

OPTIMAL REGULARITY IN ROOFTOP-LIKE OBSTACLE PROBLEM

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ABSTRACT. We study the regularity of solutions of the obstacle problem when the obstacle is smooth on each half of the unit ball but only Lipschitz across the shared boundary. We prove that the optimal regularity of these solutions is $C^{1, \frac{1}{2}}$ up to the shared boundary on each half of the unit ball. The proof uses a modification of Almgren's frequency formula.

1. INTRODUCTION

1.1. **The obstacle problem.** Given functions $\varphi : B_1 \rightarrow \mathbb{R}$ and $g : \partial B_1 \rightarrow \mathbb{R}$ satisfying $g \geq \varphi$ on ∂B_1 , consider the problem of minimizing the Dirichlet integral

$$\int_{B_1} |\nabla u|^2$$

on the closed convex set

$$\{u \in W^{1,2}(B_1) \mid u = g \text{ on } \partial B_1, u \geq \varphi \text{ in } B_1\}.$$

This problem is called the obstacle problem with φ being the obstacle. It is well-known that the problem has a unique solution u which enjoys the following properties (see [5]):

- i) u is superharmonic.
- ii) $\Delta u = 0$ on the set $\{u > \varphi\}$.
- iii) u is locally as regular as φ up to $C^{1,1}$.

In this work, we study the regularity of u when the obstacle φ is assumed to be smooth ($C^{1,1}$) up to the boundary on each half-ball B_1^\pm (see Fig. 1 for illustration). One of the motivations for the study of piecewise-smooth obstacles comes from applications to math finance, where the time-dependent version of this problem appears in the valuation of so-called American options on multiple assets, see e.g. [3, 7].

Since φ is Lipschitz in B_1 , the result mentioned above says that u is also Lipschitz in $B_{\frac{1}{2}}$ and that regularity is optimal unless extra condition is imposed on φ . However, since φ is smooth up to boundary on each B_1^\pm , it is reasonable to expect that u enjoys better regularity when restricted to these subsets. Indeed, in this work, we will prove that u is $C^{1, \frac{1}{2}}$ in $B_{\frac{1}{2}}^\pm \cup B_{\frac{1}{2}}'$ (Theorem 6.7). It can be easily seen that $C^{1, \frac{1}{2}}$ is the optimal regularity, similarly to the thin obstacle problem (see [2]): the function

$$u(x) = \operatorname{Re}(x_1 + i|x_n|)^{\frac{3}{2}}$$

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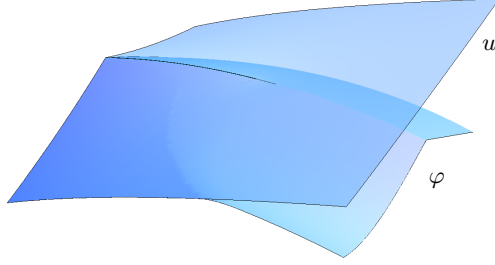


FIGURE 1. A solution of the obstacle problem with rooftop-like obstacle

is a solution of the obstacle problem with the obstacle $\varphi(x) = -C|x_n|$ for sufficiently large C .

1.2. Notation. To proceed, we fix the notations that we are going to use throughout the paper. For each

$$x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n,$$

we denote

$$x' = (x_1, x_2, \dots, x_{n-1}).$$

For any $r > 0$ and $x \in \mathbb{R}^n$, define

$$\begin{aligned} B_r(x) &= \{y \in \mathbb{R}^n \mid |y - x| < r\} \\ B_r^+(x) &= \{y \in \mathbb{R}^n \mid |y - x| < r, y_n > x_n\} \\ B_r^-(x) &= \{y \in \mathbb{R}^n \mid |y - x| < r, y_n < x_n\} \\ B_r'(x) &= \{y \in \mathbb{R}^n \mid |y - x| < r, y_n = x_n\}. \end{aligned}$$

When $x = 0$, we normally omit it and write B_r, B_r^\pm and B_r' .

Partial derivative in the direction τ is denoted by ∂_τ . Higher-order derivatives are written as $\partial_{\tau_1 \tau_2, \dots}$; ∂_ν denotes the outward normal derivative.

Since we have to work with functions which are not differentiable in the classical sense on B_1' , we introduce following notations,

$$\begin{aligned} \partial_{e_n}^+ v(x', 0) &= \lim_{x_n \rightarrow 0^+} \partial_{e_n} v(x', x_n) \\ \partial_{e_n}^- v(x', 0) &= \lim_{x_n \rightarrow 0^-} \partial_{e_n} v(x', x_n) \\ \partial_\nu^+ v(x', 0) &= -\partial_{e_n}^+ v(x', 0), \\ \partial_\nu^- v(x', 0) &= \partial_{e_n}^- v(x', 0). \end{aligned}$$

Here ∂_ν^+ and ∂_ν^- should be understood as outward normal derivative with respect to the sets $\{x_n > 0\}$ and $\{x_n < 0\}$ respectively.

We also use ∇' to denote

$$(\partial_{e_1}, \partial_{e_2}, \dots, \partial_{e_{n-1}}).$$

The frequently used set $B'_1 \cap \{u = \varphi\}$ is denoted as Σ .

1.3. Assumptions. We assume that φ is $C^{1,1}$ up to boundary in B_1^\pm and the boundary value g is bounded. This condition implies that u is $C^{0,1}$ locally in B_1 with the norm $\|u\|_{C^{0,1}(B_\rho)}$ depending only on n , $\|\varphi\|_{C^1(B_1)}$, $\|g\|_{L^\infty(\partial B_1)}$ and ρ for any $0 < \rho < 1$ (see [5]). Moreover, the optimal regularity theorem of [5] also implies that u is locally $C^{1,1}$ in B_1^\pm (but not up to B'_1).

1.4. Outline of the proof.

- In Section 2, we show that u is almost convex in every tangential direction $\tau \in \partial B'_1$, in the sense that $\partial_{\tau\tau}u$ is bounded below. The proof is in the spirit of [2].

- In Section 3, the almost-convexity is used to derive a $C^{1,\alpha}$ estimate for u . The proof follows ideas of [2, 4]. This result is used in two places. First, the extra smoothness gained enables us to prove a monotonicity formula. Secondly, it gives a uniform $C^{1,\alpha}$ estimate for appropriately defined scalings in our study of blowups of u . For this second use that we have to prove the $C^{1,\alpha}$ estimate on slightly weaker assumptions about u and φ . Precise conditions are stated at the beginning of Section 3.

- In Section 4, we obtain a monotonicity formula which, in a sense, is a truncated version of Almgren's frequency formula [1] (see Theorem 4.3). This type of monotonicity formulas has been first used in [6] and subsequently in [10] in the study of the thin obstacle problem.

- In Section 5, we derive the existence of a constant β as a consequence of the monotonicity formula such that u grows as a $C^{1,\beta}$ functions from the free-boundary $\partial\Sigma$.

- Finally, in Section 6, we show that the limit of appropriately chosen scalings of u solves a thin obstacle problem. Moreover, it is homogeneous of degree $1 + \beta$. Since the optimal regularity for solutions of thin obstacle problem is known to be $C^{1,\frac{1}{2}}$ (see [2]), we conclude that $\beta \geq \frac{1}{2}$. This in turn leads to the desired $C^{1,\frac{1}{2}}$ regularity of u up to the boundary on $B_{\frac{1}{2}}^\pm$.

2. ALMOST CONVEXITY

The first step in our proof is the almost convexity estimate on u in tangential directions $\tau \in \partial B'_1$. We mainly follow the ideas from [2, 4], suitably modified for our problem.

Lemma 2.1. *There exists a constant C depending on $\|\varphi\|_{C^{1,1}(B_1^\pm \cup B'_1)}$, $\|g\|_{L^\infty(\partial B_1)}$ and dimension n such that*

$$\inf_{B_{\frac{1}{2}}} \partial_{\tau\tau}u \geq C$$

for any unit vector $\tau \in \partial B'_1$.

Remark 2.2. The inequality is understood in the sense of distributions. Namely, if ψ is any nonnegative function in $W_0^{1,1}(B_{\frac{1}{2}})$, then

$$-\int_{B_{\frac{1}{2}}} (\partial_{\tau}u) (\partial_{\tau}\psi) \geq C \int_{B_{\frac{1}{2}}} \psi.$$

In fact, since u is locally $C^{1,1}$ in $B_1 \setminus B'_1$, the inequality is also equivalent to saying that $\partial_{\tau\tau}u \geq C$ a.e. in $B_{\frac{1}{2}}$.

Proof. Without loss of generality we may assume that φ is defined in a slightly larger ball, say $B_{1+\epsilon_0}$. Then for $0 < \epsilon < \epsilon_0$ define the mollifications of φ as follows. Fix a smooth radial function ζ on \mathbb{R}^n with support in B_1 such that

$$\int_{\mathbb{R}^n} \zeta = 1.$$

and let

$$\varphi_\epsilon(x) = \int_{B_1} \zeta(y) \varphi(x + \epsilon y) dy.$$

Then the following properties are immediate:

- i) φ_ϵ is smooth in B_1 .
- ii) φ_ϵ converges to φ uniformly on B_1 as ϵ goes to 0.
- iii) $\|\nabla \varphi_\epsilon\|_{L^\infty(B_1)}$ and $\|\partial_{\tau\tau} \varphi_\epsilon\|_{L^\infty(B_1)}$ are bounded uniformly by respective norms of φ (here τ is any unit vector in $\partial B'_1$).

Let u_ϵ be the solution for the obstacle problem with the obstacle φ_ϵ and boundary data g on B_1 . Then it is standard that $\|u_\epsilon - u\|_{L^\infty(B_1)} \leq \|\varphi_\epsilon - \varphi\|_{L^\infty(B_1)} \rightarrow 0$. Moreover, $\|u_\epsilon\|_{C^{0,1}(B_{\frac{1}{2}})}$ are uniformly bounded by $\|\varphi\|_{C^{0,1}(B_1)}$ and $\|g\|_{L^\infty(\partial B_1)}$. Thus, it will be enough to prove the almost convexity estimate for u_ϵ with a bound from below independent of ϵ .

We denote by A_ϵ the set $\{u_\epsilon = \varphi_\epsilon\}$ and by $d_\epsilon(x)$ the distance from x to A_ϵ . We have as a consequence of a result in [5] that $u_\epsilon \in C^{1,1}$ and for almost every $x \in B_{\frac{1}{2}}$

$$(2.1) \quad \partial_{\tau\tau} u_\epsilon(x) \geq \partial_{\tau\tau} \varphi_\epsilon(x) - \sigma(d_\epsilon(x)) \geq C - \sigma(d_\epsilon(x)),$$

where σ is a modulus of continuity depending on φ and ϵ . Let ζ be a nonnegative smooth cut-off function with support in $B_{\frac{3}{4}}$ such that $\zeta = 1$ in $B_{\frac{1}{2}}$. Consider the function

$$M(x) = \zeta(x)^2 \partial_{\tau\tau} u_\epsilon(x) - \lambda |\nabla u_\epsilon(x)|^2$$

where λ is some constant to be specified later. We will estimate an essential lower bound for M in B_1 , independent of ϵ . This will imply our desired estimate, since $|\nabla u_\epsilon|$ are equibounded on $B_{\frac{1}{2}}$ because of the equiboundedness of $|\nabla \varphi_\epsilon|$ on B_1 .

Further, we only need to estimate $M(x)$ in the set $\{u_\epsilon > \varphi_\epsilon\}$. Note that u_ϵ is harmonic, hence smooth there. Since as a consequence of the estimate (2.1),

$$\liminf_{\substack{x \rightarrow A_\epsilon \\ x \in \{u_\epsilon > \varphi_\epsilon\}}} \partial_{\tau\tau} u_\epsilon(x) \geq C,$$

we only need to consider the case $M(x)$ attains its minimum value at some point in the interior of $\{u_\epsilon > \varphi_\epsilon\}$. Call that minimum point x_0 . We can also assume that $\zeta(x_0) \neq 0$, otherwise the lower bound for $M(x_0)$ is trivial. One then has

$$0 = \nabla M = 2\zeta(\nabla\zeta)\partial_{\tau\tau} u_\epsilon + \zeta^2 \nabla \partial_{\tau\tau} u_\epsilon - 2\lambda \partial_{x_i} u_\epsilon \nabla \partial_{x_i} u_\epsilon,$$

or

$$\nabla \partial_{\tau\tau} u_\epsilon = -\frac{2\nabla\zeta}{\zeta} \partial_{\tau\tau} u_\epsilon + \frac{2\lambda}{\zeta^2} \partial_{x_i} u_\epsilon \nabla \partial_{x_i} u_\epsilon.$$

We also have

$$\begin{aligned} 0 &\leq \frac{1}{2} \Delta M = (\zeta \Delta \zeta + |\nabla \zeta|^2) \partial_{\tau\tau} u_\epsilon + 2\zeta \nabla \zeta \cdot \nabla \partial_{\tau\tau} u_\epsilon - \lambda \partial_{x_i x_j} u_\epsilon \partial_{x_i x_j} u_\epsilon \\ &= (\zeta \Delta \zeta + |\nabla \zeta|^2) \partial_{\tau\tau} u_\epsilon + 2\zeta \nabla \zeta \cdot \nabla \partial_{\tau\tau} u_\epsilon - \lambda |D^2 u_\epsilon|^2. \end{aligned}$$

Here, we have used that $\Delta u_\epsilon = 0$ in a neighborhood of x_0 .

Combining the computations above, we obtain

$$0 \leq \left(\zeta \Delta \zeta - 3 |\nabla \zeta|^2 \right) \partial_{\tau\tau} u_\epsilon + 4 \frac{\lambda}{\zeta} (\nabla \zeta \cdot \nabla \partial_{x_i} u_\epsilon) \partial_{x_i} u_\epsilon - \lambda |D^2 u_\epsilon|^2,$$

or

$$\lambda |D^2 u_\epsilon|^2 \leq C_1 \frac{\lambda}{\zeta} |\nabla \partial_{x_i} u_\epsilon| |\partial_{x_i} u_\epsilon| + C_0 |\partial_{\tau\tau} u_\epsilon|,$$

where C_0 and C_1 depend on ζ only. Since $|\nabla \partial_{x_i} u_\epsilon| \leq |D^2 u_\epsilon|$ and $|\partial_{\tau\tau} u_\epsilon| \leq |D^2 u_\epsilon|$, the above inequality simplifies to

$$\lambda |D^2 u_\epsilon| \leq C_1 \frac{\lambda}{\zeta} |\partial_{x_i} u_\epsilon| + C_0.$$

Now, if $|D^2 u_\epsilon| < 1$, then $|\partial_{\tau\tau} u_\epsilon| < 1$ and we have an lower bound for $M(x_0)$. Otherwise, choosing $\lambda > 2C_0$, we have

$$\zeta |D^2 u_\epsilon| \leq 2C_1 |\nabla u_\epsilon|$$

and so,

$$M \geq -2C_1 |\nabla u_\epsilon| - \lambda |\nabla u_\epsilon|^2. \quad \square$$

Remark 2.3. Since u is superharmonic in B_1 , Lemma 2.1 implies immediately that

$$\sup_{B_{\frac{1}{2}}} \partial_{e_n e_n} u \leq C.$$

Again, this should be understood in the sense of distributions.

Next, we derive some basic facts about u as consequences of this almost-convexity property.

Lemma 2.4. *Let C_0 be a constant such that in $B_{\frac{1}{2}}$,*

$$\begin{aligned} \partial_{\tau\tau} u &> -C_0, \quad \forall \tau \in \partial B'_1 \\ \partial_{e_n e_n} u &< C_0. \end{aligned}$$

Then we have

- i) $\partial_{e_n} u(x', t) - \partial_{e_n} u(x', s) \leq C_0(t - s)$ for any $(x', t), (x', s) \in B_{\frac{1}{2}}$ with $t > s$ and $t, s \neq 0$.
- ii) $\lim_{\epsilon \rightarrow 0^+} \partial_{e_n} u(x', \epsilon)$ and $\lim_{\epsilon \rightarrow 0^+} \partial_{e_n} u(x', -\epsilon)$ exist.
- iii) For any $(x', s) \in B_{\frac{1}{2}}^+$,

$$\begin{aligned} \partial_{e_n} u(x', s) - \partial_{e_n}^+ u(x', 0) &\leq C_0 s \\ \partial_{e_n}^- u(x', 0) - \partial_{e_n} u(x', -s) &\leq C_0 s. \end{aligned}$$

- iv) $\partial_\nu^+ u(x) + \partial_\nu^- u(x) \geq 0$ (or equivalently, $\partial_{e_n}^- u(x', 0) \geq \partial_{e_n}^+ u(x', 0)$) on $B'_{\frac{1}{2}}$.
- v) $\partial_\nu^+ u(x) + \partial_\nu^- u(x) = 0$ on $B' \cap \{u > \varphi\}$.
- vi) $\partial_\tau u(x' + s\tau, x_n) - \partial_\tau u(x', x_n) > -C_0 s$ for any $(x' + s\tau, x_n), (x', x_n) \in B_{\frac{1}{2}}$ with $x_n \neq 0, s > 0$, and $\tau \in \partial B'_1$.

Proof.

i) Recall that the inequality $\partial_{e_n e_n} u < C_0$ in $B_{\frac{1}{2}}$ means that for any nonnegative smooth function η with support in $B_{\frac{1}{2}}$ one has

$$-\int_{B_{\frac{1}{2}}} \partial_{e_n} u \partial_{e_n} \eta \geq C_0 \int_{B_{\frac{1}{2}}} \eta.$$

If the test function η has the form

$$\eta(x', x_n) = G(x')F(x_n),$$

where $G \in C_0^\infty(B_\epsilon(x'))$ and $F \in C_0^\infty(s - \epsilon, t + \epsilon)$ for some ϵ small, then we have

$$-\int_{B_{\frac{1}{2}}} G(x')F'(x_n) \partial_{e_n} u \geq C_0 \int_{B_{\frac{1}{2}}} G(x')F(x_n).$$

Since $\|u\|_{C^{0,1}(B_1)} < \infty$ and $\partial_{e_n} u(x)$ is continuous in $B_1 \setminus B'_1$, we can apply this inequality to a sequence of smooth functions $G_i(x')$ converging to the Dirac's delta function concentrated at x' , and then pass to the limit to obtain

$$-\int_s^t F'(x_n) \partial_{e_n} u(x', x_n) dx_n \geq C_0 \int_s^t F(x_n) dx_n.$$

Applying this inequality to a sequence of smooth functions F_i converging to $\chi_{(s,t)}$ in $W_0^{1,1}(s - \epsilon, t + \epsilon)$ such that $F'_i(0) = 0$ (it is possible since $s, t \neq 0$) and pass to the limit to obtain

$$\partial_{e_n} u(x', t) - \partial_{e_n} u(x', s) \geq C_0(t - s).$$

ii) From i) we have $\partial_{e_n} u(x', x_n) - Cx_n$ is a decreasing function of x_n in the set $\{x_n \neq 0\}$. The existence of those limits follows trivially.

iii) Just apply the result of i) for $t = x_n$ and $s = \epsilon$ (or $s = -x_n$ and $t = -\epsilon$) and let ϵ go to 0.

iv) For any $\epsilon > 0$, we have

$$\partial_{e_n} u(x', \epsilon) - \partial_{e_n} u(x', -\epsilon) \leq 2C_0\epsilon.$$

Passing ϵ to 0 we obtain the inequality.

v) When $u(x) > \varphi(x)$, u is harmonic in a neighborhood of x and hence smooth at x .

vi) The proof is similar to i). □

3. $C^{1,\alpha}$ ESTIMATE

The next step in the proof is the $C^{1,\alpha}$ regularity of u up to the boundary in B_1^\pm . We will mainly follow the ideas in [2]. Alternatively, one may use the approach in [12].

Later, in Section 6, we are going to apply the result of this section not just to u but also to some rescalings of u to obtain a uniform $C^{1,\alpha}$ bound. Since the respective rescaled obstacles are not necessarily uniformly bounded in $C^{1,1}(B_1^\pm \cup B'_1)$, we need to relax our requirement on u and φ a little bit. More precisely, we will prove the $C^{1,\alpha}$ estimate using only the following conditions on u and φ that have been established in the previous section: there exists a positive constant C_0 such

that

$$\begin{aligned}
(3.1) \quad & \Delta u \leq 0 \text{ in } B_{\frac{1}{2}} \\
(3.2) \quad & \Delta u \geq -C_0 \text{ in } B_{\frac{1}{2}}^{\pm} \\
(3.3) \quad & \partial_{\tau\tau} u > -C_0 \text{ and } \partial_{e_n e_n} u < C_0 \text{ in } B_{\frac{1}{2}} \\
(3.4) \quad & \|D^2 \varphi\|_{L^\infty(B_{\frac{1}{2}}^{\pm})} < C_0 \\
(3.5) \quad & u(x) \geq \varphi \text{ on } B'_1 \\
(3.6) \quad & \partial_\nu^+ u(x) + \partial_\nu^- u(x) \geq 0 \text{ on } B'_{\frac{1}{2}} \\
(3.7) \quad & \partial_\nu^+ u(x) + \partial_\nu^- u(x) = 0 \text{ on } B'_{\frac{1}{2}} \cap \{u > \varphi\} \\
(3.8) \quad & \|u\|_{C^{0,1}(B_{\frac{1}{2}})} \leq C_0 \\
(3.9) \quad & \|\nabla' \varphi\|_{B'_1} \leq C_0.
\end{aligned}$$

It is very important that we do not assume any bound on $|\partial_{e_n} \varphi|$. The regularity exponent α and the norm $C^{1,\alpha}(B_{\frac{1}{2}}^{\pm})$ will depend only on C_0 and the dimension n . In the rest of this section, unless stated otherwise, all constants C are understood to depend on C_0 and n only.

We observe that if u satisfies these properties, then so does

$$u^*(x', x_n) = \frac{u(x', x_n) + u(x', -x_n)}{2}.$$

On the other hand, if u^* is $C^{1,\alpha}$ up to the boundary in $B_{\frac{1}{2}}^{\pm}$, then $u(\cdot, 0)$ is $C^{1,\alpha}$ in $B'_{\frac{1}{2}}$ since u and u^* agree on B'_1 . Together with the fact that $0 \geq \Delta u \geq -C_0$ in $B_{\frac{1}{2}}^{\pm}$, the boundary regularity theorem (see e.g. [11, Theorem 8.34]) implies a $C^{1,\alpha}$ estimate for u in $B_{\frac{1}{2}}^{\pm}$. Thus, for the purpose of this section, we may assume without loss of generality that u is an even function with respect to x_n . A consequence of this assumption is that properties (3.6) and (3.7) above can be simplified to

$$\begin{aligned}
(3.6') \quad & \partial_\nu^+ u(x) \geq 0 \text{ on } B'_{\frac{1}{2}} \\
(3.7') \quad & \partial_\nu^+ u(x) = 0 \text{ for all } x \text{ on } B'_{\frac{1}{2}} \cap \{u > \varphi\}.
\end{aligned}$$

As the first step, we will show that u is $C^{1,\alpha}$ at any point in the set $\partial\Sigma$. To simplify the presentation, we will assume $0 \in \partial\Sigma$ and prove that

$$|u(x) - u(0) - x \cdot \nabla u(0)| \leq C|x|^{1+\alpha}$$

for some positive constants C and α .

Since we can subtract the linear function

$$L(x', x) = \varphi(0) + x' \cdot \nabla' \varphi(0)$$

from both φ and u without affecting the validity of (3.1)–(3.9) (with possibly $2C_0$ in the place of C_0), we can assume without loss of generality that

$$\varphi(0) = 0 \quad \text{and} \quad \nabla' \varphi(0) = 0.$$

We start with an auxiliary lemma.

Lemma 3.1. *For any $(x', x_n) \in B_1$ and $r > 0$ such that $x_n^2 + (r + |x'|)^2 \leq 1/4$, there exists a half-ball in \mathbb{R}^{n-1} of radius r with center at (x', x_n) , denoted by $HB_r(x', x_n)$, such that for all $y' \in HB_r(x', x_n)$,*

$$u(y', x_n) \geq u(x', x_n) - \frac{C_0}{2}r^2$$

Proof. We consider the case $x_n \neq 0$ first. If $\nabla' u(x', x_n) \neq 0$, define

$$HB_r(x', x_n) = \{y' \in \mathbb{R}^{n-1} \mid |y' - x'| \leq r, (y' - x') \cdot \nabla' u(x', x_n) \geq 0\}.$$

On the other hand, if $\nabla' u(x', x_n) = 0$, let $HB_r(x', x_n)$ be any half-ball of radius r with center at (x', x_n) . In either cases, the inequality

$$u(y', x_n) \geq u(x', x_n) - \frac{C_0}{2}r^2$$

is an elementary consequence of the fact that $\partial_\tau u(x', x_n) \geq 0$ and $\partial_{\tau\tau} u \geq -C_0$ where τ is the unit vector parallel to $y' - x'$.

When $x_n = 0$, we choose a sequence of $x_n^i \rightarrow 0$ such that the sequence of symmetry axes of $HB_r(x', x_n^i)$ converges. Then, we can take as $HB_r(x', 0)$ the limit of $HB_r(x', x_n^j)$ as $j \rightarrow \infty$. \square

For each $r > 0$, define

$$\begin{aligned} \Gamma_r^+ &= B'_r \times [0, r/2n] \\ \Gamma_r^- &= B'_r \times [-r/2n, 0] \\ \Gamma_r &= B'_r \times [-r/2n, r/2n]. \end{aligned}$$

Our next result says that a lower bound on $\partial_{e_n} u$ is enough to imply a bound on u around 0.

Lemma 3.2. *There exists a constant C such that for any $r > 0$ and $\delta > C_0 r$, the condition*

$$\inf_{\Gamma_r^+} \partial_{e_n} u \geq -\delta$$

implies

$$\operatorname{osc}_{\Gamma_{r/2}^+} u \leq Cr\delta.$$

Proof. From $\partial_\nu^+ u(x', 0) \geq 0$ and $\partial_{e_n e_n} u \leq C_0$, one has

$$\partial_{e_n} u(x', x_n) \leq C_0 x_n < \delta$$

for all $(x', x_n) \in \Gamma_r^+$. Therefore,

$$\sup_{\Gamma_r^+} |\partial_{e_n} u| \leq \delta.$$

Consequently, for any fixed $x' \in B'_{\frac{1}{2}}$,

$$\operatorname{osc}_{0 \leq x_n \leq r/2n} u(x', x_n) \leq \frac{1}{2n} r \delta.$$

Thus, it is enough to show that

$$-Cr\delta \leq u(x', 0) \leq Cr\delta$$

in $B'_{r/2}$ for some C . Due to the normalizing assumption $\varphi(0,0) = 0$, $\nabla'\varphi(0,0) = 0$ and $|D^2\varphi| < C_0$, we have on B'_r ,

$$u(x',0) \geq \varphi(x',0) \geq -\frac{C_0}{2}r^2 \geq -\frac{1}{2}r\delta.$$

Thus, we only need to prove the right hand side inequality $u(x',0) \leq Cr\delta$. Assume that $u(x'_0,0) > Cr\delta$ for some $x'_0 \in B'_{r/2}$ and $C > 0$. We want to derive a contradiction when C is large. First, by Lemma 3.1, we have that for all $y' \in HB_r(x'_0,0)$,

$$u(y',0) \geq Cr\delta - \frac{C_0}{2}r^2 > \frac{1}{2}Cr\delta$$

and consequently

$$u(y',x_n) \geq u(y',0) - x_n \|\nabla u\|_{L^\infty} \geq \frac{1}{2}Cr\delta - x_n\delta \geq \frac{1}{2}Cr\delta - \frac{1}{2n}r\delta \geq \frac{1}{4}Cr\delta$$

for all $0 \leq x_n \leq r/2n$, as long as $C > \max\{C_0, 2\}$.

Next, denote by V the solution of the problem

$$\begin{cases} \Delta_{x'}V(x') = \delta(x') & \text{in } B'_r \\ V = 0 & \text{on } \partial B'_r. \end{cases}$$

Applying Green's formula for V and u in $B'_r(0, x_n)$, one has that for any $x_n \in (0, r/2n)$

$$u(0, x_n) - \int_{B'_r} (\Delta_{x'}u(x', x_n))V dx' = \int_{\partial B'_r} u(x', x_n) ds.$$

Since

$$\begin{aligned} u(0, x_n) &\leq u(0) + \frac{1}{2n}r\delta = \frac{1}{2n}r\delta, \\ u(x', x_n) &\geq u(x', 0) - \frac{1}{2n}r\delta \geq -\frac{C_0}{2}r^2 - \frac{1}{2n}r\delta, \\ u(y', x_n) &\geq \frac{1}{4}Cr\delta, \quad \text{for all } y' \in HB_r(x'_0, 0), \end{aligned}$$

one has

$$\int_{B'_r} (\Delta_{x'}u)V dx' \leq \left(\frac{1}{2n} + \frac{1}{2} - a(n)C \right) r\delta,$$

where $a(n)$ is a positive dimensional constant. Therefore,

$$\begin{aligned} \frac{1}{2n} \left(\frac{1}{n} + \frac{C_0}{2} - a(n)C \right) r^2\delta &\geq \int_0^{r/2n} \int_{B'_r} (\Delta_{x'}u)V dx' dx_n \\ &= \int_0^{r/2n} \int_{B'_r} (\Delta u)V dx' dx_n - \int_0^{r/2n} \int_{B'_r} (\partial_{e_n e_n} u)V dx' dx_n \\ &\geq \int_0^{r/2n} \int_{B'_r} (-\partial_{e_n e_n} u)V dx' dx_n \quad (\Delta u, V \leq 0) \\ &= \int_{B'_r} V[\partial_{e_n} u(x', 0) - \partial_{e_n} u(x', r/2n)] dx' \\ &\geq \int_{B'_r} |V|(-2\delta) dx' \\ &\geq -b(n)r^2\delta, \end{aligned}$$

where $b(n)$ is another positive dimensional constant. We can choose C large enough to obtain a contradiction here. That contradiction implies that for that value of C , we cannot have

$$u(x', 0) > Cr\delta.$$

The desired conclusion then follows. \square

Next, we show a lower bound on $\partial_{e_n} u$.

Lemma 3.3. *There exist $C > 0$ and $\mu \in (0, 1)$ such that for any positive integer k ,*

$$\inf_{\Gamma_{2^{-k}}^+} \partial_{e_n} u \geq -C\mu^k.$$

Proof. We prove by induction. Clearly, since $|\nabla u| < C_0$ in $B_{\frac{1}{2}}$, given any $\mu \in (0, 1)$, we can find C such that the conclusion holds for $k = 1$. Assuming that it holds for $k = k_0$, we will prove that it holds for $k = k_0 + 1$.

To simplify the notation, we also use r in the place of 2^{-k_0} in the rest of the proof. Let $\{x'_j\}$ be a sequence in $B' \cap \{u > \varphi\}$ such that $x'_j \rightarrow 0$. For each j , define the auxiliary function

$$w_j(x', x_n) = u(x', x_n) - \varphi(x'_j, 0) - (x' - x'_j) \cdot \nabla' \varphi(x'_j, 0) - 2C_0 \left(|x' - x'_j|^2 - nx_n^2 \right).$$

Consider the function w_j in $\Gamma_{r/2}^+$. Since

$$\Delta w_j = \Delta u + 4C_0 > 0$$

when $x_n > 0$, w_j must attain its maximum value on the boundary $\partial\Gamma_{r/2}^+$. On the other hand, at the point $(x'_j, 0)$ we have

$$\begin{aligned} w(x'_j, 0) &= 0 \\ \partial_{e_n} w(x'_j, 0) &= 0, \end{aligned}$$

and therefore, by the Hopf boundary principle, 0 cannot be the maximum value for w_j . Hence the maximum value of w_j must be positive. Since $\nabla' \varphi(0) = 0$ and $|D^2 \varphi| \leq C_0$, we have $|\nabla' \varphi(x', 0)| \geq C_0 |x'|$. Thus, on the set $B'_{r/2} \cap \{u = \varphi\}$, we have

$$w_j(x', 0) = -(x' - x'_j) \cdot \nabla' \varphi(x'_j, 0) - 2C_0 |x' - x'_j|^2 < 0,$$

while on $B'_{r/2} \cap \{u > \varphi\}$,

$$\partial_\nu^+ w_j(x', 0) = \partial_\nu^+ u(x', 0) = 0,$$

hence w_j cannot attain its maximum value on B'_r . In other words, w_j must attain its positive maximum value on $\partial\Gamma_{r/2}^+ \setminus B'_{r/2}$. Passing j to ∞ , we derive that the function

$$w = u(x', x_n) - 2C_0 \left(|x'|^2 - nx_n^2 \right)$$

must be nonnegative at some point on

$$\partial\Gamma_{r/2} \setminus B'_{r/2} = \partial B'_{r/2} \times [0, r/4n] \cup B'_{r/2} \times \{r/4n\}.$$

We consider two cases separately.

1) w is nonnegative at some point $(x'_0, r/4n) \in B'_{r/2} \times \{r/4n\}$. Then, at that point

$$u(x'_0, r/4n) \geq 2C_0(|x'|^2 - n(r/4n)^2) \geq -\frac{C_0}{8n}r^2.$$

For all $y' \in HB_{r/2}(x'_0, r/4n)$ we then have

$$(3.10) \quad u(y', r/4n) \geq -\frac{C_0}{8n}r^2 - \frac{C_0}{8}r^2.$$

On the other hand, for any x' , if $\partial_{e_n} u(x', 0) = c < 0$, then

$$u(x', 0) = \varphi(x', 0) \leq \frac{C_0}{2}|x'|^2 \leq \frac{C_0}{8}r^2$$

and so, together with $\partial_{e_n e_n} u < C_0$, one obtains

$$(3.11) \quad u(x', r/2n) \leq \frac{C_0}{8}r^2 + c\frac{r}{2n} + \frac{C_0}{2}(r/2n)^2.$$

Combining (3.10) and (3.11), we have for any $y' \in HB_{r/2}(x'_0, r/4n)$,

$$\partial_{e_n} u(y', 0) \geq \frac{2n}{r} \left(-\frac{C_0}{8n}r^2 - \frac{C_0}{8}r^2 - \frac{C_0}{8}r^2 - \frac{C_0}{2}(r/2n)^2 \right) = -\frac{(n^2 + n + 2)C_0}{4n}2^{-k_0}.$$

2) w is nonnegative at some point $(x'_0, x_n) \in \partial B'_{r/2} \times [0, r/4n]$. It is equivalent to

$$u(x'_0, x_n) \geq 2C_0((r/2)^2 - nx_n^2) \geq 2C_0((r/2)^2 - n(r/4n)^2) = \frac{C_0(4n-1)}{8n}r^2$$

and consequently, for $y' \in HB_{r/2}(x'_0, x_n)$,

$$u(y', x_n) \geq \frac{C_0(4n-1)}{8n}r^2 - \frac{C_0}{8}r^2 = \frac{C_0(3n-1)}{8n}r^2.$$

As shown in (3.11), if $\partial_{e_n} u(x', 0) < 0$, then

$$u(x', x_n) \leq \frac{C_0}{8}r^2 + \frac{C_0}{2}(r/2n)^2 < \frac{C_0(3n-1)}{8n}r^2.$$

Hence, we must have $\partial_{e_n} u(y', 0) = 0$ for all $y' \in HB_{r/2}(x'_0, x_n)$.

In both cases, we have reached the conclusion that there exists a point $(x'_0, x_n) \in \Gamma_{r/2}^+$ such that for all $y' \in HB_{r/2}(x'_0, x_n)$,

$$\partial_{e_n} u(y', 0) \geq -\frac{(n^2 + n + 2)C_0}{4n}2^{-k_0}.$$

If C is chosen so that

$$C > \frac{(n^2 + n + 2)C_0}{2n},$$

then as long as $\mu > \frac{1}{2}$, we have for all $y' \in HB_{r/2}(x'_0, x_n)$,

$$\partial_{e_n} u(y', 0) \geq -\frac{(n^2 + n + 2)C_0}{4n}2^{-k_0} > -\frac{1}{2}C\mu^{k_0}.$$

Denote by $G_1(x, y)$ the Green's function for Γ_1^+ , then the Green's function for Γ_r^+ is given by

$$G_r(x, y) = \frac{1}{r^{n-2}}G_1\left(\frac{x}{r}, \frac{y}{r}\right).$$

Given any fixed $x \in B'_{r/2} \times \{r/4n\}$, we apply the Representation Formula to $\partial_{e_n} u$

$$\begin{aligned} \partial_{e_n} u(x) &= - \int_{\partial\Gamma_r^+} \partial_{e_n} u(y) \partial_\nu G_r(x, y) dy - \int_{\Gamma_r^+} G_r(x, y) \Delta \partial_{e_n} u dy \\ &= - \int_{\partial\Gamma_r^+} \partial_{e_n} u(y) \partial_\nu G_r(x, y) dy + \int_{\Gamma_r^+} \partial_{e_n} G_r(x, y) \Delta u dy \\ &\geq -\theta(n) C \mu^{k_0} - C_0 \int_{\Gamma_r^+} |\partial_{e_n} G_r(x, y)| dy \\ &= -\theta(n) C \mu^{k_0} - C_0 C_2(n) r, \end{aligned}$$

where

$$C_2(n) = \max_{x \in B'_{\frac{1}{2}} \times \{1/4n\}} \int_{\Gamma_1^+} |\partial_{e_n} G_1(x, y)| dy$$

and $\theta(n) \in (\frac{1}{2}, 1)$ is a dimensional constant. Since

$$\partial_{e_n} u(x', r/4n) - \partial_{e_n} u(x', x_n) < C_0(r/4n - x_n),$$

we have for all $x' \in B'_{r/2}$ and $x_n \in (0, r/4n)$

$$\begin{aligned} \partial_{e_n} u(x', x_n) &\geq -\theta(n) C \mu^{k_0} - C_0 C_2(n) r - C_0(r/4n) \\ &= -C \mu^{k_0} \left(\theta(n) + \frac{C_0 C_2(n)}{C} (2\mu)^{-k_0} + \frac{C_0}{4nC} (2\mu)^{-k_0} \right). \end{aligned}$$

Fix μ to be some number between $\theta(n)$ and 1, then we could choose C large enough so that

$$\theta(n) + \frac{C_0 C_2(n)}{C} + \frac{C_0}{4nC} < \mu.$$

Since $2\mu > 1$, it immediately leads to

$$\partial_{e_n} u(x) > -C \mu^{k_0+1}$$

in $\Gamma_{r/2}^+$. □

Lemma 3.4. *There exist C and α such that for all $x \in B_{\frac{1}{2}}$,*

$$\begin{aligned} |\partial_{e_n} u(x)| &\leq C |x|^\alpha \\ |u(x)| &\leq C |x|^{1+\alpha}. \end{aligned}$$

Proof. The proof is elementary, following Lemmas 3.2 and 3.3. □

Remark 3.5. A consequence of this lemma is that u is differentiable at 0.

Theorem 3.6. *Suppose u , φ and C_0 satisfy (3.1)–(3.9). Then there exist $\alpha \in (0, 1)$ and $C > 0$ depending only on C_0 and dimension n such that*

$$\|u\|_{C^{1,\alpha}(B_{\frac{1}{2}}^+ \cup B'_{\frac{1}{2}})} < C.$$

Proof. The proof is similar to the one in [2], except for a small change.

We may apply the result of Lemma 3.4 to any point in the set $B'_{\frac{1}{2}} \cap \partial\Sigma$ by translating, scaling, and subtracting a linear function. Namely, at any $z \in B'_{\frac{1}{2}} \cap \partial\Sigma$, ∇u is well-defined in the classical sense and for any $x \in B_1^+$ with $|x - z| < \frac{1}{4}$, we can write that

$$(3.12) \quad |u(x) - u(z) - (x - z) \cdot \nabla u(z)| \leq C |x - z|^{1+\alpha}.$$

In order to show that u is $C^{1,\alpha}$ up to boundary in $B_{\frac{1}{2}}^+$, by the boundary regularity theorem (see e.g. [11, Theorem 8.34]), it is enough to show that $\nabla' u(x', 0)$ is a C^α function of x' since u is bounded and

$$-C_0 \leq \Delta u \leq 0$$

in B_1^+ .

Let y_1 and y_2 be any two points in $B_{\frac{1}{2}}'$. For $i = 1, 2$, let $z_i \in \partial\Sigma$ and $d_i \geq 0$ be such that

$$|z_i - y_i| = d_i = \text{dist}(y_i, \partial\Sigma).$$

We also denote $d = \text{dist}(y_1, y_2)$.

To show that $|\nabla' u(y_1) - \nabla' u(y_2)| \leq Cd^\alpha$, we consider several possibilities:

1) If both $y_1, y_2 \in \{u = \varphi\}$, then

$$|\nabla' u(y_1) - \nabla' u(y_2)| = |\nabla' \varphi(y_1) - \nabla' \varphi(y_2)| \leq Cd.$$

2) If $y_1 \in \{u > \varphi\}$ and $y_2 \in \{u = \varphi\}$, then we argue as follows. Consider the auxiliary function

$$w(x) = u(x) - u(z_1) - (x - z_1) \cdot \nabla u(z_1).$$

Then by (3.12)

$$|w| \leq Cd_1^{1+\alpha} \quad \text{in } B_{d_1}(y_1),$$

since $B_{d_1}(y_1) \subset B_{2d_1}(z_1)$. Further, from to the definition of d_1 we have that $u > \varphi$ on the set $B_{d_1}(y_1) \cap B_1'$. Consequently, we have

$$-C_0 \leq \Delta w = \Delta u \leq 0 \quad \text{on } B_{d_1}(y_1).$$

Applying the result of [11, Theorem 3.9] to the function w in $B_{d_1}(y_1)$, one obtains

$$|\nabla u(y_1) - \nabla u(z_1)| = |\nabla w(y_1)| \leq Cd_1^\alpha.$$

Now, to finish this case, note that $d > d_1$ and estimate

$$\begin{aligned} |\nabla' u(y_1) - \nabla' u(y_2)| &\leq |\nabla' u(y_1) - \nabla' u(z_1)| + |\nabla' u(z_1) - \nabla' u(y_2)| \\ &\leq |\nabla' u(y_1) - \nabla' u(z_1)| + |\nabla' \varphi(z_1) - \nabla' \varphi(y_2)| \\ &\leq Cd_1^\alpha + C(d + d_1) \\ &\leq Cd^\alpha. \end{aligned}$$

3) If $y_1, y_2 \in \{u > \varphi\}$, we consider two subcases.

a) If $4d > \max\{d_1, d_2\}$, then similarly to case 2) we have the estimates $|\nabla' u(y_i) - \nabla' u(z_i)| \leq Cd_i^\alpha$ and therefore

$$\begin{aligned} |\nabla' u(y_1) - \nabla' u(y_2)| &\leq |\nabla' u(y_1) - \nabla' u(z_1)| + |\nabla' u(y_2) - \nabla' u(z_2)| + |\nabla' u(z_1) - \nabla' u(z_2)| \\ &\leq Cd_1^\alpha + Cd_2^\alpha + C(d_1 + d_2 + d) \\ &\leq Cd^\alpha. \end{aligned}$$

b) If $4d \leq d_1 = \max\{d_1, d_2\}$, then the auxiliary function

$$w(x) = u(x) - u(z_1) - (x - z_1) \cdot \nabla u(z_1)$$

satisfies

$$-C_0 \leq \Delta w = \Delta u \leq 0 \quad \text{in } B_{d_1}(y_1).$$

Applying [11, Theorem 3.9] to w in $B_{d_1}(y_1)$, we have

$$\begin{aligned} d_1^2 \frac{|\nabla w(y_1) - \nabla w(y_2)|}{d} &\leq C(n)(\sup |w| + d_1^2 \sup |\Delta w|) \log \frac{d_1}{d} \\ &\leq C d_1^{1+\alpha} \log \frac{d_1}{d} \\ |\nabla u(y_1) - \nabla u(y_2)| &\leq C d d_1^{\alpha-1} \log \frac{d_1}{d} \\ &\leq C d^\alpha \left(\frac{d_1}{d}\right)^{\alpha-1} \log \frac{d_1}{d} \\ &\leq C d^\alpha. \end{aligned}$$

In the last step, we have used the fact that the function $t^{\alpha-1} \log t$ is bounded in $[4, \infty]$ for all $\alpha > 0$. \square

Remark 3.7. From now on, α will be reserved for the $C^{1,\alpha}$ regularity exponent in Theorem 3.6. We also denote by C_α a constant such that $\|u\|_{C^{1,\alpha}(B_{\frac{1}{2}}^+ \cup B'_{\frac{1}{2}})} \leq C_\alpha$.

4. MONOTONICITY FORMULA

We are going back to our original problem as stated in Introduction. Our aim in the rest of this work is to show that u is $C^{1,\frac{1}{2}}$ up to the boundary in $B_{\frac{1}{2}}^\pm$. We will need the full strength of the definition that u is the solution to an obstacle problem. In particular, we need the fact that $\Delta u = 0$ in $\{u > \varphi\}$. Substituting it by the conditions that u is superharmonic and $\Delta u \geq -C$ in B_1^\pm , for example, is not enough for our proof to work. Therefore, while conditions (3.1)–(3.9) are enough to guarantee a $C^{1,\alpha}$ estimate, they are not enough for our $C^{1,\frac{1}{2}}$ result.

In this section we prove a monotonicity formula that will be our main tool in establishing the optimal $C^{1,\frac{1}{2}}$ regularity of u . This type of formulas goes back to Almgren [1] in the study of multivalued harmonic functions and to Garofalo-Lin [8, 9] in the context of unique continuation. Recently, this formula has been adapted to the thin obstacle problem in [6, 10] by the means of the truncation of the growth rate of the function.

As demonstrated in the proof of Theorem 3.6, in order to show that u is $C^{1,\frac{1}{2}}$ in $B_{\frac{1}{2}}^+$, it is enough to assume that $0 \in \partial\Sigma$ and to show that

$$|u(x) - u(0) - x \cdot \nabla u(0)| \leq C |x|^{\frac{3}{2}} \quad \forall x \in B_{\frac{1}{2}}.$$

Furthermore, since subtracting a linear function from both u and φ does not effect the hypotheses or conclusions of the problem, we can assume that

$$(4.1) \quad u(0) = 0 \quad \text{and} \quad \nabla u(0) = 0.$$

Note that this assumption implies that $\varphi(0) = 0$ and $\nabla \varphi(0) = 0$ (but not $\partial_{e_n} \varphi(0) = 0$).

We start by defining the operators involved in the monotonicity formula. For each function v in B_1 and $r \in (0, 1)$ let

$$\begin{aligned} D(r, v) &= \int_{B_r} |\nabla v|^2 \\ H(r, v) &= \int_{\partial B_r} v^2 \\ G(r, v) &= \int_{B_r} v^2 \\ I(r, v) &= \int_{\partial B_r} v v_\nu = \int_{B_r} |\nabla v|^2 + \int_{B_r^+ \cup B_r^-} v \Delta v - 2 \int_{B_r'} v (\partial_\nu^+ v^*) \\ N(r, v) &= \frac{rD(r, v)}{H(r, v)}, \end{aligned}$$

whenever those quantities are well defined.

Then we have the following differentiation formulas.

Lemma 4.1. *For any $r \in (0, 1)$ we have*

$$\begin{aligned} D'(r, u) &= \frac{n-2}{r} D(r, u) + 2 \int_{\partial B_r} u_\nu^2 - \frac{2}{r} \int_{B_r^+ \cup B_r^-} (x \cdot \nabla u) \Delta u - \frac{4}{r} \int_{B_r'} (x \cdot \nabla u) \partial_\nu^+ u^* \\ H'(r, u) &= \frac{n-1}{r} H(r, u) + 2I(r, u) \\ G'(r, u) &= H(r, u) \\ I'(r, u) &= \frac{n-2}{r} I(r, u) + 2 \int_{\partial B_r} u_\nu^2 - \frac{2}{r} \int_{B_r^+ \cup B_r^-} (x \cdot \nabla u) \Delta u - \frac{4}{r} \int_{B_r'} (x \cdot \nabla u) \partial_\nu^+ u^* \\ &\quad - \frac{n-2}{r} \int_{B_r^+ \cup B_r^-} u \Delta u - \frac{2(n-2)}{r} \int_{B_r'} u \partial_\nu^+ u^* + \int_{\partial B_r} u \Delta u - 2 \int_{\partial B_r'} u \partial_\nu^+ u^* \end{aligned}$$

Proof. All computations are standard (see for example [8, 9]). The only difference here is that we have used $2\partial_\nu^+ u^*$ in place of $\partial_\nu^+ u + \partial_\nu^- u$ on B_r' . Recall that

$$u^*(x', x_n) = \frac{u(x', x_n) + u(x', -x_n)}{2}. \quad \square$$

Using the $C^{1,\alpha}$ result for u , we obtain some estimates for $H(r, u)$ and $G(r, u)$ in terms of $D(r, u)$.

Lemma 4.2. *There exists a positive constant C depending only on C_0 and n such that for all $r \in (0, 1)$*

$$\begin{aligned} H(r, u) &\leq C(rD(r, u) + r^{n+1+2\alpha}) \\ G(r, u) &\leq C(r^2 D(r, u) + r^{n+2+2\alpha}). \end{aligned}$$

Recall that α denotes the regularity exponent in $C^{1,\alpha}$ result of Theorem 3.6 and C_0 is the constant defined in (3.1)–(3.9).

Proof. For each $r \in (0, 1)$ define

$$\bar{u}(r) = \int_{\partial B_r} u.$$

Since u is superharmonic, one has

$$\bar{u}(r) \leq u(0) = 0.$$

On the other hand, from the $C^{1,\alpha}$ estimate we also have that

$$\bar{u}(r) \geq -C_\alpha r^{1+\alpha}.$$

Applying Poincaré's inequality to u on B_r , we obtain

$$\begin{aligned} \int_{\partial B_r} (u - \bar{u}(r))^2 &\leq a(n)r \int_{B_r} |\nabla u|^2 \\ \int_{\partial B_r} u^2 - 2\bar{u}(r) \int_{\partial B_r} u + \bar{u}(r)^2 &\leq a(n)r \int_{B_r} |\nabla u|^2 \\ \int_{\partial B_r} u^2 + (1 - b(n)r^{n-1})\bar{u}(r)^2 &\leq a(n)r \int_{B_r} |\nabla u|^2 \\ \int_{\partial B_r} u^2 &\leq a(n)r \int_{B_r} |\nabla u|^2 + b(n)C_\alpha^2 r^{n+1+2\alpha}, \end{aligned}$$

where $a(n)$ is a dimensional constant. This gives us the first inequality. Integrating it, we obtain the second inequality. \square

Finally, we can prove the monotonicity formula.

Lemma 4.3. *There exist $r_0 > 0$ and $C_m > 0$ depending only on C_0 and n such that*

$$\Phi(r, u) = re^{C_m r^{\frac{\alpha}{2}}} \frac{d}{dr} \log \max \{H(r, u), r^{n+2+\frac{\alpha}{2}}\}$$

is increasing for $r \in (0, r_0)$.

Proof. Note that Φ is absolutely continuous (or equivalently $W^{1,1}$) in $(0, 1)$ and therefore it is enough to show that

$$\Phi(r, u) = re^{C_m r^{\frac{\alpha}{2}}} \frac{H'(r, u)}{H(r, u)} = re^{C_m r^{\frac{\alpha}{2}}} \left(\frac{n-1}{r} + 2 \frac{I(r, u)}{H(r, u)} \right)$$

is nondecreasing in each component of the set $U = \{r \in (0, r_0) \mid H(r, u) > r^{n+2+\frac{\alpha}{2}}\}$. We will prove the monotonicity of

$$re^{C_m r^{\frac{\alpha}{2}}} \frac{I(r, u)}{H(r, u)},$$

since $(n-1)e^{C_m r^{\frac{\alpha}{2}}}$ is clearly increasing. We do it by showing that

$$\frac{\partial}{\partial r} \log \left(re^{C_m r^{\frac{\alpha}{2}}} \frac{I(r, u)}{H(r, u)} \right) = \frac{1}{r} + \frac{\alpha}{2} C_m r^{-1+\frac{\alpha}{2}} + \frac{I'(r, u)}{I(r, u)} - \frac{H'(r, u)}{H(r, u)} \geq 0$$

which is further reduced to

$$\begin{aligned} E(r) &= \frac{1}{I(r, u)} \left(-\frac{2}{r} \int_{B_r^+ \cup B_r^-} (x \cdot \nabla u) \Delta u - \frac{2}{r} \int_{B_r^+} (x' \cdot \nabla' u) \partial_\nu^+ u^* - \frac{n-2}{r} \int_{B_r^+ \cup B_r^-} u \Delta u \right. \\ &\quad \left. - \frac{n-2}{r} \int_{B_r^+} u \partial_\nu^+ u^* + \int_{\partial B_r} u \Delta u - \int_{\partial B_r^+} u \partial_\nu^+ u^* \right) \\ &\geq -Cr^{-1+\frac{\alpha}{2}} \end{aligned}$$

on the set U for some positive constant C . This reduction is pretty standard and can be seen for example in [6, 8, 10].

Using Hölder's inequality and the estimates in Lemma 4.2, we have

$$\begin{aligned}
\left| \int_{B_r^+ \cup B_r^-} (x \cdot \nabla u) \Delta u \right| &\leq \left(\int_{B_r^+ \cup B_r^-} |x|^2 |\nabla u|^2 \right)^{\frac{1}{2}} \left(\int_{B_r^+ \cup B_r^-} |\Delta u|^2 \right)^{\frac{1}{2}} \\
&\leq CD(r, u)^{\frac{1}{2}} r^{\frac{n}{2}+1} \\
\left| \int_{B_r^+ \cup B_r^-} u \Delta u \right| &\leq \left(\int_{B_r^+ \cup B_r^-} |u|^2 \right)^{\frac{1}{2}} \left(\int_{B_r^+ \cup B_r^-} |\Delta u|^2 \right)^{\frac{1}{2}} \\
&\leq CG(r, u)^{\frac{1}{2}} r^{\frac{n}{2}} \\
&\leq CD(r, u)^{\frac{1}{2}} r^{\frac{n}{2}+1} + Cr^{n+1+\alpha} \\
\left| \int_{\partial B_r} u \Delta u \right| &\leq \left(\int_{\partial B_r} |u|^2 \right)^{\frac{1}{2}} \left(\int_{\partial B_r} |\Delta u|^2 \right)^{\frac{1}{2}} \\
&\leq CH(r, u)^{\frac{1}{2}} r^{\frac{n-1}{2}} \\
&\leq CD(r, u)^{\frac{1}{2}} r^{\frac{n}{2}} + Cr^{n+\alpha}.
\end{aligned}$$

Next, we estimate terms involving $\partial_\nu^+ u^*$. First, one has $\partial_\nu^+ u^* = 0$ on $B_r' \cap \{u > \varphi\}$. On the other hand, on $B_r' \cap \{u = \varphi\}$ one has that,

$$\begin{aligned}
|\partial_\nu^+ u^*| &\leq Cr^\alpha \\
|u| &= |\varphi| \leq \frac{C_0}{2} r^2 \\
|\nabla' u| &= |\nabla' \varphi| \leq C_0 r,
\end{aligned}$$

as consequences of the $C^{1,\alpha}$ estimate for u . Combining all these estimates, we obtain

$$|E(r)| < C \frac{D(r, u)^{\frac{1}{2}} r^{\frac{n}{2}} + r^{n+\alpha}}{D(r, u) - CD(r, u)^{\frac{1}{2}} r^{\frac{n}{2}+1} - Cr^{n+1+\alpha}}.$$

When $H(r, u) > r^{n+2+\frac{\alpha}{2}}$, the first inequality in Lemma 4.2 gives

$$D(r, u) > Cr^{n+1+\frac{\alpha}{2}} - Cr^{n+2}.$$

Therefore, for some $r_0 > 0$ and $C > 0$ one has

$$|E(r)| \leq C \frac{D(r, u)^{\frac{1}{2}} r^{\frac{n}{2}} + r^{n+\alpha}}{D(r, u)} \leq Cr^{-1+\frac{\alpha}{2}}$$

on the set

$$\{r \in (0, r_0) \mid H(r, u) > r^{n+2+\frac{\alpha}{2}}\}.$$

As noted before, this completes the proof of the lemma. \square

5. $C^{1,\beta}$ REGULARITY

Due to the monotonicity of $\Phi(r, u)$ for $r \in (0, r_0)$, under the normalization conditions (4.1), there must exist some β such that

$$\lim_{r \rightarrow 0^+} \Phi(r, u) = n + 1 + 2\beta.$$

Our objective in this section is to show that u grows as $|x|^{1+\beta}$ near the origin. In other words,

$$|u(x)| \leq C |x|^{1+\beta}.$$

In fact, we are going to see in the next section that $\beta \geq \frac{1}{2}$, and combined with this growth estimate it will ultimately imply the $C^{1, \frac{1}{2}}$ regularity.

We start with an observation that the “truncation” of the growth of $H(r, u)$ in the formula for $\Phi(r, u)$ provides a natural bound for β from above.

Lemma 5.1. *We have the inequality*

$$\beta \leq \frac{1}{2} + \frac{\alpha}{4}.$$

Moreover, if

$$\beta < \frac{1}{2} + \frac{\alpha}{4}$$

then there exists $r_u > 0$ such that $H(r, u) \geq r^{n+2+\frac{\alpha}{2}}$ for $0 < r < r_u$.

Proof. If there exists a sequence $r_j \rightarrow 0$ such that $H(r_j, u) < r_j^{n+2+\frac{\alpha}{2}}$, then

$$\Phi(r_j, u) = e^{Cr_j^{\frac{\alpha}{2}}} \left(n + 2 + \frac{\alpha}{2} \right)$$

and consequently, $\beta = \frac{1}{2} + \frac{\alpha}{4}$. Therefore, we only need to consider the possibility that

$$H(r, u) \geq r^{n+2+\frac{\alpha}{2}}$$

for r close to 0. This assumption implies

$$\Phi(r, u) = r e^{Cr^{\frac{\alpha}{2}}} \frac{H'(r, u)}{H(r, u)},$$

and thus,

$$\lim_{r \rightarrow 0^+} r \frac{H'(r, u)}{H(r, u)} = n + 1 + 2\beta.$$

Given any $\epsilon > 0$, there must exist some $r_\epsilon > 0$ such that

$$r \frac{H'(r, u)}{H(r, u)} > n + 1 + 2\beta - \epsilon$$

for all $r \in (0, r_\epsilon)$. Following a straightforward computation we can derive that

$$H(r, u) \leq \frac{H(r_\epsilon)}{r_\epsilon^{n+1+2\beta-\epsilon}} r^{n+1+2\beta-\epsilon}.$$

Combining that with the assumption that

$$H(r, u) \geq r^{n+2+\frac{\alpha}{2}}$$

when r is close to 0 we have

$$n + 2 + \frac{\alpha}{2} \geq n + 1 + 2\beta - \epsilon.$$

Since this inequality holds for any $\epsilon > 0$, passing to the limit one obtains

$$\beta \leq \frac{1}{2} + \frac{\alpha}{4}. \quad \square$$

Lemma 5.2. *There exists C depending only on C_0 and n such that for all $r \in (0, \frac{1}{2})$,*

$$\begin{aligned} H(r, u) &= \int_{\partial B_r} u^2 \leq Cr^{n+1+2\beta} \\ G(r, u) &= \int_{B_r} u^2 \leq Cr^{n+2+2\beta} \\ D(r, u) &= \int_{B_r} |\nabla u|^2 \leq Cr^{n+2\beta} \end{aligned}$$

Proof. We start with the estimate on $H(r, u)$. First, we can choose C large enough depending on $\|u\|_{L^\infty(B_1)}$ and r_0 such that

$$H(r, u) \leq Cr^{n+1+2\beta}$$

for all $r \geq r_0$. Here r_0 is the constant from Lemma 4.3. For all r in the set

$$\left\{ r \mid H(r, u) \leq r^{n+1+\frac{\alpha}{2}} \right\},$$

the estimate for $H(r, u)$ is trivially satisfied since $n+1+2\beta \leq n+2+\frac{\alpha}{2}$. Any other value of r must belong to some maximal open interval

$$(r_1, r_2) \subset \left\{ r \in (0, r_0) \mid H(r, u) > r^{n+2+\frac{\alpha}{2}} \right\}.$$

Note that because either $r_2 = r_0$ or $H(r_2, u) = r_2^{n+2+\frac{\alpha}{2}}$, one always has

$$H(r_2, u) \leq Cr_2^{n+1+2\beta}.$$

We have for any $r \in (r_1, r_2)$,

$$\Phi(r, u) = re^{C_m r^{\frac{\alpha}{2}}} \frac{H'(r, u)}{H(r, u)} \geq \lim_{r \rightarrow 0^+} \Phi(r, u) = n+1+2\beta$$

and so

$$\begin{aligned} \frac{H'(r, u)}{H(r, u)} &\geq (n+1+2\beta) \frac{1}{re^{C_m r^{\frac{\alpha}{2}}}} \\ &\geq (n+1+2\beta) \frac{e^{C_m r^{\frac{\alpha}{2}}} - c \frac{\alpha}{2} C_m r^{\frac{\alpha}{2}}}{re^{C_m r^{\frac{\alpha}{2}}}} \\ &\geq (n+1+2\beta) \left(\frac{1}{r} - c \frac{\alpha}{2} C_m r^{-1+\frac{\alpha}{2}} e^{-C_m r^{\frac{\alpha}{2}}} \right) \end{aligned}$$

where c is a constant such that

$$e^x \leq 1 + c \frac{\alpha}{2} x \quad \forall x \in (0, C_m).$$

Integrating both sides from r to r_2 one obtains

$$\begin{aligned} \log \frac{H(r_2, u)}{H(r, u)} &\geq (n+1+2\beta) \left(\log \frac{r_2}{r} + ce^{-C_m r_2^{\frac{\alpha}{2}}} - ce^{-C_m r^{\frac{\alpha}{2}}} \right) \\ &\geq (n+1+2\beta) \left(\log \frac{r_2}{r} - c \right) \end{aligned}$$

or

$$\begin{aligned} H(r, u) &\leq \frac{H(r_2, u)}{r_2^{n+1+\beta}} e^{c(n+1+2\beta)} r^{n+1+2\beta} \\ &\leq Ce^{c(n+1+2\beta)} r^{n+1+2\beta}. \end{aligned}$$

This proves the estimate for $H(r, u)$. Integrating, one obtains the estimate for $G(r, u)$.

To prove the estimate for $D(r, u)$, fix a smooth radial cut-off function η such that

$$\begin{aligned} 0 &\leq \eta \leq 1 \text{ in } \mathbb{R}^n \\ \eta &= 1 \text{ in } B_{\frac{1}{2}} \\ \eta &= 0 \text{ in } \mathbb{R}^n \setminus B_{\frac{3}{4}}. \end{aligned}$$

For each $r > 0$, denote by η_r the scaled function

$$\eta_r(x) = \eta\left(\frac{x}{r}\right).$$

Applying Green's formula to η_r and u^2 one has

$$\int_{B_r^+} \eta_r \Delta u^2 - \int_{B_r^+} u^2 \Delta \eta_r = \int_{B_r^+} \eta_r \partial_\nu^+ u^2,$$

or equivalently,

$$\int_{B_r^+} \eta_r u \Delta u + \int_{B_r^+} \eta_r |\nabla u|^2 - \frac{1}{2} \int_{B_r^+} u^2 \Delta \eta_r = \int_{B_r^+} \eta_r u (\partial_\nu^+ u).$$

Doing the same for B_r^- and summing up, one obtains

$$\int_{B_r^+ \cup B_r^-} \eta_r u \Delta u + \int_{B_r} \eta_r |\nabla u|^2 - \frac{1}{2} \int_{B_r} u^2 \Delta \eta_r = 2 \int_{B_r^+} \eta_r u (\partial_\nu^+ u^*).$$

Rearranging and applying Cauchy-Schwartz inequality to the first term, we have

$$\int_{B_r} \eta_r |\nabla u|^2 \leq \frac{1}{2} \int_{B_r} u^2 \Delta \eta_r + 2 \int_{B_r^+} \eta_r u (\partial_\nu^+ u^*) + \frac{1}{2r^2} \int_{B_r} \eta u^2 + \frac{r^2}{2} \int_{B_r^+ \cup B_r^-} \eta (\Delta u)^2.$$

Using assumptions on η and the fact that

$$\Delta \eta_r = \frac{1}{r^2} \Delta \eta,$$

we can derive

$$\begin{aligned} \int_{B_{\frac{r}{2}}} |\nabla u|^2 &\leq \frac{\|\Delta \eta\|_{L^\infty} + 1}{2r^2} \int_{B_r} u^2 + \frac{r^2}{2} \int_{B_r^+ \cup B_r^-} (\Delta u)^2 + 2 \int_{B_r^+} u (\partial_\nu^+ u^*) \\ &\leq Cr^{n+2\beta} + Cr^{n+2} + Cr^{n+1+\alpha} \\ &\leq Cr^{n+2\beta}. \end{aligned}$$

Here we have used the estimate

$$\int_{B_r^+} u (\partial_\nu^+ u^*) \leq Cr^{n+1+\alpha}$$

proved in Lemma 4.3. □

Lemma 5.3. *There exists C depending only on C_0 and n such that for any $r > 0$,*

$$|u(x)| \leq Cr^{1+\beta}$$

in B_r .

Proof. First, since u is superharmonic, u^- is subharmonic. Together with the integral estimate

$$G(r, u) \leq Cr^{n+2+2\beta}$$

one has

$$u^- \leq Cr^{1+\beta}.$$

In other words,

$$u \geq -Cr^{1+\beta}.$$

To obtain the estimate from above, we cannot use that u^+ is subharmonic, since it is not generally true. However, we claim that the function $v = \max\{C_0r^2, u\} + C_0|x|^2$ is subharmonic in B_r . Away from $\{B'_1 \cap \Sigma\}$, $\Delta u \geq -C_0$ and so $u + C_0|x|^2$ is subharmonic. Therefore, we only need to study the behavior of v near $B_r \cap \Sigma$. If $x \in B_r \cap \Sigma$, then

$$u(x) = \varphi(x) < C_0r^2 \quad (\varphi(0) = 0, \nabla' \varphi(0) = 0, \partial_{\tau\tau} \varphi < C_0)$$

and so, $v = C_0r^2 + C_0|x|^2$ in a neighborhood of x . Thus, v is subharmonic in B_r . On the other hand, we easily have from the definition of v that

$$\begin{aligned} \int_{B_r} v &\leq \int_{B_r} u^+ + Cr^{n+2} \\ &\leq C(r^n G(r, u))^{\frac{1}{2}} + Cr^{n+2} \\ &\leq Cr^{n+1+\beta}. \end{aligned}$$

Therefore,

$$v \leq Cr^{1+\beta}$$

and so,

$$u \leq Cr^{1+\beta}. \quad \square$$

6. BLOWUPS AND $C^{1, \frac{1}{2}}$ REGULARITY

To complete the proof of $C^{1, \frac{1}{2}}$ regularity, we will show in this section that

$$\beta \geq \frac{1}{2}.$$

We will always assume that

$$\beta < \frac{1}{2} + \frac{\alpha}{4}$$

since otherwise, there is nothing else to do.

Under the normalizing conditions (4.1), for each $r > 0$, define the rescaling factor

$$d_r = \left(\frac{1}{r^{n-1}} H(r, u) \right)^{\frac{1}{2}}$$

and rescaled functions

$$\begin{aligned} u_r(x) &= \frac{u(rx)}{d_r} \\ \varphi_r(x) &= \frac{\varphi(rx)}{d_r}. \end{aligned}$$

The inequality $\beta \geq \frac{1}{2}$ will follow from the study of the limits of these rescalings over sequences $r = r_j \rightarrow 0^+$, which are called blowups. In fact, we will show that such blowups solve a thin obstacle

problem and are homogeneous of degree $1 + \beta$. From the known $C^{1, \frac{1}{2}}$ regularity for the thin obstacle problem, we will immediately obtain $\beta \geq \frac{1}{2}$.

First, in order to show that such limits exist, we start with a uniform $C^{1, \lambda}$ estimate on u_r as an application of Theorem 3.6. To apply that theorem, we need to verify the existence of a constant C , independent of r such that the triple (u_r, φ_r, C) satisfies (3.1)–(3.9). The only non-trivial condition to check is (3.8).

We start with relatively simple estimates for second derivatives of u_r and φ_r .

Lemma 6.1. *Define*

$$\gamma = \frac{1}{2} - \frac{\alpha}{4}.$$

There exists some positive constant r_u (possibly depending on u) such that for $0 < r < r_u$, we have

- i) u_r is superharmonic in B_1 .
- ii) $\Delta u_r \geq -C_0 r^\gamma$ in B_1^\pm .
- iii) $\partial_{\tau\tau} u_r \geq -C_0 r^\gamma$ and $\partial_{e_n e_n} u_r \leq C_0 r^\gamma$ in B_1 .
- iv) $|D^2 \varphi_r| \leq C_0 r^\gamma$ in $B_1^\pm \cup B'_1$.

Proof. Since

$$\beta < \frac{1}{2} + \frac{\alpha}{4}$$

we must have

$$H(r, u) \geq r^{n+2+\frac{\alpha}{2}}$$

when r is small. Thus,

$$d_r \geq r^{\frac{3}{2}+\frac{\alpha}{4}}$$

or

$$\frac{r^2}{d_r} \leq r^\gamma$$

for r small. All inequalities are consequences of this fact and the corresponding inequalities for u and φ . \square

Lemma 6.2. *For all $0 < r < \frac{1}{2}$ we have*

- i) $u_r \geq \varphi_r$ on B'_1 .
- ii) $\partial_\nu^+ u_r(x) + \partial_\nu^- u_r(x) \geq 0$ on B'_1 .
- iii) $\partial_\nu^+ u_r(x) + \partial_\nu^- u_r(x) = 0$ on $B'_1 \cap \{u_r > \varphi_r\}$.
- iv) $|\nabla' \varphi_r| \leq C_0 r^\gamma$ on B'_1 .

Proof. All are trivial consequences of the corresponding properties of u and φ . In the last one, we have used $\nabla' \varphi_r(0) = 0$ and $|D^2 \varphi_r| \leq C_0 r^\gamma$ in $B_1^+ \cup B'_1$. \square

In order to estimate $|\nabla u_r|$, we first show a uniform $W^{1,2}$ estimate for u_r .

Lemma 6.3. *There exists some $r_u > 0$ such that for all $0 < r < r_u$,*

- i) $G(r, u) \leq rH(r, u)$ for r small.

- ii) $\lim_{r \rightarrow 0^+} N(r, u) = 1 + \beta$.
- iii) $G(1, u_r) \leq 1$ for r small.
- iv) $\lim_{r \rightarrow 0^+} D(1, u_r) = 1 + \beta$.

Proof.

i) Since

$$\beta < \frac{1}{2} + \frac{\alpha}{4}$$

we have

$$\Phi(r, u) = r \frac{H'(r, u)}{H(r, u)}$$

for r small (say $r < r_u$ for some $r_u > 0$). The definition of β therefore becomes

$$\lim_{r \rightarrow 0^+} r \frac{H'(r, u)}{H(r, u)} = n + 1 + 2\beta.$$

In particular, $H'(r, u) > 0$ or $H(r, u)$ is an increasing function. Consequently,

$$G(r, u) = \int_0^r H(s, u) ds \leq rH(r, u).$$

ii) Plugging the formula (Lemma 4.1),

$$\begin{aligned} H'(r, u) &= \frac{n-1}{r} H(r, u) + 2I(r, u) \\ &= \frac{n-1}{r} H(r, u) + 2 \left(\int_{B_r^+} |\nabla u|^2 + \int_{B_r^+} u \Delta u - \int_{B_r'} u \partial_\nu^+ u \right) \end{aligned}$$

into the formula for $\Phi(r, u)$ above one has

$$\lim_{r \rightarrow 0^+} r \frac{D(r, u) + \int_{B_r^+ \cup B_r^-} u \Delta u - 2 \int_{B_r'} u (\partial_\nu^+ u^*)}{H(r, u)} = 1 + \beta.$$

We will show that

$$\lim_{r \rightarrow 0^+} r \frac{\int_{B_r^+ \cup B_r^-} u \Delta u - 2 \int_{B_r'} u (\partial_\nu^+ u^*)}{H(r, u)} = 0.$$

First, in order to estimate the numerator, we use the inequalities in the proof of Lemma 4.3 and the results of Lemma 5.2 to obtain

$$\begin{aligned} \left| \int_{B_r^+ \cup B_r^-} u \Delta u \right| &\leq CG(r, u)^{\frac{1}{2}} r^{\frac{n}{2}} \\ &\leq Cr^{\frac{n+2+2\beta}{2}} r^{\frac{n}{2}} \\ &= Cr^{n+1+\beta} \end{aligned}$$

and

$$\left| \int_{B_r'} u (\partial_\nu^+ u^*) \right| \leq Cr^{n+1+\alpha}.$$

On the other hand, from

$$\lim_{r \rightarrow 0^+} r \frac{H'(r, u)}{H(r, u)} = n + 1 + 2\beta,$$

we can deduce by using an argument similar to the one in the proof of Lemma 4.3 that for any $\epsilon > 0$, there exist $C > 0$ and $r_\epsilon > 0$ such that

$$H(r, u) \geq Cr^{n+1+2\beta+\epsilon}, \quad 0 < r < r_\epsilon.$$

Combining all these estimates, one then has

$$\lim_{r \rightarrow 0^+} r \frac{\int_{B_r^+ \cup B_r^-} u \Delta u - 2 \int_{B_r'} u (\partial_\nu^+ u^*)}{H(r, u)} = 0.$$

Therefore,

$$\lim_{r \rightarrow 0^+} N(r, u) = \lim_{r \rightarrow 0^+} r \frac{D(r, u)}{H(r, u)} = 1 + \beta.$$

iii) A restatement of i), since $G(1, u_r) = G(r, u)/(rH(r, u))$.

iv) A restatement of ii), since $D(1, u_r) = N(r, u)$. \square

Lemma 6.4. *There exists r_u such that $\|\nabla u_r\|_{C^{0,1}(B_1)}$ are uniformly bounded for $0 < r < r_u$.*

Proof. First, by using the same argument as in the proof of Lemma 5.3, the bounds for second derivatives of u_r in Lemma 6.1 and $W^{1,2}$ estimates in Lemma 6.3, one obtains

$$|u_r| \leq C_1$$

in $B_{\frac{3}{4}}$ where C can be made independent of r .

In the next step, we bound $\partial_{e_n} u_r$ from below in $B_{\frac{1}{2}}^+$ by using the estimate for $|u_r|$ and $\partial_{e_n e_n} u_r$. Since $\partial_{e_n e_n} u_r \leq C_0$, we have

$$u_r(x', x_n + \frac{1}{4}) - u_r(x', x_n) \leq \frac{1}{4}(\partial_{e_n} u_r(x', x_n)) + \frac{1}{32}C_0.$$

Therefore,

$$-2C_1 \leq \frac{1}{4}(\partial_{e_n} u_r(x', x_n)) + \frac{1}{32}C_0$$

which implies $\partial_{e_n} u_r \geq -C_2$ in $B_{\frac{1}{2}}^+$. On the other hand, the same argument for the negative side $B_{\frac{1}{2}}^-$ yields an upper bound

$$\partial_{e_n} u_r(x) < C_2$$

there. For the other direction of the inequality, recall that

$$\partial_\nu^+ u_r + \partial_\nu^- u_r \geq 0 \quad \text{or} \quad \partial_{e_n}^- u_r \geq \partial_{e_n}^+ u_r$$

on B_1' . From the upper bound for $\partial_{e_n} u_r$ in $B_{\frac{1}{2}}^-$, we immediately have

$$\partial_{e_n}^+ u_r(x', 0) \leq C_2.$$

Together with $\partial_{e_n e_n} u_r \leq C_0$, this leads to

$$\partial_{e_n} u_r(x) \leq C_2 + \frac{1}{2}C_0.$$

Combining together the lower and upper bounds, we can write

$$|\partial_{e_n} u_r| \leq C_2$$

in $B_{\frac{1}{2}}$, where C_2 is independent of r .

As the last step, we estimate $|\nabla' u_r|$. Since $\nabla' \varphi_r(0) = 0$ and $|D^2 \varphi_r| \leq C_0$, we have

$$|\nabla' \varphi_r| \leq C_0$$

in B_1 . Fix any $\tau \in \partial B'_1$. Consider the function $v = \max\{C_0, \partial_\tau u_r\}$. We want to show that it is subharmonic. If $x \in \{\partial_\tau u_r > C_0\}$, then

$$\partial_\tau u_r > \partial_\tau \varphi_r.$$

It implies that

$$u_r(x) \neq \varphi_r(x).$$

Therefore

$$\Delta u_r(x) = 0$$

or consequently,

$$\Delta(\partial_\tau u_r) = 0$$

in a neighborhood of x . It is enough to say that v is a subharmonic function in B_1 . On the other hand, since

$$\lim_{r \rightarrow 0^+} D(1, u_r) = \lim_{r \rightarrow 0} \int_{B_1} |\nabla u_r|^2 = 1 + \beta,$$

$D(1, u_r)$ is uniformly bounded when r is small. Together, they imply that

$$\partial_\tau u_r \leq C_3$$

in $B_{\frac{1}{2}}$ for some C_3 independent of r and τ . Since τ is an arbitrary unit vector in $\partial B'_1$, we obtain

$$|\nabla' u_r| \leq C_3. \quad \square$$

We have demonstrated a uniform $C^{1,\lambda}$ estimate on u_r . It guarantees the existence of at least one blowup. In the next step, we study the properties of this blowup. We want to show that it solves a thin obstacle problem and is also a homogeneous function.

Lemma 6.5. *Assume that $\beta < \frac{1}{2} + \frac{\alpha}{4}$. Let $\{r_j\}$ be any sequence converging to 0. If there exists a function u_0 on B_1 such that*

$$u_{r_j} \rightarrow u_0 \text{ in } C_{\text{loc}}^{1,\lambda}(B_1^\pm \cup B'_1),$$

then u_0 possesses the following properties:

- i) u_0 is not identically 0 in B_1 .
- ii) $\Delta u_0 = 0$ in B_1^\pm .
- iii) $u_0 \geq 0$, $\partial_\nu^+ u_0 + \partial_\nu^- u_0 \geq 0$, $u_0(\partial_\nu^+ u_0 + \partial_\nu^- u_0) = 0$ on B'_1 .
- iv) u_0 is homogeneous of degree $1 + \beta$.

Proof.

i) We can derive directly from the definition of u_r that

$$\int_{\partial B_1} u_{r_j}^2 = 1$$

for any j . Since we have a uniform $W^{1,2}$ bound on u_r in B_1 , see Lemma 6.3, we may assume that $u_{r_j} \rightarrow u_0$ strongly in $L^2(\partial B_1)$ and passing to the limit we obtain

$$\int_{\partial B_1} u_0^2 = 1.$$

Therefore u_0 cannot be identically 0 in B_1 .

ii) From Lemma 6.5 we have in B_1^\pm

$$0 \geq \Delta u_r \geq -C_0 r^\gamma$$

for some $\gamma > 0$. Passing r to 0, we immediately obtain $\Delta u_0 = 0$ in B_1^\pm .

iii) The facts that $u_0 \geq 0$ and $\partial_\nu^+ u + \partial_\nu^- u \geq 0$ on B_1' are elementary consequences of inequalities

$$\begin{aligned} u(x', 0) &\geq \varphi(x', 0) \geq -C_0 |x'|^2 \\ \partial_\nu^+ u(x', 0) + \partial_\nu^- u(x', 0) &\geq 0. \end{aligned}$$

It is then enough for us to show that

$$\partial_\nu^+ u_0(x) + \partial_\nu^- u_0(x) = 0$$

when $x \in B_1' \cap \{u_0 > 0\}$. For any such x' , there must exist some $c > 0$ and K such that for all $j \geq K$,

$$u_{r_j}(x) > c$$

which is equivalent to

$$\begin{aligned} u(r_j x) &> cr_j^{-\frac{n-1}{2}} H(r_j, u)^{\frac{1}{2}} \\ &> cr_j^{\frac{3}{2} + \frac{\alpha}{4}}. \end{aligned}$$

By choosing a bigger K if necessary, one has for all $j \geq K$,

$$u(r_j x) > cr_j^{\frac{3}{2} + \frac{\alpha}{4}} > \frac{C_0}{2} r_j^2 \geq \frac{C_0}{2} r_j^2 |x'|^2 \geq \varphi(r_j x).$$

Therefore,

$$\partial_\nu^+ u(r_j x) + \partial_\nu^- u(r_j x) = 0.$$

Passing j to ∞ , one has

$$\partial_\nu^+ u_0(x) + \partial_\nu^- u_0(x) = 0.$$

iv) From the definition of $N(r, v)$ and u_0 we observe that

$$N(r, u_0) = \lim_{r_j \rightarrow 0} N(r, u_{r_j}) = \lim_{r_j \rightarrow 0} N(rr_j, u).$$

Thus, we have

$$N(r, u_0) = 1 + \beta$$

for any $r > 0$. The conclusion then follows immediately from the lemma below. \square

Lemma 6.6. *Assume that v is a function on B_1 with the following properties*

- i) v is not identically 0 in B_1 .
- ii) $\Delta v = 0$ in $B_1^+ \cup B_1^-$.
- iii) $v \geq 0$, $\partial_\nu^+ v + \partial_\nu^- v \geq 0$, $v(\partial_\nu^+ v + \partial_\nu^- v) = 0$ on B_1' .

Then $N(r, v)$ is nondecreasing as a function of r . Moreover, if $N(r, v) = k$ for all $r \in (0, 1)$, then v is homogeneous of degree k .

Proof. The proof in [10] works for symmetric function v but can be easily modified for this non-symmetric case. We just have to substitute $\partial_\nu^+ v$ by $\partial_\nu^+ v + \partial_\nu^- v$ on B_1' . \square

Lemma 6.7. *We must have*

$$\beta \geq \frac{1}{2}.$$

Proof. Combining Lemmas 6.1, 6.2, and 6.4 with Theorem 3.6 we can deduce that u_r are uniformly bounded in $C^{1,\lambda}(B_1^\pm \cup B'_1)$ for some $\lambda > 0$ when r is small. Because of this, we can always find a sequence $\{r_j\}$ converging to 0 such that u_{r_j} converges to some function u_0 in $C^{1,\lambda}(B_1^\pm \cup B'_1)$. Properties of u_0 as stated in Lemma 6.5 means that u_0 is a solution to a thin obstacle problem as studied in [2]. We know from that work that u_0 is $C^{1,\frac{1}{2}}$ in $B_{\frac{1}{2}}$. However, our u_0 is also non-zero and homogeneous of degree $1 + \beta$. Therefore, we must have

$$\beta \geq \frac{1}{2}. \quad \square$$

Theorem 6.8. *Let u be a solution of the obstacle problem in B_1 with rooftop-like obstacle $\varphi \in C^{1,1}(B_1^\pm \cup B'_1)$ and boundary values $g \in L^\infty(\partial B_1)$. Then there exists $C > 0$ depending only on n , $\|\varphi\|_{C^{1,1}(B_1^\pm \cup B'_1)}$ and $\|g\|_{L^\infty(\partial B_1)}$ such that*

$$\|u\|_{C^{1,\frac{1}{2}}(B_{\frac{1}{2}}^\pm \cup B'_{\frac{1}{2}})} \leq C.$$

Proof. From Lemmas 5.3 and 6.7 we have

$$|u(x)| \leq C|x|^{\frac{3}{2}}$$

for all $x \in B_{\frac{1}{2}}$ where C is a constant depending on C_0 and n . Combining this growth estimate with arguments used in Theorem 3.6, we obtain the $C^{1,\frac{1}{2}}$ estimate for u in $B_{\frac{1}{2}}^\pm \cup B'_{\frac{1}{2}}$. Finally, notice that the constant C_0 depends only on n , $\|\varphi\|_{C^{1,1}(B_1^\pm \cup B'_1)}$ and $\|g\|_{L^\infty(\partial B_1)}$. \square

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