathcal{N}_{\mu r}(S)$ with a small fixed $\mu\in(0,1)$, and where $\omega_r$ denotes the caloric measure with respect to $\Psi_r(y,s)\cap D$. Then by Theorem 4.3, Harnack inequality, and the maximum principle we obtain

$$\begin{aligned}u(x,t)&\leq C\max\{u(\overline{A}_r^+(y,s)),u(\overline{A}_r^-(y,s))\}\omega_r^{(x,t)}(\alpha_r)\\ v(x,t)&\geq C\min\{v(\underline{A}_r^+(y,s)),v(\underline{A}_r^-(y,s))\}\omega_r^{(x,t)}(\beta_r),\end{aligned}$$

which combined with (6.5) completes the proof. □

**Theorem 6.4** (Global comparison theorem)**.** *Let $u,v$ be two positive caloric functions in $D$, vanishing continuously on $S$, and let $(x_0,t_0)$ be a fixed point in $D$. If $\delta>0$, then there exists $C=C(n,L,\delta)>0$, such that*

$$\frac{u(x,t)}{v(x,t)}\leq C\frac{u(x_0,t_0)}{v(x_0,t_0)},\quad \text{for all } (x,t)\in D\cap\{t>-1+\delta^2\}. \tag{6.6}$$

*Proof.* It is an easy consequence of Theorems 6.2 and 6.3. □

Now we show the doubling properties of the caloric measure at the lateral boundary points by using the properties of the kernel functions we showed in Section 5. The idea of the proof is similar to that of Lemma 2.2 in [Wu79], but with a more careful inspection of the different types of boundary points.

To proceed, we will need to define the time-invariant corkscrew points at $(y,s)$ on the lateral boundary, in addition to future and past corkscrew points. Namely, for $(y,s)\in S$ we let

$$\begin{aligned}A_r(y,s)&=(y(1-r),s), &&\text{if } \Psi_r(y,s)\cap E_f=\emptyset\\ A_r^\pm(y,s)&=(y'',y_{n-1}+r/2,\pm r/2,s), &&\text{if } \Psi_r(y,s)\cap E_f\neq\emptyset.\end{aligned}$$

**Theorem 6.5** (Doubling at the lateral boundary points)**.** *For $0<r<1/4$ and $(y,s)\in S$ with $s\geq -1+8r^2$, there exist $\varepsilon_0=\varepsilon_0(n,L)>0$ small and $C=C(n,L)>0$ such that for any $r<\varepsilon_0$ we have:*

(i) *If $(y,s)\in E_f$ and $\Psi_{2r}(y,s)\cap G_f\neq\emptyset$, then*

$$C^{-1}r^nG(X,T;A_r^\pm(y,s))\leq\omega^{(X,T)}(\Delta_r(y,s))\leq Cr^nG(X,T;A_r^\pm(y,s)); \tag{6.7}$$

(ii) *If $(y,s)\in\mathcal{N}_r(E_f)\cap\partial_pD$ and $\Psi_{2r}(y,s)\cap G_f=\emptyset$, then*

$$C^{-1}r^nG(X,T;A_r^+(y,s))\leq\vartheta_+^{(X,T)}(\Delta_r^+(y,s))\leq Cr^nG(X,T;A_r^+(y,s)), \tag{6.8}$$

$$C^{-1}r^nG(X,T;A_r^-(y,s))\leq\vartheta_-^{(X,T)}(\Delta_r^-(y,s))\leq Cr^nG(X,T;A_r^-(y,s)); \tag{6.9}$$

(iii) *If $(y,s)\in\partial_pD\setminus\mathcal{N}_r(E_f)$, then*

$$C^{-1}r^nG(X,T;A_r(y,s))\leq\omega^{(X,T)}(\Delta_r(y,s))\leq Cr^nG(X,T;A_r(y,s)). \tag{6.10}$$

*Moreover, there is a constant $C = C(n, L) > 0$, such that*

$$(i) \quad \textit{For } (y,s) \in S \cap \{s \geq -1 + 8r^2\},$$

$$\omega^{(X,T)}(\Delta_{2r}(y,s)) \leq C\omega^{(X,T)}(\Delta_r(y,s))u(x,t); \tag{6.11}$$

$$(ii) \quad \textit{For } (y,s) \in \mathcal{N}_r(E_f) \cap S \cap \{s \geq -1 + 8r^2\},$$

$$\vartheta_+^{(X,T)}(\Delta_{2r}^+(y,s)) \leq C\vartheta_+^{(X,T)}(\Delta_r^+(y,s)),$$

$$\vartheta_-^{(X,T)}(\Delta_{2r}^-(y,s)) \leq C\vartheta_-^{(X,T)}(\Delta_r^-(y,s)). \tag{6.12}$$

*Proof.* We start by showing the estimates from above in (6.7) and (6.8).

*Case 1:* $(y, s) \in E_f$ and $\Psi_{2r}(y, s) \cap G_f \neq \emptyset$. By Lemma 2.3 there is $(\tilde{y}, \tilde{s}) \in G_f$ such that

$$\Psi_r(y,s) \cap D \subset \Psi_{4r}(\tilde{y},\tilde{s}) \cap D \subset \Psi_{8r}(y,s) \cap D.$$

It is not hard to check by (5.26) that $F_r^0(\Delta_{4r}(\tilde{y}, \tilde{s})) \subset D$. Moreover, the parabolic distance between $F_r^0(\Delta_{4r}(\tilde{y}, \tilde{s}))$ and $\partial_p D$, and the $t$ coordinate distance from $F_r^0(\Delta_{4r}(\tilde{y}, \tilde{s}))$ down to $A_r^{\pm}$ are greater than $cr$ for some universal $c$ which only depends on $n$ and $L$. Therefore, by the estimate of Green's function as in [Wu79] we have

$$G(x,t; A_r^{\pm}(y,s)) \geq C(n,L)r^{-n}, \quad (x,t) \in F_r^0(\Delta_{4r}(\tilde{y},\tilde{s}))$$

Applying the maximum principle to $F_r^0(D_r^0)$, we have

$$G(x,t; A_r^{\pm}(y,s)) \geq C(n,L)r^{-n}\omega_r^{0^{F_r^{0^{-1}}(x,t)}}(\Delta_{4r}(\tilde{y},\tilde{s})).$$

In particular,

$$G(X,T; A_r^{\pm}(y,s)) \geq C(n,L)r^{-n}\omega_r^{0^{F_r^{0^{-1}}(X,T)}}(\Delta_{4r}(\tilde{y},\tilde{s})).$$

Let $(X_r, T_r) := F_r^{0^{-1}}(X, T)$ and take $(X', T') \in D$ with $T' = T - 1/4$, $X' = X$ in particular $T' > 1/4 + T_r$. Then we obtain by the Harnack inequality that

$$G(X,T; A_r^{\pm}(y,s)) \geq C(n,L)r^{-n}\omega_r^{0^{(X',T')}}(\Delta_{4r}(\tilde{y},\tilde{s})). \tag{6.13}$$

By Lemma 5.11(i), for $0 < r < \min\{1/4, \rho_0\}$ there exists $C = C(n, L)$ independent of $r$ such that

$$\omega_r^{0^{(X',T')}}(\Delta_{4r}(\tilde{y},\tilde{s})) \geq C\omega^{(X',T')}(\Delta_{4r}(\tilde{y},\tilde{s})). \tag{6.14}$$

By Theorem 5.7 for each $(\tilde{y}, \tilde{s}) \in G_f$

$$K_0(X',T';\tilde{y},\tilde{s}) = \lim_{r\to 0} \omega^{(X',T')}(\Delta_{4r}(\tilde{y},\tilde{s}))/\omega^{(X,T)}(\Delta_{4r}(\tilde{y},\tilde{s})) > 0,$$

and by Corollary 5.9 for $(X', T')$ fixed $K_0(X', T'; \cdot, \cdot)$ is continuous on $\partial_p D$. Therefore, in the compact set $G_f$ there exists $c > 0$ only depending on $n, L$ such that $K_0(X', T'; \tilde{y}, \tilde{s}) \geq c > 0$ for any $(\tilde{y}, \tilde{s}) \in G_f$. Hence by the Radon-Nikodym theorem for $0 < r < \min\{1/4, \rho_0\}$ we have

$$\omega^{(X',T')}(\Delta_{4r}(\tilde{y},\tilde{s})) \geq \frac{c}{2}\omega^{(X,T)}(\Delta_{4r}(\tilde{y},\tilde{s})) \geq \frac{c}{2}\omega^{(X,T)}(\Delta_r(y,s)). \tag{6.15}$$

Combining (6.13), (6.14) and (6.15) we obtain the estimate from above in (6.7) for Case 1.

*Case 2:* $(y, s) \in \mathcal{N}_r(E_f) \cap \partial_p D$ and $\Psi_{2r}(y, s) \cap G_f = \emptyset$.

In this case $\Psi_{2r}(y,s)\cap D$ splits into a disjoint union of $\Psi_{2r}(y,s)\cap D_\pm$. We use $F_r^+$ and $F_r^-$ defined in (5.3) and (5.4), and apply the same arguments as in Case 1 in $D_r^+$ and $D_r^-$. Then

$$\omega_r^{\pm(X,T)}(\Delta_r^\pm(y,s))\le Cr^nG(X,T;A_r^\pm(y,s)).$$

Taking $0<r<\delta_0$, where $\delta_0=\delta_0(n,L)$ is the constant in Lemma 5.11(ii), we have

$$\vartheta_\pm^{(X,T)}(\Delta_r(y,s))\le 2\omega_r^{\pm(X,T)}(\Delta_r(y,s))\le Cr^nG(X,T;A_r^\pm(y,s)).$$

*Case 3:* $(y,s)\in\partial_pD\setminus\mathcal{N}_r(E_f)$. We argue similarly to Case 1 and 2.

Taking $\varepsilon_0=\min\{\rho_0,\delta_0,1/4\}$, we complete the proof of the estimates from above in (6.7)–(6.10).

The proof of the estimate from below in (6.7)–(6.10) is the same as in [Wu79]. For (6.7) it is a consequence of Lemma 4.4 and the maximum principle. (6.8) and (6.9) follow from (5.12) and the maximum principle. The doubling properties of caloric measure $\omega^{(x,t)}$ and $\theta_\pm^{(x,t)}$ are easy consequences of (6.7)–(6.10) and Proposition 5.2(ii) for $0<r<\varepsilon_0/2$. For $r>\varepsilon_0/2$ we use Lemma 4.4 and (5.12).

□

Theorem 6.5 implies the following backward Harnack principle.

**Theorem 6.6** (Backward boundary Harnack principle)**.** *Let $u$ be a positive caloric function in $D$ vanishing continuously on $S$ and let $\delta>0$. Then there exists a positive constant $C=C(n,L,\delta)$ such that for $(y,s)\in\partial_pD\cap\{s>-1+\delta^2\}$ and for $0<r<r(n,L,\delta)$ sufficiently small we have*

$$\begin{aligned}&C^{-1}u(\underline{A}_r^+(y,s))\le u(\overline{A}_r^+(y,s))\le Cu(\underline{A}_r^+(y,s)),\\ &C^{-1}u(\underline{A}_r^-(y,s))\le u(\overline{A}_r^-(y,s))\le Cu(\underline{A}_r^-(y,s)),\quad \textit{if } (y,s)\in\mathcal{N}_r(E_f);\end{aligned}$$

*and*

$$C^{-1}u(\underline{A}_r(y,s))\le u(\overline{A}_r(y,s))\le Cu(\underline{A}_r(y,s)),\quad \textit{if } (y,s)\notin\mathcal{N}_r(E_f). \tag{6.16}$$

*Proof.* Once we have Theorem 6.5, which is an analogue of Lemma 2.2 in [Wu79], we can proceed as Theorem 4 in [FGS84] to show the above backward Harnack principle.

□

*Remark* 6.7. From (6.7) and using the same proof as in Theorem 6.6 we can conclude that for any positive caloric function $u$ vanishing continuously on $S$ and $(y,s)\in G_f$ there exists $C=C(n,L,\delta)>0$ such that

$$\begin{aligned}&C^{-1}u(\overline{A}_r^-(y,s))\le u(\overline{A}_r^+(y,s))\le Cu(\overline{A}_r^-(y,s)),\\ &C^{-1}u(\underline{A}_r^-(y,s))\le u(\underline{A}_r^+(y,s))\le Cu(\underline{A}_r^-(y,s)).\end{aligned}$$

## 7. Various versions of boundary Harnack

In the applications, it is very useful to have a local version of the backward Harnack for solutions vanishing only on a portion of the lateral boundary $S$. For the parabolically Lipschitz domains this was proved in [ACS96] as a consequence of the (global) backward Harnack principle.

To state the results, we will use the following corkscrew point associated with $(y,s)\in G_f$: for $0<r<1/4$, let

$$\begin{aligned}\overline{A}_r(y,s)&=(y'',y_{n-1}+4nLr,0,s+2r^2),\\ \underline{A}_r(y,s)&=(y'',y_{n-1}+4nLr,0,s-2r^2),\\ A_r(y,s)&=(y'',y_{n-1}+4nLr,0,s).\end{aligned}$$

When $(y,s)=(0,0)$ we simply write $\overline{A}_r$, $\underline{A}_r$ and $A_r$, in addition to $\Psi_r$, $\Delta_r$, $\overline{A}_r^{\pm}$, $\underline{A}_r^{\pm}$.

**Theorem 7.1.** *Let $u$ be nonnegative caloric in $D$, continuously vanishing continuously on $E_f$. Let $m=u(\underline{A}_{3/4})$, $M=\sup_D u$, then there exists a constant $C=C(n,L,M/m)$, such that for any $0<r<1/4$ we have*

$$u(\overline{A}_r)\le Cu(\underline{A}_r). \tag{7.1}$$

*Proof.* Using Theorems 6.6 and 6.5 and following the lines of Theorem 13.7 in [CS05] we have

$$u(\overline{A}_{2r}^{\pm})\le Cu(\underline{A}_{2r}^{\pm}),\quad 0<r<1/4,$$

for $C=C(n,L,M/m)$. Then (7.1) follows from Theorem 6.6 and an observation that there is a Harnack chain with a constant $\mu=\mu(n,L)$ and length $N=N(n,L)$ joining $\overline{A}_r$ to $\overline{A}_{2r}^{\pm}$ and $\underline{A}_{2r}^{\pm}$ to $\underline{A}_r$. □

Theorem 7.1 implies the boundary Hölder regularity of the quotient of two negative caloric functions vanishing on $E_f$. The proof of the following corollary is the same as for Corollary 13.8 in [CS05] and is therefore omitted.

**Theorem 7.2.** *Let $u_1$, $u_2$ be nonnegative caloric functions in $D$ continuously vanishing on $E_f$. Let $M_i=\sup_D u_i$ and $m_i=u_i(\underline{A}_{3/4})$ with $i=1,2$. Then we have*

$$C^{-1}\frac{u_1(A_{1/4})}{u_2(A_{1/4})}\le\frac{u_1(x,t)}{u_2(x,t)}\le C\frac{u_1(A_{1/4})}{u_2(A_{1/4})},\quad \text{for } (x,t)\cap\Psi_{1/8}\cap D, \tag{7.2}$$

*where $C=C(n,L,M_1/m_1,M_2/m_2)$. Moreover, if $u_1$ and $u_2$ are symmetric in $x_n$, then $u_1/u_2$ extends to a function in $C^\alpha(\Psi_{1/8})$ for some $0<\alpha<1$, where the exponent $\alpha$ and the $C^\alpha$ norm depend only on $n,L,M_1/m_1,M_2/m_2$.* □

*Remark* 7.3. The symmetry condition in the latter part of the theorem is important to guarantee the continuous extension of $u_1/u_2$ to the Euclidean closure $\overline{\Psi_{1/8}\setminus E_f}=\overline{\Psi_{1/8}}$, since the limits at $E_f\setminus G_f$, as we approach from different sides, may be different. Without the symmetry condition, one may still prove that $u_1/u_2$ extends to a $C^\alpha$ function on the completion $(\Psi_{1/8}\setminus E_f)^*$ with respect to the inner metric.

For a more general application, we need to have a boundary Harnack inequality for $u$ satisfying a nonhomogeneous equation with bounded right hand side but additionally with a nondegeneracy condition. The method we use here is similar as the one used in the elliptic case ([CSS08]).

**Theorem 7.4.** *Let $u$ be a nonnegative function in $D$, continuously vanishing on $E_f$, and satisfying*

$$|\Delta u-\partial_t u|\le C_0\quad \text{in } D \tag{7.3}$$

$$u(x,t)\ge c_0 d(x,t)^\gamma\quad \text{in } D, \tag{7.4}$$

*where* $d(x,t) = \operatorname{dist}_p((x,t); E_f)$, $0 < \gamma < 2$, $c_0 > 0$, $C_0 \geq 0$. *Then there exists* $C = C(n, L, \gamma, C_0, c_0) > 0$ *such that for* $0 < r < 1/4$ *we have*

$$u(x,t) \leq C u(\overline{A}_r), \quad (x,t) \in \Psi_r. \tag{7.5}$$

*Moreover, if* $M = \sup_D u$, *then there exists a constant* $C = C(n, L, \gamma, C_0, c_0, M)$, *such that for any* $0 < r < 1/4$ *we have*

$$u(\overline{A}_r) \leq C u(\underline{A}_r). \tag{7.6}$$

*Proof.* Let $u^*$ solve the heat equation in $\Psi_{2r} \cap D$ and equal to $u$ on $\partial_p(\Psi_{2r} \cap D)$. Then by the Carleson estimate we have $u^*(x,t) \leq C(n,L) u^*(\overline{A}_r)$ for $(x,t) \in \Psi_r$.

On the other hand, we have

$$\begin{aligned} u^*(x,t) + C(|x|^2 - t - 8r^2) &\leq u(x,t) && \text{on } \partial_p(\Psi_{2r} \cap D) \\ (\Delta - \partial_t)(u^*(x,t) + C(|x|^2 - t - 8r^2)) &\geq C(2n-1) \\ &\geq (\Delta - \partial_t) u(x,t) && \text{in } \Psi_{2r} \cap D \end{aligned}$$

for $C \geq C_0/(2n-1)$. Hence, by the comparison principle we have $u^* - u \leq Cr^2$ in $\Psi_{2r} \cap D$ for $C = C(C_0, n)$. Similarly, $u - u^* \leq Cr^2$ and hence $|u - u^*| \leq Cr^2$ in $\Psi_{2r} \cap D$. Consequently,

$$u(x,t) \leq C(n,L)(u(\overline{A}_r) + C(C_0,n) r^2), \quad (x,t) \in \Psi_r. \tag{7.7}$$

Next note that by the nondegeneracy condition (7.4)

$$u(\overline{A}_r) \geq c_0 r^\gamma \geq c_0 r^2, \quad r \in (0,1). \tag{7.8}$$

Thus, combining (7.7) and (7.8), we obtain (7.5).

The proof of (7.6) follows in a similar manner from Theorem 7.1 for $u_*$. □

*Remark* 7.5. In fact, the nondegeneracy condition (7.4) is necessary. An easy counterexample is $u(x,t) = x_{n-1}^2 x_n^2$ in $\Psi_1$ and $E_f = \{(x,t) : x_{n-1} \leq 0, x_n = 0\} \cap \Psi_1$. Then $u(\overline{A}_r) = 0$ for $r \in (0,1)$ but obviously $u$ does not vanish in $\Psi_r \cap D$.

We next state a generalization of the local comparison theorem.

**Theorem 7.6.** *Let* $u_i$, $i = 1, 2$, *be nonnegative functions in* $D$, *continuously vanishing on* $E_f$, *and satisfying*

$$\begin{aligned} |\Delta u_i - \partial_t u_i| &\leq C_0 \quad \text{in } D \\ u_i(x,t) &\geq c_0 d(x,t)^\gamma \quad \text{in } D, \end{aligned}$$

*where* $d(x,t) = \operatorname{dist}_p((x,t); E_f)$, $0 < \gamma < 2$, $c_0 > 0$, $C_0 \geq 0$. *Let also* $M = \max\{\sup_D u_1, \sup_D u_2\}$. *Then there exists a constant* $C = C(n, L, \gamma, C_0, c_0, M) > 0$ *such that*

$$C^{-1} \frac{u_1(A_{1/4})}{u_2(A_{1/4})} \leq \frac{u_1(x,t)}{u_2(x,t)} \leq C \frac{u_1(A_{1/4})}{u_2(A_{1/4})}, \quad (x,t) \in \Psi_{1/8} \cap D. \tag{7.9}$$

*Moreover, if* $u_1$ *and* $u_2$ *are symmetric in* $x_n$, *then* $u_1/u_2$ *extends to a function in* $C^\alpha(\overline{\Psi_{1/8}})$ *for some* $0 < \alpha < 1$, *with* $\alpha$ *and* $C^\alpha$ *norm depending only on* $n, L, \gamma, C_0, c_0, M$.

To prove this theorem, we will also need the following two lemmas, which are essentially Lemmas 11.5 and 11.8 in [DGPT13]. The proofs are therefore omitted.

**Lemma 7.7.** *Let $\Lambda$ be a subset of $\mathbb{R}^{n-1}\times(-\infty,0]$, and $h(x,t)$ a continuous function in $\Psi_1$. Then for any $\delta_0>0$ there exists $\varepsilon_0>0$ depending only on $\delta_0$ and $n$ such that if*

i) $h\geq 0$ *on* $\Psi_1\cap\Lambda$,
ii) $(\Delta-\partial_t)h\leq\varepsilon_0$ *in* $\Psi_1\setminus\Lambda$,
iii) $h\geq-\varepsilon_0$ *in* $\Psi_1$,
iv) $h\geq\delta_0$ *in* $\Psi_1\cap\{|x_n|\geq\beta_n\}$, $\beta_n=1/(32\sqrt{n-1})$

*then* $h\geq 0$ *in* $\Psi_{1/2}$. □

**Lemma 7.8.** *For any $\delta_0>0$ there exists $\varepsilon_0>0$ and $c_0>0$ depending only on $\delta_0$ and $n$ such that if $h$ is a continuous function on $\Psi_1\cap\{0\leq x_n\leq\beta_n\}$, $\beta_n=1/(32\sqrt{n-1})$, satisfying*

i) $(\Delta-\partial_t)h\leq\varepsilon_0$ *in* $\Psi_1\cap\{0<x_n<\beta_n\}$
ii) $h\geq 0$ *in* $\Psi_1\cap\{0<x_n<\beta_n\}$,
iii) $h\geq\delta_0$ *on* $\Psi_1\cap\{x_n=\beta_n\}$,

*then*

$$h(x,t)\geq c_0x_n\quad\text{in }\Psi_{1/2}\cap\{0<x_n<\beta_n\}.$$ □

*Proof of Theorem 7.6.* We first note that arguing as in the proof of Theorem 7.4 and using Theorem 7.1, we will have that

$$u_i(x,t)\leq Cu_i(A_{1/4}),\quad (x,t)\in\Psi_{1/8},\tag{7.10}$$

for $C=C(n,L,\gamma,C_0,c_0,M)$. Next, dividing $u_i$ by $u_i(A_{1/4})$, we can assume $u_i(A_{1/4})=1$. Then, consider the rescalings

$$u_{i\rho}(x,t)=\frac{u_i(\rho x,\rho^2t)}{\rho^\gamma},\quad \rho\in(0,1),\quad i=1,2.$$

It is immediate to verify that $u_{i\rho}$ satisfy for $(x,t)\in\Psi_{1/(8\rho)}\cap D$,

$$|(\Delta-\partial_t)u_{i\rho}(x,t)|\leq C_0\rho^{2-\gamma},\tag{7.11}$$

$$u_{i\rho}(x,t)\geq c_0\operatorname{dist}((x,t),E_{f_\rho})^\gamma,\tag{7.12}$$

$$u_{i\rho}(x,t)\leq\frac{C}{\rho^\gamma},\quad C\text{ is the constant in (7.10)},\tag{7.13}$$

where $f_\rho(x'',t)=(1/\rho)f(\rho x'',\rho^2t)$ is the scaling of $f$. By (7.12) there exists $c_n>0$ such that

$$u_{i\rho}(x,t)\geq c_0c_n,\quad (x,t)\in\Psi_{1/(8\rho)}\cap\{|x_n|\geq\beta_n\}.\tag{7.14}$$

Consider now the difference

$$h=u_{2\rho}-su_{1\rho},$$

for a small positive $s$, specified below. By (7.11), (7.14), (7.13) one can choose a positive $\rho=\rho(n,L,\gamma,C_0,c_0,M)<1/16$ and $s=s(\rho,n,c_0,C)>0$ such that

$$\begin{aligned}
&h(x,t)\geq c_0c_n-s\cdot\frac{C}{\rho^\gamma}\geq\frac{c_0c_n}{2}, && (x,t)\in\Psi_{1/(8\rho)}\cap\{|x_n|\geq\beta_n\},\\
&h(x,t)\geq -s\cdot\frac{C}{\rho^\gamma}\geq-\varepsilon_0, && (x,t)\in\Psi_{1/(8\rho)},\\
&|(\Delta-\partial_t)h(x,t)|\leq C_0\rho^{2-\gamma}\leq\varepsilon_0, && (x,t)\in\Psi_{1/(8\rho)}\cap D,
\end{aligned}$$

where $\varepsilon_0 = \varepsilon_0(c_0, c_n, n)$ is the constant in Lemma 11.5. Thus by Lemma 11.5, $h > 0$ in $\Psi_{1/2} \cap D$, which implies

$$\frac{u_1(x,t)}{u_2(x,t)} \leq \frac{1}{s}, \quad (x,t) \in \Psi_{\rho/2} \cap D. \tag{7.15}$$

By moving the origin to any $(z, h) \in \Psi_{1/8} \cap E_f$ we will therefore obtain the bound

$$\frac{u_1(x,t)}{u_2(x,t)} \leq C(n, L, \gamma, C_0, c_0, M) \tag{7.16}$$

for any $(x,t) \in \Psi_{1/8} \cap \mathcal{N}_{\rho/2}(E_f) \cap D$. On the other hand, for $(x,t) \in \Psi_{1/8} \setminus \mathcal{N}_{\rho/2}(E_f)$ the estimate (7.16) will follow from (7.4) and (7.10). Hence (7.16) holds for any $(x,t) \in \Psi_{1/8} \cap D$, which gives the bound from above in (7.9). Changing the roles of $u_1$ and $u_2$ we get the bound from below.

The proof of $C^\alpha$ regularity follows by iteration from (7.9) similarly to the proof of Corollary 13.8 in [CS05]; however, we need to make sure that at every step the nondegeneracy condition is satisfied. We will only verify the Hölder continuity of $u_1/u_2$ at the origin, the rest being standard.

For $k \in \mathbb{N}$ and $\lambda > 0$ to be specified below let

$$l_k = \inf_{\Psi_{\lambda^k} \cap D} \frac{u_1}{u_2}, \quad L_k = \sup_{\Psi_{\lambda^k} \cap D} \frac{u_1}{u_2}.$$

We then know that $1/C \leq l_k \leq L_k \leq C$ for $\lambda \leq 1/8$. Let also

$$\mu_k = \frac{u_1(\underline{A}_{\lambda^k/4})}{u_2(\underline{A}_{\lambda^k/4})} \in [l_k, L_k].$$

Then there are two possibilities:

$$\text{either} \quad L_k - \mu_k \geq \frac{1}{2}(L_k - l_k) \quad \text{or} \quad \mu_k - l_k \geq \frac{1}{2}(L_k - l_k).$$

For definiteness, assume that we are in the latter case, the former cases being treated similarly. Then consider two functions

$$v_1(x,t) = \frac{u_1(\lambda^k x, \lambda^{2k} t) - l_k u_2(\lambda^k x, \lambda^{2k} t)}{u_1(\underline{A}_{\lambda^k/4}) - l_k u_2(\underline{A}_{\lambda^k/4})}, \quad v_2(x,t) = \frac{u_2(\lambda^k x, \lambda^{2k} t)}{u_2(\underline{A}_{\lambda^k/4})}.$$

In $\Psi_1 \setminus E_{f_{\lambda^k}}$, we will have

$$|(\Delta - \partial_t) v_1(x,t)| \leq \frac{\lambda^{2k}(1 + l_k) C_0}{u_1(\underline{A}_{\lambda^k/4}) - l_k u_2(\underline{A}_{\lambda^k/4})},$$
$$|(\Delta - \partial_t) v_2(x,t)| \leq \frac{\lambda^{2k} C_0}{u_2(\underline{A}_{\lambda^k/4})}.$$

To proceed, fix a small $\eta_0 > 0$, to be specified below. Then from the nondegeneracy of $u_2$, we immediately have

$$|(\Delta - \partial_t) v_2(x,t)| \leq C \lambda^{(2-\gamma)k} < \eta_0,$$

if we take $\lambda$ small enough. For $v_1$, we have a dichotomy:

$$\text{either} \quad |(\Delta - \partial_t) v_1(x,t)| \leq \eta_0 \quad \text{or} \quad \mu_k - l_k \leq C \lambda^{(2-\gamma)k}.$$

In the latter case, we obtain

$$L_k - l_k \leq 2(\mu_k - l_k) \leq C \lambda^{(2-\gamma)k}. \tag{7.17}$$

In the former case we notice that both functions $v = v_1$, $v_2$ satisfy

$$v \geq 0, \quad v(\underline{A}_{1/4}) = 1 \quad \text{and} \quad |(\Delta - \partial_t)v(x,t)| \leq \eta_0 \quad \text{in } \Psi_1 \setminus E_{f_{\lambda^k}}$$

and that $v$ vanishes continuously on $\Psi_1 \cap E_{f_{\lambda^k}}$. We next establish a nondegeneracy property for such $v$. Indeed, first note that by the parabolic Harnack inequality, see Theorems 6.17 and 6.18 in [Lie96], for small enough $\eta_0$, we will have that

$$v \geq c_n \quad \text{on } \Psi_{1/8} \cap \{|x_n| \geq \beta_n/8\}.$$

Then, by invoking Lemma 7.8, we will obtain that

$$v(x,t) \geq c_n|x_n| \quad \text{in } \Psi_{1/16} \setminus E_{f_{\lambda^k}}. \tag{7.18}$$

We further claim that

$$v(x,t) \geq c \operatorname{dist}_p((x,t), E_{f_{\lambda^k}}) \quad \text{in } \Psi_{1/32} \setminus E_{f_{\lambda^k}}. \tag{7.19}$$

To this end, for $(x,t) \in \Psi_{1/32} \setminus E_{f_{\lambda^k}}$ let $d = \sup\{r : \Psi_r(x,t) \cap E_{f_{\lambda^k}} = \emptyset\}$ and consider the box $\Psi_d(x,t)$. Without loss of generality assume $x_n \geq 0$. Then let $(x_*, t_*) = (x', x_n + d, t - d^2) \in \partial_p \Psi_d(x,t)$. From (7.18) we have that

$$v(x_*, t_*) \geq c_n(x_n + d) \geq c_n d$$

and applying the parabolic Harnack inequality, we obtain

$$v(x,t) \geq c_n v(x_*, t_*) - C_n \eta_0 d^2 \geq c_n d,$$

provided $\eta_0$ is sufficiently small. Hence, (7.19) follows.

Having the nondegeneracy, we also have the bound from above for functions $v_1$ and $v_2$. Indeed, by Theorem 7.4 for $v_1$ and $v_2$ we have

$$\begin{aligned} \sup_{\Psi_1} v_1 \leq C v_1(\overline{A}_{1/4}) &= C \frac{u_1(\overline{A}_{\lambda^k/4}) - l_k u_2(\overline{A}_{\lambda^k/4})}{u_1(\underline{A}_{\lambda^k/4}) - l_k u_2(\underline{A}_{\lambda^k/4})} \\ &\leq C \frac{u_2(\overline{A}_{\lambda^k/4})}{u_2(\underline{A}_{\lambda^k/4})} \frac{L_k - l_k}{\mu_k - l_k} \leq C \end{aligned} \tag{7.20}$$

and

$$\sup_{\Psi_1} v_2 \leq C v_2(\overline{A}_{1/4}) = C \frac{u_2(\overline{A}_{\lambda^k/4})}{u_2(\underline{A}_{\lambda^k/4})} \leq C, \tag{7.21}$$

where we have also invoked the second part of Theorem 7.4 for $u_2$.

We thus verified all conditions necessary for applying the estimate (7.9) to functions $v_1$ and $v_2$. Particularly, the inequality from below, applied in $\Psi_{8\lambda} \setminus E_{f_{\lambda^k}}$, will give

$$\inf_{\Psi_\lambda \setminus E_{f_{\lambda^k}}} \frac{v_1}{v_2} \geq c \frac{v_1(A_{2\lambda})}{v_2(A_{2\lambda})} \geq c\lambda$$

for a small $c > 0$, or equivalently

$$l_{k+1} - l_k \geq c\lambda(\mu_k - l_k) \geq \frac{c\lambda}{2}(L_k - l_k).$$

Hence, we will have

$$L_{k+1} - l_{k+1} \leq L_k - l_k - (l_{k+1} - l_k) \leq \left(1 - \frac{c\lambda}{2}\right)(L_k - l_k). \tag{7.22}$$

Summarizing, (7.17) and (7.22) give a dichotomy: for any $k \in \mathbb{N}$,

$$\text{either} \quad L_k - l_k \leq C\lambda^{(2-\gamma)k} \quad \text{or} \quad L_{k+1} - l_{k+1} \leq (1 - c\lambda/2)(L_k - l_k).$$

This clearly implies that

$$L_k - l_k \le C\beta^k \quad \text{for some } \beta \in (0,1),$$

for any $k \in \mathbb{N}$, which is nothing but the Hölder continuity of $u_1/u_2$ at the origin. □

We next want to prove a variant of Theorem 7.6 but with $\Psi_r$ replaced with their lower halves

$$\Theta_r = \Psi_r \cap \{t \le 0\}.$$

**Theorem 7.9.** *Let $u_i$, $i = 1, 2$, be nonnegative functions in $\Theta_1 \setminus E_f$, continuously vanishing on $\Theta_1 \cap E_f$, and satisfying*

$$\begin{aligned} |\Delta u_i - \partial_t u_i| &\le C_0 \quad \textit{in } \Theta_1 \setminus E_f \\ u_i(x,t) &\ge c_0 \operatorname{dist}((x,t), E_f) \quad \textit{in } \Theta_1 \setminus E_f, \end{aligned}$$

*for some $c_0 > 0$, $C_0 \ge 0$. Let also $M = \max\{\sup_D u_1, \sup_D u_2\}$. Moreover, if $u_1$ and $u_2$ are symmetric in $x_n$, then $u_1/u_2$ extends to a function in $C^\alpha(\overline{\Theta}_{1/8})$ for some $0 < \alpha < 1$, with $\alpha$ and $C^\alpha$ norm depending only on $n, L, \gamma, C_0, c_0, M$.*

The idea is that the functions $u_i$ can be extended to $\Psi_\delta$, for some $\delta > 0$, while still keeping the same inequalities, including the nondegeneracy condition.

**Lemma 7.10.** *Let $u$ be a nonnegative continuous function on $\Theta_1$ such that*

$$\begin{aligned} u &= 0 \quad \textit{in } \Theta_1 \cap E_f \\ |(\Delta - \partial_t)u| &\le C_0 \quad \textit{in } \Theta_1 \setminus E_f \\ u(x,t) &\ge c_0 \operatorname{dist}_p(x,t; E_f) \quad \textit{in } \Theta_1 \setminus E_f. \end{aligned}$$

*for some $C_0 \ge 0$, $c_0 > 0$. Then, there exists positive $\delta$ and $\tilde{c}_0$ depending only on $n$, $L$, $c_0$ and $C_0$, and a nonnegative extension $\tilde{u}$ of $u$ to $\Psi_\delta$ such that*

$$\begin{aligned} \tilde{u} &= 0 \quad \textit{in } \Psi_\delta \cap E_f \\ |(\Delta - \partial_t)\tilde{u}| &\le C_0 \quad \textit{in } \Psi_\delta \setminus E_f \\ \tilde{u}(x,t) &\ge \tilde{c}_0 \operatorname{dist}_p((x,t), E_f) \quad \textit{in } \Psi_\delta \setminus E_f. \end{aligned}$$

*Moreover, we will also have that $\sup_{\Psi_\delta} \tilde{u} \le \sup_{\Theta_1} u$.*

*Proof.* We first continuously extend the function $u$ from the parabolic boundary $\partial_p \Theta_{1/2}$ to $\partial_p \Psi_{1/2}$ by also keeping it nonnegative and bounded above by the same constant. Further, put $u = 0$ on $E_f \cap (\Psi_{1/2} \setminus \Theta_{1/2})$. Then extend $u$ to $\Psi_{1/2}$ by solving the Dirichlet problem for the heat equation in $(\Psi_{1/2} \setminus \Theta_{1/2}) \setminus E_f$, with already defined boundary values. We still denote it the extended function by $u$.

Then it is easy to see that $u$ is nonnegative in $\Psi_{1/2}$, $\sup_{\Psi_{1/2}} u \le \sup_{\Theta_1} u$, $u$ vanishes on $\Psi_{1/2} \cap E_f$ and $|(\Delta - \partial_t)u| \le C_0$ in $\Psi_{1/2} \setminus E_f$. Note that we still have the nondegeneracy property $u(x,t) \ge c_0 \operatorname{dist}_p((x,t), E_f)$ for in $\Theta_{1/2} \setminus E_f$, so it remains to prove the nondegeneracy for $t \ge 0$. We will be able to do it in a small box $\Psi_\delta$, as a consequence of Lemma 7.8.

For $0 < \delta < 1/2$ consider the rescalings

$$u_\delta(x,t) = \frac{u(\delta x, \delta^2 t)}{\delta}, \quad (x,t) \in \Psi_{1/(2\delta)}.$$

Then we have

$$|(\Delta-\partial_t)u_\delta|\leq C_0\delta,\quad \text{in } \Psi_1\setminus E_{f_\delta}$$
$$u_\delta(x,t)\geq c_0|x_n|\quad \text{in } \Theta_1,$$

where $f_\delta(x'',t)=(1/\delta)f(\delta x'',\delta^2 t)$ is the rescaling of $f$. Then by using the parabolic Harnack inequality (see Theorems 6.17 and 6.18 in [Lie96]) in $\Theta_1^\pm$, we obtain that

$$u_\delta(x,t)\geq c_n c_0 - C_n C_0\delta > c_1\quad \text{on } \{|x_n|=\beta_n/2\}\cap\Psi_{1/2}.$$

Further, choosing $\delta$ small and applying Lemma 7.8, we deduce that

$$u_\delta(x,t)\geq c_2|x_n|\quad \text{in } \Psi_{1/4}.$$

Then, repeating the arguments based on the parabolic Harnack inequality, as for the inequality (7.19), we obtain

$$u(x,t)\geq C\operatorname{dist}_p((x,t),E_{f_\delta}),\quad \text{in } \Psi_{1/8}.$$

Scaling back, this gives

$$u(x,t)\geq C\operatorname{dist}_p((x,t),E_f),\quad \text{in } \Psi_{\delta/8}.$$ □

*Proof of Theorem 7.9.* Extend functions $u_i$ as is Lemma 7.10 and apply Theorem 7.6. If we repeat this at every $(y,s)\in\Theta_{1/8}\cap G_f$, we will obtain the Hölder regularity of $u_1/u_2$ in $\mathcal{N}_{\delta/8}(\Theta_{1/8}\cap G_f)\cap\{t\leq 0\}$. For the remaining part of $\Theta_{1/8}$, we argue as in the proof of localization property Lemma 2.3 cases 1) ans 2), and use the corresponding results for parabolically Lipschitz domains. □

7.1. **Parabolic Signorini problem.** In this subsection we discuss an application of the boundary Harnack principle in the parabolic Signorini problem. The idea of such applications goes back to the paper Athanasopoulos and Caffarelli [AC85]. The particular result that we will discuss here, can be found also in [DGPT13], with the same proof based on our Theorem 7.9.

In what follows, we will use $H^{\ell,\ell/2}$, $\ell>0$, to denote the parabolic Hölder classes, as defined for instance in [LSU67].

For a given function $\varphi\in H^{\ell,\ell/2}(Q_1')$, $\ell\geq 2$, known as the *thin obstacle*, we say that a function $v$ solves the *parabolic Signorini problem* if $v\in W_2^{2,1}(Q_1^+)\cap H^{1+\alpha,(1+\alpha)/2}(\overline{Q_1^+})$ , $\alpha>0$, and

$$(\Delta-\partial_t)v=0\quad \text{in } Q_1^+, \tag{7.23}$$

$$v\geq\varphi,\quad -\partial_{x_n}v\geq 0,\quad (v-\varphi)\partial_{x_n}v=0\quad \text{on } Q_1'. \tag{7.24}$$

This kind of problems appears in many applications, such as thermics (boundary heat control), biochemistry (semipermeable membranes and osmosis), and elastostatics (the original Signorini problem). We refer to the book [DL76] for the derivation of such models as well as for some basic existence and uniqueness results.

The regularity that we impose on the solutions (7.23)–(7.24) is also well known in the literature, see e.g. [Ath82, Ura85, AU96]. It was proved recently in [DGPT13] that one can actually take $\alpha=1/2$ in the regularity assumptions on $v$, which is the optimal regularity as can be seen from the explicit example

$$v(x,t)=\operatorname{Re}(x_{n-1}+ix_n)^{3/2},$$

which solves the Signorini problem with $\varphi = 0$. One of the main objects of study in the Signorini problem is the *free boundary*

$$G(v) = \partial_{Q_1'}(\{v > \varphi\} \cap Q_1'),$$

where $\partial_{Q_1'}$ is the boundary in the relative topology of $Q_1'$.

As the initial step in the study, we make the following reduction. We observe that the difference

$$u(x,t) = v(x,t) - \varphi(x',t)$$

will satisfy

$$(\Delta - \partial_t)u = g \quad \text{in } Q_1^+, \tag{7.25}$$

$$u \geq 0, \quad -\partial_{x_n} u \geq 0, \quad u\partial_{x_n} u = 0 \quad \text{on } Q_1', \tag{7.26}$$

where $g = -(\Delta_{x'} - \partial_t)\varphi \in H^{\ell-2,(\ell-2)/2}$. That is, one can make the thin obstacle equal to 0 at the expense of getting a nonzero right-hand side in the equation for $u$. For our purposes, this simple reduction will be sufficient, however, to take the full advantage of the regularity of $\varphi$, when $\ell > 2$, one may need to subtract an additional polynomial from $u$ to guarantee the decay rate

$$|g(x,t)| \leq M(|x|^2 + |t|)^{(\ell-2)/2}$$

near the origin, see Proposition 4.4 in [DGPT13]. With the reduction above, the free boundary $G(v)$ becomes

$$G(u) = \partial_{Q_1'}(\{u > 0\} \cap Q_1').$$

Further, it will be convenient to consider the even extension of $u$ in $x_{n-1}$ variable to the entire $Q_1$, i.e., by putting $u(x', x_n, t) = u(x', -x_n, t)$. Then such an extended function will satisfy

$$(\Delta - \partial_t)u = g \quad \text{in } Q_1 \setminus \Lambda(u),$$

where $g$ has also been extended by even symmetry in $x_n$, and where

$$\Lambda(u) = \{u = 0\} \cap Q_1',$$

the so-called *coincidence set*.

As shown in [DGPT13], a successful study of the properties of the free boundary near $(x_0, t_0) \in G(u) \cap Q_{1/2}'$ can be made by considering the rescalings

$$u_r(x,t) = u_r^{(x_0,t_0)}(x,t) = \frac{u(x_0 + rx, t_0 + r^2 t)}{H_u^{(x_0,t_0)}(r)^{1/2}},$$

for $r > 0$ and then studying the limits of $u_r$ as $r = r_j \to 0+$ (so-called blowups). Here

$$H_u^{(x_0,t_0)}(r) := \frac{1}{r^2} \int_{t_0 - r^2}^{t_0} \int_{\mathbb{R}^n} u(x,t)^2 \psi^2(x) \Gamma(x_0 - x, t_0 - t) dx dt,$$

where $\psi(x) = \psi(|x|)$ is a cutoff function that equals 1 on $B_{3/4}$. Then a point $(x_0, t_0) \in G(u) \cap B_{1/2}$ is called regular, if $u_r$ converges in the appropriate sense to

$$u_0(x,t) = c_n \operatorname{Re}(x_{n-1} + i x_n)^{3/2},$$

as $r = r_j \to 0+$, after a possible rotation of coordinate axes in $\mathbb{R}^{n-1}$. See [DGPT13] for more details. Thus, let $\mathcal{R}(u)$ be the set of regular points of $u$. The following result has been proved in [DGPT13].

**Proposition 7.11.** *Let $u$ be a solution of the parabolic Signorini problem* (7.25)–(7.26) *in $Q_1^+$ with $g \in H^{1,1/2}(Q_1^+)$. Then the regular set $\mathcal{R}(u)$ is a relatively open subset of $G(u)$. Moreover, if $(0,0) \in \mathcal{R}(u)$, then there exists $\rho = \rho_u > 0$ and a parabolically Lipschitz function $f$ such that*

$$G(u) \cap Q'_\rho = \mathcal{R}(u) \cap Q'_\rho = G_f \cap Q'_\rho$$
$$\Lambda(u) \cap Q'_\rho = E_f \cap Q'_\rho.$$

*Furthermore, for any $0 < \eta < 1$, we can find $\rho > 0$ such that*

$$\partial_e u \geq 0 \quad \text{in } Q_\rho,$$

*for any unit direction $e \in \mathbb{R}^{n-1}$ such that $e \cdot e_{n-1} > \eta$ and moreover*

$$\partial_e u(x,t) \geq c \operatorname{dist}_p((x,t), E_f) \quad \text{in } Q_\rho,$$

*for some $c > 0$.* □

We next show that an application of Theorem 7.9 implies the following result.

**Theorem 7.12.** *Let $u$ be as in Proposition 7.11 and $(0,0) \in \mathcal{R}(u)$. Then there exist $\delta < \rho$ such that $\nabla'' f \in H^{\alpha,\alpha/2}(Q'_\delta)$ for some $\alpha > 0$, i.e., $\mathcal{R}(u)$ has Hölder continuous spatial normals in $Q'_\delta$.*

*Proof.* We will work in parabolic boxes $\Theta_\delta = \Psi_\delta \cap \{t \leq 0\}$ instead of cylinders $Q_\delta$. For a small $\varepsilon > 0$ let $e = (\cos\varepsilon)e_{n-1} + (\sin\varepsilon)e_j$ for some $j = 1, \dots, n-2$ and consider two functions

$$u_1 = \partial_e u \quad \text{and} \quad u_2 = \partial_{e_{n-1}} u.$$

Then by Proposition 7.11, the conditions of Theorem 7.9 are satisfied (after a rescaling), provided $\cos\varepsilon > \eta$. Thus, if we fix such $\varepsilon > 0$, then we will have that for some $\delta > 0$ and $0 < \alpha < 1$

$$\frac{\partial_e u}{\partial_{e_{n-1}} u} \in H^{\alpha,\alpha/2}(\Theta_\delta).$$

This gives that

$$\frac{\partial_{e_j} u}{\partial_{e_{n-1}} u} \in H^{\alpha,\alpha/2}(\Theta_\delta), \quad j = 1, \dots, n-2.$$

Hence the level surfaces $\{u = \sigma\} \cap \Theta'_\delta$ are given as graphs

$$x_{n-1} = f_\sigma(x'', t), \quad x'' \in \Theta''_\delta,$$

with uniform in $\sigma > 0$ estimate on $\|\nabla'' f_\sigma\|_{H^{\alpha,\alpha/2}(\Theta''_\delta)}$. Consequently, this implies that

$$\nabla'' f \in H^{\alpha,\alpha/2}(\Theta''_\delta),$$

and completes the proof of the theorem. □

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