TWO-PHASE SEMILINEAR FREE BOUNDARY PROBLEM WITH A DEGENERATE PHASE

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Abstract. We study minimizers of the energy functional

$$\int_D [|\nabla u|^2 + \lambda (u^+)^p] \, \mathrm{d}x$$

for $p \in (0,1)$ without any sign restriction on the function u. The main result states that in dimension two the free boundaries $\Gamma^+ = \partial\{u > 0\} \cap D$ and $\Gamma^- = \partial\{u < 0\} \cap D$ are C^1-regular, provided $1 - \epsilon_0 < p < 1$.

1. Introduction and Main results

1.1. **The problem.** In this paper we study a two-phase free boundary problem, obtained by minimizing the functional

$$J(u) := \int_D \left(|\nabla u|^2 + 2F(u) \right) \mathrm{d}x \tag{1.1}$$

in an open subset $D \subset \mathbb{R}^n$, where F is a Hölder continuous function

$$F(u) := \lambda \, (u^+)^p, \quad \lambda > 0, \quad 0 < p < 1. \tag{1.2}$$

(Hereafter we denote $u^\pm = \max\{\pm u, 0\}$.) By a *minimizer*, we understand $u \in W^{1,2}(D)$ such that

$$J(u) \le J(v), \quad \text{for any } v \in u + W^{1,2}_0(D).$$

The existence of minimizers with a given Sobolev trace boundary data $u_0 \in W^{1,2}(D)$ follows easily by the direct methods of the calculus of variations, however there is generally no uniqueness as J is not convex. The minimizers satisfy

$$\begin{aligned} \Delta u &= p\lambda \, u^{p-1} && \text{in } \Omega^+(u) := \{u > 0\} \\ \Delta u &= 0 && \text{in } \Omega^-(u) := \{u < 0\} \end{aligned}$$

and our objective is to study the interfaces or *free boundaries*

$$\Gamma^\pm(u) := \partial\Omega^\pm(u) \cap D.$$

One of the main difficulties in this problem is related to the lack of nondegeneracy in the negative phase in the sense that there is no apriori bound on how slowly u^- can grow near Γ^-. On the other hand, it is relatively easy to show that $u^+(x)$ grows as $\operatorname{dist}(x, \Gamma^+)^\beta$ away from Γ^+, $\beta = 2/(2-p)$.

2000 *Mathematics Subject Classification.* Primary 35R35.

Key words and phrases. Regularity of the free boundary, semilinear equation, monotonicity formula.

N. Matevosyan is partially supported by the WWTF (Wiener Wissenschafts, Forschungs und Technologiefonds) project No. CI06 003 and by award No. KUK-I1-007-43, made by King Abdullah University of Science and Technology (KAUST).

A. Petrosyan is supported in part by NSF grant DMS-0701015.

Recently, a similar problem has been considered by Lindgren and Petrosyan [LP08] with

$$F(u) = \lambda_+(u^+)^p + \lambda_-(u^-)^p, \quad \lambda_\pm > 0, \quad 0 < p < 1.$$

The main difference is that this problem has nondegeneracy in both phases, which plays a major role in the study of the free boundary. In particular, for the above problem it has been shown that $\Gamma^\pm$ are C^1 regular in dimension $n = 2$. The regularity in higher dimension is completely open.

The corresponding one-phase problem (i.e., nonnegative minimizers of u) has been studied in a series of papers by Phillips [Phi83a, Phi83b] and Alt and Phillips [AP86]. In the latter paper it has been proved that that there exists a singular set $\Sigma \subset \Gamma^+$ of $(n-1)$-Hausdorff measure zero such that $\Gamma^+ \setminus \Sigma$ is C^∞ (actually real analytic). Moreover, they have shown that when the dimension $n = 2$ then the singular set $\Sigma = \emptyset$, i.e. the free boundary is fully regular.

1.2. **Main result.** The main result in this paper states that despite the lack of nondegeneracy in the negative phase, the free boundary is still fully regular for p near 1 in dimension two.

Theorem I. *Let u be a minimizer of* (1.1) *in dimension $n = 2$. Then there exists an absolute constant $\epsilon_0 > 0$ such that $\Gamma^+(u)$ and $\Gamma^-(u)$ are locally $C^{1,\alpha}$ curves, provided $1 - \epsilon_0 < p < 1$. Moreover, $\Gamma^-(u) \subset \Gamma^+(u)$.*

The last part of the theorem basically says that the fattening of $\{u = 0\}$ cannot occur between the phases $\Omega^\pm$, but only as the "dead core" in Ω^+.

2. Preliminaries

2.1. **Optimal regularity and the Euler-Lagrange equation.** Perhaps the first question associated with the variational problem (1.1) is the optimal regularity of the minimizers. While studying the one-phase free boundary problem, Phillips [Phi83a] has established that the nonnegative minimizers of J satisfy

$$u \in C^{1,\beta-1}_{\mathrm{loc}}(D), \qquad \beta = \frac{2}{2-p}.$$

This is the best regularity possible, as one can see from the one-dimensional example $u(x_1) = C_0(x_1^+)^\beta$ for suitably chosen $C_0 = C_0(\lambda, p) > 0$. Later, Giaquinta and Giusti [GG84] have extended this optimal regularity result for general minimizers of J with no restrictions on their sign.

The Euler-Lagrange equation associated with the variational problem (1.1) is

$$\Delta u = p\lambda(u^+)^{p-1}\chi_{\{u>0\}} \quad \text{in } D. \tag{2.1}$$

However, since even the local integrability of $(u^+)^{p-1}\chi_{\{u>0\}}$ in D is apriori unknown, we have to specify in which sense we understand (2.1). But first notice that since $\Omega^\pm(u) = \{\pm u > 0\}$ are open, the equation (2.1) is clearly satisfied in the classical sense in $\Omega^\pm(u)$, and moreover, u is real analytic there. In general, we understand (2.1) in the sense of *domain variation*:

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} J(u(x+\epsilon\phi(x))) = \int_D [2\nabla u \cdot D\phi \nabla u - (\operatorname{div}\phi)(|\nabla u|^2 + 2F(u))]\,\mathrm{d}x = 0,$$

for any $\phi \in C_0^\infty(D, \mathbb{R}^n)$.

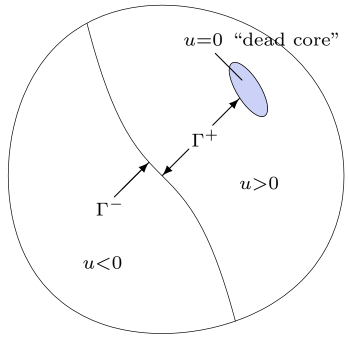

FIGURE 1. $\Gamma^+ = \partial\{u > 0\} \cap D$, $\Gamma^- = \partial\{u < 0\} \cap D$

2.2. **Structure of the free boundary.** Here we briefly discuss the structure of the free boundary (see also Fig. 1).

1° We start with an observation that $\Gamma^- \subset \Gamma^+$. Indeed, if $x_0 \in \Gamma^- \setminus \Gamma^+$, then $u \leq 0$ in $B_\delta(x_0)$ for some small $\delta > 0$. This implies that the term $F(u)$ in the integrand of (1.1) vanishes identically in $B_\delta(x_0)$, and consequently that the minimizer u is harmonic there. But then by the maximum principle, $u \equiv 0$ in $B_\delta(x_0)$, which contradicts to the fact that $x_0 \in \partial\{u < 0\}$. Hence, $\Gamma^- \subset \Gamma^+$. This essentially means that no thickening of the level set $\{u = 0\}$ can occur between Γ^- and Γ^+, nor that there could be “dead cores” of $\{u = 0\}$ in $\{u \leq 0\}$.

2° However, it is possible that $\Gamma^+ \setminus \Gamma^-$ is nonempty, e.g. when u is nonnegative. In fact, for any $x_0 \in \Gamma^+ \setminus \Gamma^-$ there exists a ball $B_\delta(x_0) \subset D$ such that $u \geq 0$ in $B_\delta(x_0)$. We call such x_0 a *one-phase* free boundary point. The analysis of the regularity of the free boundary near one-phase points is reduced to the case already studied by Alt and Phillips [AP86]. In particular, in dimension $n = 2$, δ can be chosen so small that $B_\delta(x_0) \cap \Gamma^+$ will be a real-analytic surface.

3° We say that x_0 is a *two-phase* free boundary point, if

$$x_0 \in \Gamma^+ \cap \Gamma^- (= \Gamma^-).$$

We distinguish two types of two-phase points. The first kind is so-called *branching points*, where the condition

$$|\nabla u(x_0)| = 0$$

is satisfied. This terminology is reminiscent of the fact that in similar two-phase free-boundary problems this condition holds automatically at $x_0 \in \Gamma^+ \cap \Gamma^- \cap \overline{\{u = 0\}^\circ}$, i.e., at points x_0 where the free boundary branches out to $\Gamma^\pm$. By 1) above, the branching in this narrow sense can never occur in our case; however, apriori we may not exclude the existence of two-phase points with vanishing gradient. *In fact, the proof of Theorem I consists in showing that such points don't exist when* $n = 2$ *and* $1 - \epsilon_0 < p < 1$.

The second kind of two-phase points are the *non-branching points* is where

$$|\nabla u(x_0)| > 0.$$

Since $u \in C^{1,\beta-1}_{\mathrm{loc}}(D)$, the implicit function theorem implies that for such points there exists a small δ such that $B_\delta(x_0) \cap \Gamma^+ = B_\delta(x_0) \cap \Gamma^-$ is a graph of a $C^{1,\alpha}$ function.

3. Rescalings and Blowups

3.1. **Rescalings.** One of the key ideas in studying the infinitesimal properties of the free boundary is to make an infinite "zoom-in" (or "blowup") at a free boundary point.

More specifically, given a minimizer u of (1.1), $x_0 \in \Gamma^+ \cup \Gamma^- (= \Gamma^+)$ and $r > 0$ define the *rescaling*

$$u_{x_0,r}(x) := \frac{u(x_0 + rx)}{r^\beta}, \quad \beta = \frac{2}{2-p}$$

for $x \in D_{x_0,r} = \frac{1}{r}(D - x_0)$. We will use the notation u_r for $u_{x_0,r}$ if $x_0 = 0$. If $x_0 \in (\Gamma^+ \cup \Gamma^-) \cap K$ for $K \Subset D$ and is such that $|\nabla u(x_0)| = 0$, we have the uniform estimates

$$|u_{x_0,r}(x)| \leq C_K |x|^\beta, \quad \text{for } |x| \leq \frac{\delta}{r},$$

where $\delta = \frac{1}{2}\operatorname{dist}(K, \partial\Omega)$. This follows from the optimal $C^{1,\beta-1}_{\mathrm{loc}}$-regularity of u. Hence, for a fixed x_0, we may extract a sequence $r_j \to 0$ such that

$$u_{x_0,r_j} \to u_0 \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^n),$$

where $u_0 \in C^{1,\beta-1}(\mathbb{R}^n)$. We will call u_0 a *blowup* of u at x_0. It is a simple exercise to show that u_0 is a global minimizer of functional J, i.e. it minimizes J on every subdomain $U \subset \mathbb{R}^n$ among the functions in $W^{1,2}(U)$ with the same trace on ∂U as u. Note that the blowup is not defined at free boundary points x_0 where $|\nabla u(x_0)| > 0$, i.e. at non-branching points. Moreover, at points where blowups exist, it is not clear apriori if the blowup is unique. Namely, taking a different subsequence $r'_j \to 0$ may result in convergence of u_{x_0,r'_j} to a different blowup u'_0. This may happen, e.g. when the free boundary "spirals" near x_0, see Fig. 2.

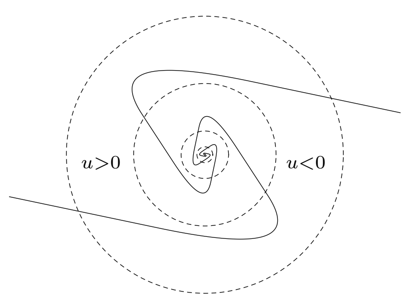

Figure 2. Possible nonuniqueness of blowups: spiraling free boundary

3.2. **Nondegeneracy.** Another possibility that needs to be ruled out is that u_0 vanishes identically in $\mathbb{R}^n$. This is accomplished with the help of the following nondegeneracy lemma.

Lemma 3.1 (Nondegeneracy). *Let u be a minimizer of* (1.1) *and let $x_0 \in \Omega^+ \cup \Gamma^+$. Then for any $r > 0$ with $B_r(x_0) \Subset D$ we have*

$$\sup_{\Omega^+ \cap \partial B_r(x_0)} u \geq c_0 \, r^\beta,$$

with $c_0 = c_0(\lambda, n, p) > 0$.

In fact, we will need the following more refined version of the nondegeneracy lemma.

Lemma 3.2 (Nondegeneracy in connected components). *Let u be a minimizer of* (1.1) *and let $x_0 \in D$. For any $B_r(x_0) \Subset D$, let V be a connected component of $\Omega_r^+ = \Omega^+ \cap B_r(x_0)$ such that $V \cap B_{r/2}(x_0) \neq \emptyset$. Then*

$$\sup_{\partial V \cap \partial B_r(x_0)} u \geq c_0 \, r^\beta$$

with $c_0 = c_0(\lambda, n, p) > 0$.

Proof. The proof is a slight modification of the one in [LP08], which in turn is similar to the one in [Phi83a], which follows the original idea in [CR76].

Fix a connected component V of $\Omega^+ \cap B_r(x_0)$. Then we can write

$$\partial V = E \cup F,$$

where

$$\begin{aligned} E &:= \partial V \cap B_r(x_0) \subset \Gamma^+ \\ F &:= \partial V \cap \partial B_r(x_0) \subset \partial B_r(x_0). \end{aligned}$$

Next, pick $y_0 \in V \cap B_{r/2}(x_0)$ and consider

$$w(x) = |u(x)|^{2/\beta} - c|x - y_0|^2, \quad x \in D$$

for some constant $c > 0$ to be specified later. Then by a direct computation, we have

$$\Delta w = (2/\beta)p\lambda + (2/\beta)(2/\beta - 1)|\nabla u|^2 |u|^{-p} - 2nc \quad \text{in } V.$$

Hence, by choosing $c = p\lambda/\beta n$, we make $\Delta w \geq 0$ in V. Since $w(y_0) > 0$ and w is subharmonic, there must exist $z_0 \in \partial V$ such that $w(z_0) > 0$. On the other hand, since $w \leq 0$ on $E \subset \Gamma^+$, necessarily $z_0 \in F$, which gives that

$$\sup_F w > 0.$$

By construction we have that $|x - y_0| \geq r/2$ for any $x \in F \subset B_r(x_0)$ and therefore we obtain that

$$\sup_F u^{2/\beta} > cr^2/4,$$

which completes the proof of the lemma with $c_0 = (c/4)^{\beta/2}$. □

3.3. **Homogeneity of blowups.** The next proposition characterizes the blowups of solutions.

Proposition 3.3 (Homogeneity of blowps). *Let u be a minimizer of* (1.1) *and $x_0 \in (\Gamma^+ \cup \Gamma^-) \cap \{|\nabla u| = 0\}$. Then any blowup u_0 of u at x_0 is a homogeneous function of degree β with respect to the origin, i.e.,*

$$u_0(rx) = r^\beta u_0(x), \quad x \in \mathbb{R}^n, \; r > 0.$$

The proof of this proposition is based on the monotonicity formula due to Weiss [Wei98]:

Lemma 3.4 (Weiss's monotonicity formula)**.** *Let u be a minimizer of* (1.1) *and*

$$W(r,x_0)=\frac{1}{r^{n+2\beta-2}}\int_{B_r(x_0)}[|\nabla u|^2+2F(u)]\,\mathrm{d}x-\frac{\beta}{r^{n+2\beta-1}}\int_{\partial B_r(x_0)}u^2(x)\,\mathrm{d}\sigma,$$

for $r>0$ such that $B_r(x_0)\Subset D$. Then W is monotonically increasing with respect to r. Moreover, $W(r,x_0)=0$ for $0<r<r_0$ iff u is a homogeneous function of degree β with respect to x_0 in $B_{r_0}(x_0)$.

Sometimes we will use the abbreviated notation $W(r)$ for $W(r,x_0)$ if the point x_0 is clear from the context, and more expanded notation $W(r,x_0,u)$, if we want to specify the function u.

Proof. For the complete proof we refer to the original paper of Weiss [Wei98]. Here we just indicate that using the identity

$$W(r,x_0,u)=W(1,0,u_{x_0,r}),$$

where $u_{x_0,r}(x)=u(x_0+rx)/r^\beta$, one can show that

$$W'(r)=\frac{2}{r^{n+2\beta-1}}\int_{\partial B_r(x_0)}((x-x_0)\cdot\nabla u-\beta u)^2\,\mathrm{d}\sigma.$$

The last part of the lemma follows from the fact that $(x-x_0)\cdot\nabla u-\beta u=0$ in $B_{r_0}(x_0)$ is equivalent to the homogeneity of u. □

By using this monotonicity formula, one can give a quick proof of Proposition 3.3.

Proof of Proposition 3.3. Let $u_{x_0,r_j}\to u_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then for any $\rho>0$, we have

$$W(\rho,0,u_0)=\lim_{j\to\infty}W(\rho,0,u_{x_0,r_j})=\lim_{j\to\infty}W(r_j\rho,x_0,u)=W(0+,x_0,u).$$

Hence $W(\rho,0,u_0)$ is constant in ρ, which implies that u_0 is homogeneous of degree β. □

3.4. **Classification of homogeneous global minimizers.** Since the blowups of minimizers u are homogeneous of degree β, it would be desirable to obtain the classification of such global minimizers. This poses a challenging open problem even in the one-phase case in higher dimensions. In dimension $n=2$, the problem is much simpler and, loosely speaking, reduces to identifying the solutions of an ODE with period 2π.

Proposition 3.5 (Classification of blowups)**.** *Let u_0 be a homogeneous global minimizer of J in dimension $n=2$. Then after a suitable rotation of coordinate axes*

$$u_0(x)=C_0(x_1^+)^\beta,$$

where $C_0=C_0(p)>0$.

The proof is based on the analysis of positivity and negativity sets of u_0, which are unions of cones. This is done in the next two lemmas. For convenience, we use the polar coordinates in the statement of these lemmas.

Lemma 3.6 (Positive solutions in angles)**.** *Let $u(r,\theta) = r^\beta f(\theta)$ be a* positive *solution of $\Delta u = p\lambda u^{p-1}$ in the cone $\mathcal{C}_\gamma = \{(r,\theta) : r > 0,\ \theta \in (0,\gamma)\}$, vanishing continuously on $\partial\mathcal{C}_\gamma$: $u(r,0) = u(r,\gamma) = 0$. Suppose also that $u \in C^1\left(\overline{\mathcal{C}_\gamma}\right)$. Then*

$$\frac{\pi}{\beta} \le \gamma \le \pi.$$

Furthermore, if $f'(0) = 0$ or $f'(\gamma) = 0$ then $\gamma = \pi$. Conversely, if $\gamma = \pi$ then necessarily $f'(0) = f'(\pi) = 0$. Moreover, in this case $f(\theta) = C_0(\sin\theta)^\beta$ for some $C_0 = C_0(\lambda, p)$.

Proof. See the proof of [LP08, Lemma 4.2]. □

Lemma 3.7 (Negative solutions in angles)**.** *Let $u(r,\theta) = r^\beta f(\theta)$ be a negative harmonic function in the cone $\mathcal{C}_\gamma$, continuously vanishing on $\partial\mathcal{C}_\gamma$. Then*

$$\gamma = \frac{\pi}{\beta}$$

and $f(\theta) = -C\sin(\beta\theta)$ for some $C > 0$. In particular, $|f'(0)| = |f'(\gamma)| > 0$.

Proof. Proof is a simple exercise. □

Proof of Proposition 3.5. Consider two cases:

1° 0 is a positive one-phase point, i.e., $0 \in \Gamma^+(u_0) \setminus \Gamma^-(u_0)$. In this case $u_0 \ge 0$. Consider then the positivity set $\Omega^+(u_0)$. From the homogeneity, the connected components of Ω^+ are cones. Lemma 3.6 implies that the cones have openings between π/β and π. In fact, since $|\nabla u_0| = 0$ on Γ^+ for nonnegative solutions, the openings of the components of Ω^+ are exactly π. Hence, there are either two, or just one components of Ω^+ of opening π, which after a rotation, correspond to

$$u_0(x) = C_0|x_1|^\beta$$

and

$$u_0(x) = C_0(x_1^+)^\beta,$$

respectively. The former case is actually impossible, since for nonnegative minimizers the zero set $\{u_0 = 0\}$ must have nonzero Lebesgue density at free boundary points, see [Phi83b].

2° 0 is a two-phase point, i.e., $0 \in \Gamma^+(u_0) \cap \Gamma^-(u_0)$. In this case both Ω^+ and Ω^- are nonempty. By Lemmas 3.6–3.7 each component of $\Omega^\pm$ is a cone of opening between π/β and π. Since $\beta < 2$ there could be no more than 3 different components in $\Omega^\pm$.

If there are three components, then we have two possibilities: either there are two components of the same sign sharing a common side, or the set $\{u = 0\}$ has a nonempty interior. In both cases, $|\nabla u| = 0$ on one side of at least two of the components. By Lemmas 3.6–3.7, these components must be positive and have opening π. This doesn't leave space for the third component.

Hence, there are precisely two components, one necessarily in Ω^+, the other in Ω^-, since we assume that 0 is a two-phase free boundary point. Since the negative component must have opening $\pi/\beta < \pi$ and the positive one at most π, the zero set $\{u = 0\}$ will have nonempty interior and therefore $|\nabla u| = 0$ on one side of both components. But then again by Lemmas 3.6–3.7, both components must be positive, which is a contradiction. Thus, 0 cannot be a two-phase free boundary point. □

4. Proof of the main theorem

The main ingredient in the proof of Theorem I is the following fact:

Proposition 4.1. *Let u be a minimizer of* (1.1). *Then there exist no free boundary branching points, i.e.,* $\Gamma^+ \cap \Gamma^- \cap \{|\nabla u| = 0\} = \emptyset$, *provided p is close enough to* 1.

In fact, we are going to reformulate this proposition as follows.

Proposition 4.2. *Let u be a minimizer of* (1.1) *in B_1 and suppose that*

$$0 \in \Gamma^+, \quad |\nabla u(0)| = 0.$$

If p is close enough to 1, *then there exists a small $r = r_u > 0$ such that $u \geq 0$ in B_r.*

The proof of this proposition is subdivided into four steps.

1° We start by showing that the free boundary becomes flatter as we approach to the origin.

Claim 1. *For any $\sigma > 0$, there exists an $r_\sigma > 0$ such that for all $0 < r < r_\sigma$ there exists a direction $e_r \in \mathbb{R}^2$, $|e_r| = 1$ such that*

$$u > 0 \ \textit{in} \ \{x \cdot e_r > \sigma r\} \cap B_r =: D^+_{\sigma,r},$$
$$u \leq 0 \ \textit{in} \ \{x \cdot e_r < -\sigma r\} \cap B_r =: D^-_{\sigma,r}.$$

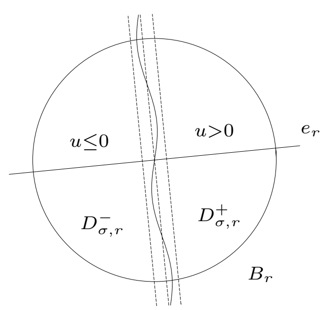

Figure 3. Claim 1

Proof. The proof follows from the fact that any blowup is a halfspace solution and that u_+ is nondegenerate. Indeed, suppose that the claim fails. Then there exist a sequence $r_j \to 0+$ such that for the rescalings

$$u_{r_j}(x) = u_{0,r_j}(x) = \frac{u(r_j x)}{r_j^\beta}$$

at least one of the conditions

$$u_{r_j} > 0 \text{ in } \{x \cdot e > \sigma\} \cap B_1,$$
$$u_{r_j} \leq 0 \text{ in } \{x \cdot e < -\sigma\} \cap B_1$$

fails for any choice of the direction e. Passing to a subsequence, if necessary, we may assume that $u_{r_j} \to u_0$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2)$. By Propositions 3.3 and 3.5 we have that $u_0(x) = C_0(x_1^+)^\beta$, after a suitable rotation of coordinate axes. Then we claim that

$$u_{r_j} > 0 \text{ in } \{x_1 > \sigma\} \cap B_1,$$
$$u_{r_j} \le 0 \text{ in } \{x_1 < -\sigma\} \cap B_1$$

for sufficiently large j. Indeed, the first inequality follows from the uniform convergence of $u_{r_j} \to u_0$. The second one is a direct corollary of the nondegeneracy (see Lemma 3.1): if $\{u_{r_j} > 0\} \cap \{x_1 < -\sigma\} \cap B_1 \ne \emptyset$, then $\sup_{\{x_1<-\sigma/2\}\cap B_2} u_{r_j} \ge c_0(\sigma/2)^\beta$ and consequently $\sup_{\{x_1<-\sigma/2\}\cap B_2} u_0 \ge c_0(\sigma/2)^\beta$, which is a contradiction. Thus, we also arrive at a contradiction with the assumption made on u_{r_j}. This proves the claim. □

2° Since u is harmonic in $\{u \le 0\}^\circ$, by the maximum principle, in $D^-_{\sigma,r} = \{x \cdot e_r < -\sigma r\}$ there are two possibilities:

i° $u \equiv 0$ in $D^-_{\sigma,r}$
ii° $u < 0$ in $D^-_{\sigma,r}$

and in principle different alternatives may hold for different r. However, our next claim says that the same alternative holds for all small r.

Claim 2. *If $u < 0$ in $D^-_{\sigma,r}$ for some $r = r_0 < r_\sigma$, then also $u < 0$ in $D^-_{\sigma,r}$ for all $0 < r \le r_0$ provided σ is small enough.*

Proof. Suppose $r_0 \ge r \ge r_0(1-\sigma)$. From Claim 1 we know that $D^-_{\sigma,r}$ cannot intersect D^+_{σ,r_0}. But then $D^-_{\sigma,r}$ must intersect D^-_{σ,r_0}, since otherwise $D^-_{\sigma,r}$ should be contained in a strip $|x \cdot e_{r_0}| < \sigma r_0$, which is too narrow if σ is small. This yields that $u < 0$ in $D^-_{\sigma,r}$. By iteration, the claim follows. □

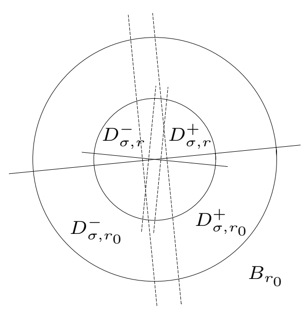

FIGURE 4. Claim 2

3° Now let us show, that

Claim 3. *If the exponent $p \in (0,1)$ in (1.1) is sufficiently close to 1, the alternative $u < 0$ in $D^-_{\sigma,r}$ for $r < r_\sigma$ is not possible for small σ.*

Proof. Let $\sigma = \epsilon/N$, with $\epsilon > 0$ small and N large to be specified later. For some $r_0 < r_{\epsilon/N}$ consider the sequence of radii

$$r_n = r_0(1-\epsilon)^n$$

as well as a sequence of points

$$P_n = -\frac{1}{2} r_n e_{r_n},$$

where e_r, $r = r_n$, are as in Claim 1.

The proof consists of the repetitive application of the Harnack inequality, to obtain an estimate for $|u(P_n)|$ from below, which will contradict to the growth estimate $|u(x)| \leq C|x|^\beta$. More detailed outline is as follows:

Step 1: Estimate the distance between P_n and P_{n+1}.

Step 2: Apply the Harnack inequality with precise constants to u in $D^-_{\frac{\epsilon}{N},r}$ at points P_n and P_{n+1} to obtain

$$-u(P_{n+1}) > -\lambda u(P_n), \tag{4.1}$$

which implies $-u(P_n) > \alpha\lambda^n$, where $\alpha = -u(P_0)$

Step 3: On the other hand, we have that

$$|u(P_n)| \leq C r_n^\beta = C r_0^\beta (1-\epsilon)^{\beta n}.$$

Thus, if we show that (4.1) holds with $\lambda > (1-\epsilon)^\beta$ then we will arrive at a contradiction.

We now start implementing this strategy.

Step 1: *Distance between P_n and P_{n+1}.*

Actually, we will estimate the distance between P_0 and P_1, since the general case can be easily obtained from this by scaling.

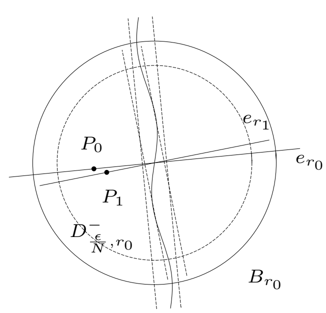

FIGURE 5. Claim 3: Step 1

We start by estimating the rotation of the unit vector e_r. Namely, we want to find a control of

$$\theta = \text{angle}(e_{r_0}, e_{r_1}).$$

Since $D^+_{\frac{\epsilon}{N},r_1}$ cannot intersect $D^-_{\frac{\epsilon}{N},r_0}$, using elementary geometry we obtain that

$$\sin\theta \le \frac{\epsilon/N}{1-\epsilon},$$

$$\cos\theta \ge \sqrt{1-\frac{\epsilon^2/N^2}{(1-\epsilon)^2}}.$$

Then, recalling that

$$P_0 = -\frac{1}{2}r_0 e_{r_0}, \quad P_1 = -\frac{1}{2}r_0(1-\epsilon)e_{r_1},$$

we obtain

$$\begin{aligned}\operatorname{dist}(P_0,P_1)^2 &= \frac{r_0^2}{4}\left(1+(1-\epsilon)^2-2(1-\epsilon)\cos\theta\right)\\ &\le \frac{r_0^2}{4}\left(1+(1-\epsilon)^2-2\sqrt{(1-\epsilon)^2-\frac{\epsilon^2}{N^2}}\right)\\ &= \frac{r_0^2}{4}\left[\left(1+\frac{1}{N^2}\right)\epsilon^2+O(\epsilon^3)\right].\end{aligned}$$

This implies that

$$\operatorname{dist}(P_0,P_1) < \frac{r_0\epsilon}{2}(1+\delta),$$

where $\delta=\delta_{\epsilon,N}>0$ can be made arbitrarily small if $\epsilon>0$ is sufficiently small and N is large.

Step 2: *The Harnack constant in dimension* 2.

To simplify the notations in this step we will assume $r_0=1$ and $e_{r_0}=(0,-1)$. This will not affect the generality, since the estimates that we are going to use will be scale and rotation invariant. We start by observation that

$$B_{\frac{1}{2}-\frac{\epsilon}{N}(P_0)} \subset D^-_{\frac{\epsilon}{N},1}$$

and

$$P_1 \in B_{\frac{\epsilon}{2}(1+\delta)}(P_0).$$

Now, if $u<0$ satisfies

$$\Delta u = 0 \ \text{ in } B_R(P_0),$$

for a certain $R>0$ then the Harnack inequality for balls in $\mathbb{R}^2$ gives

$$\frac{R-r}{R+r} \le \frac{u\,(P_0)}{u\,(P_1)} \le \frac{R+r}{R-r}.$$

In our case

$$R=\frac{1}{2}-\frac{\epsilon}{N} \quad \text{and} \quad r=\frac{\epsilon}{2}\,(1+\delta)\,.$$

However, it turns out that this estimate is not enough for our purposes. Since $u<0$ actually in a larger set $D^-_{\frac{\epsilon}{N},1}$, we can improve this estimate as follows. Consider a bijective conformal mapping

$$\phi: D^-_{\frac{\epsilon}{N},1} \to B_R \quad \text{such that } \phi(P_0)=0.$$

Since $B_R(P_0)\subset D^-_{\frac{\epsilon}{N},1}$, then using complex notations, we have

$$|\phi'(P_0)|<1.$$

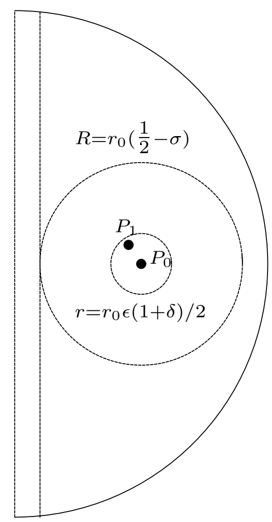

FIGURE 6. Claim 3: Step 2

Consequently, for small r we have

$$\phi\left(B_r(P_0)\right) \subset B_{(1-\kappa)r},$$

for some $\kappa > 0$, which can be regarded as an absolute constant if ϵ/N is small. Applying the Harnack inequality to $u \circ \phi^{-1}$ for balls $B_{(1-\kappa)r} \subset B_R$, we obtain that for our u we have

$$\frac{u(P_0)}{u(P_1)} \leq \frac{R + (1-\kappa)r}{R - (1-\kappa)r}.$$

Thus, we get

$$-u(P_1) \geq -\lambda u(P_0)$$

with

$$\lambda := \frac{R - (1-\kappa)r}{R + (1-\kappa)r} = \frac{1 - \frac{2\epsilon}{N} - (1-\kappa)(1+\delta)\epsilon}{1 - \frac{2\epsilon}{N} + (1-\kappa)(1+\delta)\epsilon}.$$

Similarly, we obtain that

$$-u(P_{n+1}) \geq -\lambda u(P_n)$$

for any n and as a corollary that

$$|u(P_n)| \geq \lambda^n |u(P_0)|.$$

Step 3: *Estimating* λ.

Using that $(1+x)/(1-y) = (1+x)(1+y+O(y^2)) = 1+x+y+O(x^2+y^2)$, we find that

$$\begin{aligned}
\lambda &= \frac{1 - \frac{2\epsilon}{N} - (1-\kappa)(1+\delta)\epsilon}{1 - \frac{2\epsilon}{N} + (1-\kappa)(1+\delta)\epsilon} \\
&= 1 - \frac{2\epsilon}{N} - (1-\kappa)(1+\delta)\epsilon + \frac{2\epsilon}{N} - (1-\kappa)(1+\delta)\epsilon + O\left(\epsilon^2\right) \\
&= 1 - 2(1-\kappa)(1+\delta)\epsilon + O\left(\epsilon^2\right).
\end{aligned}$$

Now recall that κ is an absolute constant and δ can be chosen as small as we wish. Thus, we can make

$$(1-\kappa)(1+\delta) < 1-\kappa/2,$$

and consequently

$$\lambda \geq 1 - 2(1-\kappa/2)\epsilon + O(\epsilon^2) \geq (1-\epsilon)^{2-\kappa/2},$$

provided ϵ is sufficiently small. This implies that

$$|u(P_n)| \geq \lambda^n |u(P_0)| \geq c|P_n|^{2-\kappa/2},$$

with $c > 0$. On the other hand, we have the growth estimate

$$|u(P_n)| \leq C|P_n|^\beta$$

and to arrive at a contradiction we note that

$$\beta = \frac{2}{2-p} > 2 - \kappa/2$$

if p is close enough to 1. □

4° So far we have proved that for $r \leq r_0$

$$u > 0 \text{ in } D^+_{\sigma,r}$$
$$u \equiv 0 \text{ in } D^-_{\sigma,r},$$

provided p is close enough to 1.

In this step, rescaling and rotating if necessary, we will assume that $r_0 = 1$ and $e_{r_0} = (1, 0)$.

Claim 4. *Under the assumption above, $u \geq 0$ in $B_{\frac{1}{4}}$.*

Proof. Assume the contrary and let W be a connected component of $\{u < 0\} \cap B_1$ such that $W \cap B_{\frac{1}{4}} \neq \emptyset$. Observe that W must be contained in the strip $\{|x_1| < \sigma\}$.

We then consider two cases.

i° $W \Subset B_{\frac{3}{4}}$. Here we will have a contradiction to the maximum principle, since u is harmonic in W and must vanish on ∂W.

ii° $W \cap \partial B_{\frac{3}{4}} \neq \emptyset$. Then we can find a point $P \in W$ such that $|P| = \frac{1}{2}$. Take a narrow horizontal box R containing P and connecting it to $\{x_1 < -\sigma\}$. Then there exists a point $Q \in R$ such that $u(Q) > 0$. Otherwise $u \leq 0$ in R and thus $u \leq 0$ in $\{x_1 < -\sigma\} \cup R \cup W$. However, $u \equiv 0$ in $\{x_1 < -\sigma\}$, $u < 0$ in W and u is harmonic in the union $\{x_1 < -\sigma\} \cup R \cup W$. This contradicts to the strong maximum principle.

Thus, there exists $Q \in R$ with $u(Q) > 0$ that lies “between” W and $\{x_1 < -\sigma\}$. Taking R narrow enough, we can guarantee that $\frac{3}{8} \leq |Q| \leq \frac{5}{8}$. Now consider the intersection

$$B_{\frac{1}{8}}(Q) \cap \{u > 0\}.$$

Since W is connected and intersects both $\partial B_{\frac{1}{4}}$ and $\partial B_{\frac{3}{4}}$, there exists a connected component V of the above set such that

$$Q \in V \subset \{|x_1| < \sigma\}.$$

Note that here we have strongly used the topological properties of $\mathbb{R}^2$.

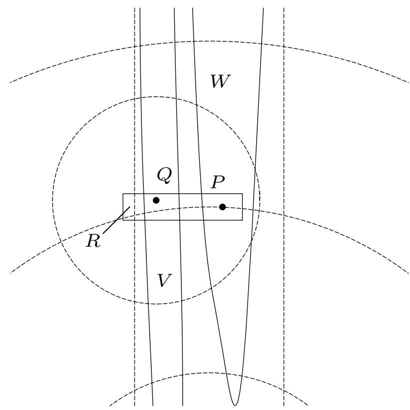

FIGURE 7. Claim 4: alternative ii°

Next, we apply the nondegeneracy lemma in connected components to arrive at a contradiction. Namely, by Lemma 3.2 we must have

$$\sup_{\partial B_{\frac{1}{8}}(Q)\cap V} u \geq c_0 \left(\frac{1}{8}\right)^{\beta}.$$

On the other hand, since $u \equiv 0$ in $\{x_1 < -\sigma\}$, and $u \in C^{1,\beta-1}$, we have

$$|u| \leq C_0\sigma^{\beta} \quad \text{in } \{|x_1| < \sigma\} \cap B_1.$$

which is a contradiction if σ is small.

This proves the claim. □

Proof of Proposition 4.2. Note that Claim 4 is just the rescaled version of Proposition 4.2. □

We are now ready to prove the main theorem.

Proof of Theorem I. By Proposition 4.1, we have that $|\nabla u|$ does not vanish on $\Gamma^- = \Gamma^- \cap \Gamma^+$, provided p is close enough to 1. Hence, by the implicit function theorem, Γ^- is locally a C^1 graph. On the other hand, $u \geq 0$ in a neighborhood of any point on $\Gamma^+ \setminus \Gamma^-$ and therefore using the result of [AP86] in dimension 2 we conclude that $\Gamma^+ \setminus \Gamma^-$ is C^1 regular as well. □

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