PARABOLIC OBSTACLE PROBLEMS APPLIED TO FINANCE
A FREE-BOUNDARY-REGULARITY APPROACH

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Abstract. The parabolic obstacle problem within financial mathematics is the main object of discussion in this paper. We overview various aspects of the problem, ranging from the fully nonlinear case to the specific equations in applications. The focus will be on the free boundary and its behavior close to initial state, and the fixed boundary. In financial terms these can be translated as the behavior of the exercise boundary for American option, close to maturity and close to the barrier.

From free-boundary-regularity point of view, such problems have not been considered earlier.

1. Introduction

1.1. **Background.** The parabolic obstacle problem refers to finding the smallest supersolution (for a given parabolic operator, and given domain and boundary data) over a given function (obstacle).

This problem appears in applications, such as phase transitions (melting and crystallization), mathematical biology (tumor growth), and American type contracts in finance. Of obvious reasons, the latter application has gained grounds in the recent past.

Although many financial problems involve linear equations, there are many related problems with nonlinear governing equations, that require more delicate analysis. It is our aim here to give an account of some new ideas and techniques, for nonlinear parabolic obstacle problem from free boundary regularity point of view. In doing so, we will present some general results describing the optimal regularity of the solutions as well as describing the behavior of the free boundary at initial state and also close to a fixed boundary. These, general results are then exemplified in terms of applications in finance.

A highlight of this paper is the specific application of our results for the valuation of the American put option of minimum of two underliers, see last section.

1.2. **Mathematical formulation.** Let us denote by Q^+ the upper-half of the unit cylinder in $\mathbf{R}^{n+1}$, and let $\psi(x,t)$ be parabolically $C^{0,\alpha}$.

Set

$$H(u) = F(D^2u, Du, u, x, t) - D_t u, \tag{1.1}$$

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where F is a fully nonlinear uniformly elliptic operator with certain homogeneity properties for which the regularity theory of viscosity solutions [Wan92a, Wan92b, Wan92c] applies. For more precise assumptions on F, see Section 1.3 in [Sha06]. Let u solve the parabolic obstacle problem

$$(u-\psi)H(u)=0, \quad u\geq\psi \quad \text{in } Q^+, \tag{1.2}$$

$$H(u)\leq 0, \tag{1.3}$$

with boundary datum

$$u(x,t)=g(x,t)\geq\psi(x,t) \qquad \partial_p Q^+. \tag{1.4}$$

(Here the notation ∂_p stands for the parabolic boundary, see [Wan92a].)

Let us set

$$\mathcal{E}(u)=\{(x,t)\in Q^+ :\ u(x,t)=\psi(x,t)\},$$
$$\mathcal{C}(u)=\{(x,t)\in Q^+ : u(x,t)>\psi(x,t)\}.$$

The obstacle ψ in general has singularities, usually representing a change in the nature of the contract in applications in finance. It should be remarked that in the applications in this paper we exclusively consider the American put option. Examples of obstacles that appear in finance are given below (here E is a constant and it denotes the exercise price).

Obstacle ψ	Applications
$(E-x_1)^+$	1-d contract, American put
$\min\{(E-x_1)^+,(E-x_2)^+\}$	min option, American put.

Another type of contract that also is of interest in the market is that of barrier type options, referring to termination/start of a contract, before the time of maturity. Here we will also discuss this issue from the obstacle problem point of view. However, we will only consider the case of down-and-out barriers (termination of a contract with a rebate) for American put option.

1.3. **Smooth obstacles.** For the smooth obstacle case, there is a vast literature treating various aspects such as existence and regularity of the solutions, as well as certain geometric inheritance. In the linear case one can even make a further simplification of the problem by considering the function $U=u-\psi$, and after some standard analysis one comes to the conclusion that

$$H(U)=f\chi_{U>0}, \qquad U\geq 0, \tag{1.5}$$

where $f=-H(\psi)$, and the free boundary is now $\partial\{U>0\}$.

The study of the obstacle problem, in one dimension and in the case of Stefan problem $\partial_t U\geq 0$ (which is a result of specific data, e.g. if the obstacle is time-independent) has been successfully carried out by several people; see [Fri75], [vM74a, vM74b].

In higher dimensions, besides existence theory and some partial results on geometry, there are hardly any other results. Concerning the regularity of the free boundary, the only known general result is [CPS04]. For the case of Stefan problem see also [Caf77]. Although both these papers treat the Laplacian case, one can easily generalize part of the results to more general, but still linear, operators. Cf. also a recent paper by Blanchet et al. [BDM05].

1.4. **Non-smooth obstacles.** When the obstacle is non-smooth, even in the simple Laplacian case, one can not reformulate the problem into (1.5) and hence there are not many know techniques to be used in the study of the obstacle problem. Of course, even if the obstacle is smooth, but the operator is non-linear, then we still are in situation that earlier techniques are not applicable. A complete study of the problem is yet to come.

In this note we will present some results for the non-smooth case, with focus on applications to American type options in finance.

2. Applications to Finance and Optimal Stopping

The above obstacle problem, appears naturally in the valuation of American type claims, in financial market. The obstacle is the so-called payoff function and the optimal configuration u is the value of the option. For a good background study we refer the reader to the paper by M. Broadie and J. Detemple [BD97].

Let us denote $\mathbf{S}_t = (S_t^1, S_t^2)$ to be the price vector of underlying assets at time t. The price S_t^i follows the standard Brownian motion

$$dS_t^i = (r - \delta_i)S_t^i dt + \sigma_i dW_t^i, \qquad i = 1, 2$$

where r is the constant interest rate, δ_i is the dividend rate of the i-th stock, and σ_i is the volatility of the price of the corresponding asset. The notation W_t^i also stands for the standard Brownian motion, over a probability filtered space $(\Omega, \mathcal{F}, P)$, with P as the risk-neutral measure.

The value function V of the American option is given by

$$V(S,t) = \sup_\tau E\left(e^{-r(T-t)}\psi(\mathbf{S}_\tau)\right) \tag{2.1}$$

with the stopping time τ varying over all $\mathcal{F}_t$-adapted random variables, and $\psi(S)$ the option payoff. Here $(\mathcal{F}_t)_{t\geq 0}$ denotes the P completion of the natural filtration associated to $(W_t^i)_{t\geq 0}$. This completion comes from the so-called completeness of markets, so that one has a unique solution to the problem.

Stochastic analysis can now be used to show that V satisfies a variational inequality, here written in the complementary form,

$$\mathcal{L}V + \partial_t V \leq 0, \qquad (\mathcal{L}V + \partial_t V)(\psi - V) = 0, \qquad V \geq \psi\ ,$$

a.e. on $\mathbb{R}^n \times [0,T)$, and with condition

$$V(x,T) = \psi(x,T).$$

Here the elliptic operator $\mathcal{L}$ is given by

$$\mathcal{L}V = (r-\delta_1)S^1\frac{\partial V}{\partial S^1} + (r-\delta_2)S^2\frac{\partial V}{\partial S^2} +$$
$$+\frac{1}{2}\left((\sigma_1 S^1)^2\frac{\partial^2 V}{\partial (S^1)^2} + \sigma_1\sigma_2 S^1 S^2\frac{\partial^2 V}{\partial S^1 \partial S^2} + (\sigma_2 S^2)^2\frac{\partial^2 V}{\partial (S^2)^2}\right) - rV\ .$$

The backward parabolic equation, can be turned into forward equation by a change of variable, and hence we arrive at the case of the parabolic obstacle problem in this paper.

In general, the obstacle ψ is non-smooth at some point x^0, at time of maturity $t = T$. Examples of such obstacles can be found in [BD97].

3. Parabolic notation

For a point $X = (x, t) \in \mathbf{R}^n \times \mathbf{R}$ and $r > 0$, we consider three kinds of parabolic cylinders:

$$\begin{aligned} Q_r(X) &= B_r(x) \times (t - r^2, t + r^2), \\ Q_r^-(X) &= B_r(x) \times (t - r^2, t], && \text{(lower cylinder)} \\ Q_r^+(X) &= B_r(x) \times [t, t + r^2). && \text{(upper cylinder)} \end{aligned}$$

When $X = (0, 0)$ we don't indicate the center. For a cylinder $Q = B \times I$, where I is an interval with endpoints $a < b$, then we define

$$\begin{aligned} \partial_x Q &= \partial B \times I, && \text{(lateral boundary)} \\ \partial_b Q &= B \times \{a\}, && \text{(bottom)} \\ \partial_p Q &= \partial_x Q \cup \partial_b Q && \text{(parabolic boundary)} \end{aligned}$$

For $X = (x, t)$ we denote $|X| = \sqrt{x^2 + |t|}$. Then $|X - Y|$ is the parabolic distance between X and Y.

Given $0 < \alpha \leq 1$, the parabolic Hölder space $C^{0,\alpha}(\Omega)$ is defined as the subspace of $C(\Omega)$ consisting of functions f such that the norms

$$\|f\|_{C^{0,\alpha}(\Omega)} = \|f\|_{C(\Omega)} + \sup_{X,Y \in \Omega, X \neq Y} \frac{|f(X) - f(Y)}{|X - Y|^\alpha}$$

are finite. Next, the parabolic spaces $C^k(\Omega)$ for integers k are defined as the space of continuous functions f for which the derivatives $D_x^i D_t^j f$ with $|i| + 2j \leq k$ are also continuous, with appropriately defined norm. (Here i is a multi-index and j is an integer.) For integer k, $0 < \alpha \leq 1$, and $f \in C(\Omega)$ let

$$[f]_{k,\alpha}(X) = \inf_{P_k} \sup_{r>0} \frac{1}{r^\alpha} \sup_{Q_r^-(X) \cap \Omega} |u - P_k|,$$

where $P_k(X) = \sum_{|i|+2j \leq k} a_{i,j} x^i t^j$ are polynomials of parabolic order $\leq k$. Then the higher parabolic Hölder spaces $C^{k,\alpha}(\Omega)$ are defined as the subspaces of $C^k(\Omega)$ consisting of functions f such that the norms

$$\|f\|_{C^{k,\alpha}(\Omega)} = \|f\|_{C^k(\Omega)} + \sup_{X \in \Omega} [f]_{k,\alpha}(X)$$

are finite.

4. Regularity of the solution

A compactness-based technique developed in [Sha06] allows to prove the following result, which essentially says that the solution of the obstacle problem is as regular as the obstacle, up to $C^{1,1}$, similarly to the classical obstacle problem.

Theorem 4.1 (Optimal interior regularity)**.** *Let u be a solution to the obstacle problem in Q_1^- with obstacle $\psi \in C^{k,\alpha}\left(Q_1^-\right)$ for $k = 0, 1$ and $0 < \alpha \leq 1$. Then $u \in C^{k,\alpha}_{\mathrm{loc}}(Q_1^-)$ and for any $K \subset\subset Q_1^-$*

$$\|u\|_{C^{k,\alpha}(K)} \leq C\left(K, F, k, \alpha, n, \|u\|_{L^\infty(Q_1^-)}, \|\psi\|_{C^{k,\alpha}(Q_1^-)}\right).$$

The next two results deal with the boundary regularity of the solutions.

Theorem 4.2 (Lateral boundary regularity). *Let u be as in Theorem 4.1 with $k = 0, 1$, $0 < \alpha \le 1$, and boundary values $g \in C^{2,\epsilon}(Q_1^- \cup \partial_x Q_1^-)$ for some $\epsilon > 0$. Then $u \in C^{k,\alpha}_{\mathrm{loc}}\left(Q_1^- \cup \partial_x Q_1^-\right)$ and for any $K \subset\subset Q_1^- \cup \partial_x Q_1^-$*

$$\|u\|_{C^{k,\alpha}(K)} \le C\left(K, F, k, \alpha, n, \epsilon, \|g\|_{C^{2,\epsilon}\left(Q_1^- \cup \partial_x Q_1\right)}, \|\psi\|_{C^{k,\alpha}\left(Q_1^-\right)}\right).$$

Theorem 4.3 (Initial regularity). *Let u be as in Theorem 4.1 and assume also that it takes boundary values $g \in C^{k,\alpha}(Q_1^- \cup \partial_b Q_1)$. Then there exists $\beta = \beta(k, \alpha)$, $0 < \beta < 1$ such that $u \in C^{k,\beta}_{\mathrm{loc}}(Q_1^- \cup \partial_b Q_1^-)$ and for any $K \subset\subset Q_1^- \cup \partial_b Q_1^-$*

$$\|u\|_{C^{k,\beta}(K)} \le C\left(K, F, k, \alpha, n, \|g\|_{C^{k,\alpha}\left(Q_1^-\right)}, \|\psi\|_{C^{k,\alpha}\left(Q_1^-\right)}\right).$$

In applications to finance, where u represents the value function for a certain derivative, usually one can represent the u as the supremum of the expected value over a class of stopping times. From such representations one can derive the optimal regularity of the solution by computations; cf. [CC03].

Sketch of the proof. The proof of Theorem 4.1 is an consequence from the following growth estimate.

Lemma 4.4. *Let u be as in Theorem 4.1 and assume that $\|u\|_{L^\infty\left(Q_1^-\right)} \le 1$, $\|\psi\|_{C^{k,\alpha}\left(Q_1^-\right)} \le 1$. Then there exists $C = C(k, \alpha, n)$ such that*

$$\sup_{X \in Q_r^-(Z)} |u(X) - u(Z)| \le C r^\alpha, \quad \text{if} \quad k = 0$$

and

$$\sup_{X \in Q_r^-(Z)} |u(X) - u(Z) - D_x\psi(Z)(x - z)| \le C r^{1+\alpha}, \quad \text{if} \quad k = 1,$$

for any $Z = (z, s) \in \partial\mathcal{E}(u) \cap Q_{1/2}^-$ and $0 < r < 1/2$.

The proof of this lemma is based on a compactness argument developed in [Sha06].

In the case $k = 0$ define

$$S_j = S_j(u, Z) = \sup_{X \in Q_{2^{-j}}(Z)} |u(X) - u(Z)|, \quad j = 1, 2 \ldots,$$

for any point $Z \in \partial\mathcal{E}(u) \cap Q_{1/2}^-$. The statement of the lemma will follow once we prove that

$$S_j \le C 2^{-\alpha j}, \quad j = 1, 2 \ldots$$

for a universal constant C. The proof is by establishing the recursive relation

$$S_{j+1} \le \max\left\{\frac{C}{2^{-j\alpha}}, \frac{S_j}{2^\alpha}, \frac{S_{j-1}}{2^{2\alpha}}, \ldots, \frac{S_1}{2^{j\alpha}}\right\}.$$

Assuming that this inequality fails for any value of C, we will have a sequence of solutions u_i in Q_1^-, $i = 1, 2, \ldots$, with obstacles ψ_i as above, integer k_i and a point $Z_i = (z_i, s_i) \in \partial\mathcal{E}(u_i)$ such that

$$S_{k_i+1} \ge \max\left\{\frac{i}{2^{-k_i\alpha}}, \frac{S_{k_i}}{2^\alpha}, \frac{S_{k_i-1}}{2^{2\alpha}}, \ldots, \frac{S_1}{2^{k_i\alpha}}\right\}.$$

Then we consider the rescalings

$$\tilde{u}_i(x,t) = \frac{u_i(z_i + 2^{-k_i}x, s_i + 2^{-2k_i}t) - u_i(Z_i)}{S_{k_i+1}(u_i, Z_i)}, \quad (x,t) \in Q^-_{2^{k_i}}(-Z_i).$$

It is easy to see that $\tilde{u}_i$ is a solution of the obstacle problem with appropriately modified obstacle $\tilde{\psi}_i$. Next, without loss of generality we may assume that Z_i converges to $Z_0 \in \overline{Q^-_{1/2}}$. Then $\tilde{u}_i$ are uniformly bounded in Q^-_1 and $\tilde{\psi}_i \to 0$ locally uniformly in Q^-_1. Then a barrier argument can be used to show that

$$\tilde{u}_i \to 0 \quad \text{uniformly on } Q^-_{1/2}.$$

However, this contradicts to the fact that

$$\sup_{Q^-_{1/2}} |\tilde{u}_i(X)| = 1.$$

The proof in this case $k = 1$ follows the same scheme, except we need to redefine

$$S_j = S_j(u, Z) = \sup_{X \in Q_{2^{-j}}(Z) \cap Q^-_1} |u(X) - u(Z) - D_x\psi(Z)(x - z)|$$

and

$$\tilde{u}_i(x,t) = \frac{u_i(z_i + 2^{-k_i}x, s_i + 2^{-2k_i}t) - u_i(Z_i) - 2^{-k_i}D_x\psi(Z_i)x}{S_{k_i+1}(u_i, Z_i)}$$

and establish that

$$S_j \le C2^{-(1+\alpha)j}, \quad j = 1, 2, \ldots.$$

The proofs of Theorems 4.2 and 4.3 are little more involved as they require boundary analysis, but also follow a similar scheme (see Theorem 1.3 in [Sha06]).

5. Regularity of the free boundary

In this section we discuss results concerning the regularity of the free boundary. The results are stated in the framework of fully nonlinear operators, however, some of the results are currently known only for the heat equation, while some others are known for a fairly general class of operators.

5.1. Interior regularity for smooth obstacles.

Theorem 5.1 (Interior spatial regularity backward in time)**.** *Let u be a solutions of the obstacle problem, with $\psi \in C^\infty(Q_1)$. Suppose for some point $Z \in \partial\mathcal{E}(u) \cap Q_{1/2}$, with $H(\psi)(Z) = -c_0 < 0$, we have that the set*

$$\mathcal{E}^-_r(u; Z) := \mathcal{E}(u) \cap Q^-_r(Z) \tag{5.1}$$

is thick enough for some $r = r_1$ in the sense that

$$\delta^-_r(u; Z) := \frac{|\mathcal{E}^-_r(u; Z)|}{|Q^-_r(Z)|} > \sigma(r)$$

for a certain modulus of continuity σ. Then the free boundary $\partial\mathcal{E}$ is locally in $Q^-_{r_0}(Z)$ a C^1-graph in some of the space directions, and for some r_0. Moreover, the modulus of continuity σ as well as r_0 can be chosen uniformly for ψ in appropriately defined classes.

This result is currently known only for the heat operator, see [CPS04]. A related result for a fully nonlinear elliptic obstacle-type problem can be found in [LS01].

Remark 5.2. We note here that the possibility of having "flat in time" free boundaries as in the example

$$u(x,t) = -\frac{|x|^2}{2n} + \left(\frac{1}{2} - t\right)^+, \quad \psi(x,t) = -\frac{x^2}{2n}$$

for the operator $F = \Delta$ prevents from having the analogue of this theorem for forward in time spatial regularity of the free boundary. However, under the assumption $D_t u \geq D_t \psi$ in Q_1, which mathematically corresponds to the Stefan problem case, we can replace the thickness of $\mathcal{E}_r^-(u;Z)$ by that of

$$\mathcal{E}_r(u;Z) = \mathcal{E}(u) \cap Q_r(Z)$$

or even with the thickness of any of its t-cuts in $Q_r(Z)$ and obtain the spatial regularity of $\partial\mathcal{E}$ in $Q_{r_0}(Z)$.

To illustrate the idea of the proof, we state here one of the intermediate steps in the case $F = \Delta$ in [CPS04].

Lemma 5.3 (Convexity of global solutions). *Let w be a nonnegative solution of $\Delta w - D_t w = c_0 \chi_{\{w>0\}}$ in $\mathbf{R}^n \times \mathbf{R}^-$ such that*

$$|w(X)| \leq M(1 + |X|^2), \quad X \in \mathbf{R}^n \times \mathbf{R}^-$$

for some constant M. Then $D_{ee} w \geq 0$ for any spatial direction e and $D_t w \leq 0$. In particular, the t-slices

$$\mathcal{E}(t) = \{x : w(x,t) = 0\}$$

are convex and shrink as t decreases.

The proof of this lemma is based on the following argument. Assuming that $D_{ee} w$ is negative at some points, we take a minimizing sequence X_n such that

$$D_{ee} w(X_n) \to m = \inf_{\{w>0\}} D_{ee} w < 0.$$

Note that m is finite by the optimal regularity theorem (see Theorem 4.1 above). Then consider the rescalings

$$w_n(X) = \frac{1}{d_n^2} w(x_n + d_n x, t_n + d_n^2 t),$$

were d_n is the parabolic distance of the point X_n to the coincidence set $\{w = 0\}$. Over a subsequence, w_n will converge to a solution w_0 of $\Delta w_0 - D_t w_0 = c_0 \chi_{\{w_0>0\}}$ in $\mathbf{R}^n \times \mathbf{R}^-$, which will have the following properties:

$$\Delta w_0 - D_t w_0 = 1 \quad \text{in} \quad Q_1^-, \qquad D_{ee} w_0(0,0) = \min_{Q_1^-} D_{ee} w_0.$$

The function $v = D_{ee} w$ itself will satisfy the heat equation in Q_1^- and therefore by the minimum principle $D_{ee} w = m = const$ in Q_1^-. Moreover, this equality continues to hold in the parabolic connected component of the set $\{w > 0\}$. Assuming that $e = e_1$, this implies the representation

$$w_0(x_1, 0, \ldots, 0) = \frac{m}{2} x_1^2 + b x_1 + c$$

for some constants b and c, which will hold at least for $0 < x_1 < 1$ and will continue to hold as long as $w_0 > 0$. However, since $m < 0$ there will be the first $x_1 = x_1^*$, where w_0 will become 0. This will imply the representation

$$w_0(x_1, 0, \ldots, 0) = \frac{m}{2}(x_1 - x_1^*)^2, \quad 0 < x_1 < x_1^*,$$

since both w_0 and $D_1 w_0$ should vanish at $x_1 = x_1^*$. However, this contradict to nonnegativity of w, as $m < 0$.

A similar argument proves also that $D_t w \le 0$.

5.2. **Behavior near the fixed boundary.**

Theorem 5.4 (Touch with the fixed boundary)**.** *Let u be a solutions of the obstacle problem in Q_1 with $\psi, g \in C^\infty(\overline{Q_1})$. Suppose for a boundary point $Z = (z, 0)$ with $z \in \partial B_1$ and $H\psi(Z) < 0$, we have that $Z \in \partial\mathcal{E}$, i.e. there exists $Z^j \to Z$ with $Z^j \in \mathcal{C}(u)$. Suppose further $|g(X) - g(Z)| = o(|X - Z|^2)$ for $X \in \partial_p Q_1$. Then we have that the set*

$$\mathcal{E}_r(u) := \mathcal{E}(u) \cap Q_r(Z) \cap Q_1 \tag{5.2}$$

is thin in the following sense

$$\mathcal{E}_r(u) \subset \{(x,t) : \ -(x - z) \cdot z \le |X - Z|\sigma(|X - Z|)\} \cap Q_{r_0}(Z) \cap Q_1.$$

Here σ is a modulus of continuity, which along with he constant r_0, depends only on the appropriately defined classes of ψ and g.

This result is currently known only for the heat operator, see [AUS03].

Remark 5.5. In contrast to the interior regularity, the boundary condition forces the free boundary to touch the fixed boundary parabolically-tangentially both forward and backward in time.

The proof in the case $F = \Delta$ in [AUS03] is based on the following classification of global solutions.

Lemma 5.6 (Global solutions with zero fixed boundary data)**.** *Let w be a nonnegative solution of $\Delta w - D_t w = c_0 \chi_{\{w>0\}}$ in $\mathbf{R}^n_+ \times \mathbf{R}$ and $w = 0$ on $\partial\mathbf{R}^n_+ \times \mathbf{R}$, where $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n : x_1 > 0\}$. Then $w(x,t) = c_0 (x_1^+)^2/2$ for all $(x,t) \in \mathbf{R}^n_+ \times \mathbf{R}$.*

5.3. **Initial behavior.**

In the special case when $\psi = g$ (of interest in applications to finance) and when ψ has a structure

$$\psi(x,t) = \psi^a(x,t) = (x_1^+)^a \psi_1(x,t) + \psi_2(x,t)$$

for a certain $a > 0$ with continuous ψ_1, ψ_2 such that

$$\psi_1(0,0) = 1, \quad \psi_2(X) = o(|X|^a),$$

one can characterize the initial behavior of the free boundary as follows.

We consider the cases $a \ge 1$ and $0 < a < 1$ separately, as they offer different geometric behaviors for the coincident set.

Theorem 5.7. *Let u be a solution of the obstacle problem with $\psi = g = \psi^a$ as above for a certain $a \ge 1$. Then there exists $r_0 > 0$, and a modulus of continuity σ such that*

$$\mathcal{E}(u) \cap Q^+_{r_0} \subset \{(x,t) : \ t \le |x|^2 \sigma(|x|)\} \cap Q^+_{r_0}. \tag{5.3}$$

Here r_0, and σ depend on the class of the obstacles $\psi = \psi_a$ only.

Theorem 5.8. *For $0 < a < 1$, there exist positive constants r_0, c_a, and a modulus of continuity σ, such that if u is a solution to the obstacle problem with $\psi = g = \psi^1$ as above then*

$$\mathcal{E}(u) \cap Q^+_{r_0} \subset P_\sigma \cup T_\sigma \tag{5.4}$$

where

$$P_\sigma := \{(x,t) : \ x_1 > 0, \ t \le (c_a + \sigma(|x|))x_1^2)\},$$

and

$$T_\sigma := \{(x,t) : \ t \le \sigma(|x|)|x|^2)\}.$$

These results correspond to Theorems 1.4, 1.6 in [Sha06] and known for a class of uniformly parabolic fully nonlinear operators H; see [Sha06] for precise assumptions on H.

Again, as before, the proof relies on a classification of a certain class of global solution.

Lemma 5.9 (Global solutions with $\psi = g = (x_1^+)^a$). *Let u be a global solutions of the obstacle problem in $\mathbf{R}^n \times \mathbf{R}^+$ with $\psi(x,t) = g(x,t) = (x_1^+)^a$.*

(i) *If $a \ge 1$, then $\mathcal{E}(u) = \emptyset$.*
(ii) *If $0 < a < 1$, then there exists a constant $c_a > 0$ such that*

$$\mathcal{E}(u) = \{(x,t) : 0 < t \le c_a(x_1^+)^2\}.$$

6. Specific applications to finance

6.1. **Regularity theory.** In applications of the regularity theory one needs to verify the thickness conditions in Theorem 5.1. However, in general the equations and ingredients (the obstacle and the boundary data) are such that one usually has monotonicity in the t, for the solution u. Such monotonicity in general reflects the fact that usually the value of the option becomes smaller in time (the time is reversed in applications) since the possibility of the change of the value of the underlier is becoming less. Hence, in general, $D_t u \ge 0$ in our formulations. Another feature, reflected in applications, is the convexity of u in space directions, $D_{ee} u \ge 0$, and also the monotonicity of u in some of its space directions . Such conclusion to hold requires that the governing operator, the obstacle and boundary data to satisfy certain conditions. Let us illustrate this by an example that arise in the valuation of the early exercise contacts in incomplete markets; see [OZ03]. The equation in this problem (after inverting time) are given by

$$\min\left\{u_t - Lu + \frac{1}{2}\gamma(1-\rho^2)a^2 u_x^2, \ u - g(x)\right\} = 0, \qquad u(x,0) = g(x),$$

where

$$L = \frac{1}{2}a^2 D^2_{xx} + \left(b - \rho\frac{\mu}{\sigma}\right) D_x.$$

Observe that x is one space dimensional. Here, with correct assumptions on the ingredients $a, \gamma, \cdots$ one may use standard comparison principle (use sliding in t-direction) to conclude that $D_t u \ge 0$. To obtain the regularity of the free boundary we need to make further assumptions on the ingredients, and especially on the obstacle (payoff) g.

We may replace the function u with $v = u - g$, which satisfies (this requires some work)

$$\Delta v - D_t v = \tilde{g}\chi_{\{v>0\}}, \qquad v \ge 0.$$

Here $\tilde{g}$ contains all extra terms that is given rise to after reformulation. It is also not hard to verify that $\tilde{g}(Z) > 0$, for any free boundary point, and that $\tilde{g}$ is H^α.

6.2. **Close to maturity.** The study of the exercise region, for one underlier (one space dimension), close to maturity has been much in focus, for American put as well as American call option. These results give accurate description of the exercise region close to maturity. More precisely for the American put, with payoff (obstacle) $(E-x)_+$ one has that the early exercise boundary can be represented by a graph $t = h(x)$ with $h(x) \approx (x-E)_+^2/\log|E-x|$.

The methods for all such results rely heavily on one space dimension, and representations of the solutions and pure computational methods (see [CC03]).

Theorems 5.7–5.8 above give good descriptions of the exercise region close to maturity, for a more general equations as well as a general payoff function. The methods are purely geometric and do not take into account the space dimensions. The main differences (in one space dimension) with the existing results is that our modulus of continuity σ in Theorems 5.7–5.8 is not give explicitly, while in classical results its known to be $1/\log|r|$. In this regard our result is not a good replacement for classical results in one dimension, unless one has nonlinear equations to treat, or if one has ingredients that are not "clean" so that the computations have to take into account error terms.

Here we want to exemplify our technique for a simple model of American type contract for the min-put option

$$\psi(x,t) = \min\{(E-x_1)_+, (E-x_2)_+\}, \tag{6.1}$$

where E denotes the exercise price for both underlier x_1, x_2. We refer the reader to [DFT03] for the corresponding call option on the minimum of two assets

$$(\min\{x_1, x_2\} - E))_+.$$

It should be remarked that the local behavior of the free boundary close to the point $(E, E, 0)$, and along the set $(x_1, x_1, 0)$, in both put and call case seem to be pretty much the same!

Let us now consider a forward parabolic operator H, and analyze the behavior of the exercise region close to initial state (i.e. close to maturity for corresponding finance problem), for the min-option.

According to Theorem 4.1 we have that our solution u is a Lipschitz function up to the time $t = 0$. Suppose also that the free boundary touches the initial state at the point $(E, E, 0)$; this is the case of the American put option. As one can easily show that for the European option the value goes below the obstacle for small times and close to (E, E).

From both theoretical and numerical point of view, the behavior of the exercise region close to this point is more difficult to analyze. For other points like $(E, s, 0)$, with $s < E$, one can use the one-dimensional analogy.

Following the lines of the proof of Theorems 1.4, 1.6 in [Sha06] we want to classify global solutions of the corresponding problem for the min-option. This would give us a good information about the behavior of the local solutions close to maturity. We consider a translation of the point $(E, E, 0)$ to the origin, and replace x_i with $-x_i$ so that in the global setting the the obstacle becomes $\min\{(x_1)_+, (x_2)_+\}$. The blow-up operator H_0 also as before has the strong minimum principle. This in particular implies that the only possible points where the solution and the obstacle touch could be on the set $\{x_1 > 0, x_2 > 0,\ x_1 = x_2\} \cap \{t > 0\}$.

We know that the global solution to the obstacle problem is unique (otherwise we take the minimum of the two solutions, which also is a solution) and hence by scaling we see that the solution is parabolically homogeneous of degree one, i.e. $u_0(rx, r^2t) = ru_0(x,t)$.

Next, let us analyze the solution to the Cauchy problem in $t > 0$ with initial datum $\psi_0 = \min\{(x_1)_+, (x_2)_+\}$. If this solution goes below the obstacle at some point, then it obviously implies that the solution to the obstacle problem, with this obstacle must touch the obstacle, and consequently at the set $\{x_1 = x_2 > 0\}$. Now it is not hard to realize that the solution to the above mentioned Cauchy problem has the property that $D_t u = \Delta u$, where $\Delta u < 0$ on the the set $\{t = 0,\ x_1 = x_2 > 0\}$. Hence, for very small t values our solution is pushed down from its position along the set $\{x_1 = x_2 > 0\}$. Therefore we conclude that the global solution to the obstacle problem must touch the obstacle at the set $\{x_1 = x_2 > 0\}$.

Let us now describe the exercise region (coincidence set), $u_0 = \psi_0$. We know from the discussion above that the exercise region is a subset of the set $G := \{t > 0,\ x_1 = x_2 > 0\}$. The question is whether it coincides with G. If it where so, then one has by uniform Lipschitz regularity of u_0 that the solution is bounded close to the set $(0, 0, t)$ for all $t > 0$. Since u_0 is also monotone non-decreasing (this is simple sliding and comparison principle) we may look at the limit $v_0(x) := \lim_{t\to\infty} u_0(x,t)$, which is a solution to the global elliptic obstacle problem in $\mathbf{R}^2$ with obstacle ψ_0.

It is not hard to show that such a solution does not exists. Indeed, due to homogeneity of degree one, $v_0(x) = r\phi_0(\theta)$, where (r, θ) denote the polar coordinates in the plane, we can apply Laplacian in polar coordinates to arrive at

$$\phi_0 + D^2_{\theta\theta}\phi_0 = 0, \qquad -3\pi/4 \le \theta < \pi/4, \quad \pi/4 < \theta \le 5\pi/4.$$

Next using the simple fact that ϕ_0 is symmetric across the line $x_1 = x_2$, and it is non-increasing for $5\pi/4 \le \theta \le \pi/4$, we should have

$$D^2_{\theta\theta}\phi_0(5\pi/4) \ge 0.$$

The above two equations imply $\phi_0(5\pi/4) = 0$, i.e. $\{\theta = 5\pi/4, r < 0\}$ is in the exercise region. This is a contradiction to what we have above.

This particular analysis implies that the global solution to the parabolic equation, must be so that as time increase then the graph of the solution function to the obstacle problem detaches from the obstacle and eventually it becomes infinity at every point, i.e.

$$\lim_{t\to\infty} u_0(x,t) = \infty$$

for all $x \in \mathbf{R}^2$. This along with the parabolic homogeneity implies, in particular, that the exercise region for the global solution can be represented as follows

$$\mathcal{E}(u_0) = \{0 < x_1 = x_2, \quad t \le c(x_1)^2\},$$

for some positive constant c.

Using ideas of the proof of Theorems 14, 1.6 in [Sha06], the conclusion from above analyzes is an accurate description of the free boundary close to maturity. We summarize this in the following theorem.

Theorem 6.1. *Let u be a solution to the American put option, with the payoff*

$$\psi(x,t) = \min\{(E - x_1)_+, (E - x_2)_+\},$$

and exercise time T. Denote by L the line $\{x_1 = x_2\}$ in the x_1x_2-plane. Then the exercise region $\mathcal{E}(u)$ along the line L lies above a parabola like region. More exactly, there are constants c_0, $r_0 > 0$ and a modulus of continuity σ such that

$$\mathcal{E}(u) \cap \{(s, s, T - t)\} \cap Q^-_{r_0}(E, E, T) \subset$$

$$\{(T - t) > (c_0 - \sigma(|x_1 - E|))(x_1 - E)^2_+\} \cap \{(s, s, T - t)\} \cap Q^-_r(E, E, T),$$

where $Q^+_{r_0}(E, E, T)$ is the lower half-cylinder $B_{r_0}(E, E) \times \{-r_0^2 + T < t < T\}$.

In financial terms the above theorem tells us that immediate exercise is optimal at a given point (x_1, x_1) close to maturity. Similarly, Theorem 5.7 tells us that at points close to $((s, as, E)$, with $a > 0$ and $a \neq 1$ exercise is suboptimal. There is a sub-optimality constant depending on a, above.

6.3. **Barrier Options.** In the literature, Barrier options (for American type contracts) are usually considered so that one avoids the free boundary, e.g. for the American put one considers up-an-out options, or down-and-in, and for American call one considers down-and-out options, and up-and-in.

Here we want to give some insight into the case of American down-and-out, with an appropriate rebate, which is to stay above the payoff. For simplicity of the argument let us again look at a reversed time, so that we are in the situation of our problem obstacle problem.

We consider a general situation as follows. In the obstacle problem, above, we assume that the boundary value $u = g(0, t) \geq \psi(0, t)$ is prescribed; for simplicity we assume only the one space-dimensional case. We also assume that in finite time the free boundary meets the t-axis. In application one may of course replace the boundary $x_1 = 0$ with any other boundary $x_1 = s$ and s below the exercise price E, so that the free boundary of the American type contract, starting at E moves towards $x_1 = s$, and meets this line in finite time. Naturally the constant s should be taken so that it stays above the exercise price at maturity. In mathematical terms (with forward parabolic equation) this means that we assume that the free boundary should appear on the right hand side of the line $\{x_1 = s\}$ and after some time it heats this line.

The question that we raise here, is how the free boundary approaches the fixed boundary $\{x_1 = s\}$. The qualitative feature we are looking for is whether the free boundary approaches the fixed one in a tangential or non-tangential way.

In [AUS03], the authors prove that, in the one-space dimension, the free boundary touches the fixed one only in a non-tangential way. They show this using an explicit barrier. However, in the same paper, the authors also show that the free boundary touches the fixed one in a parabolically-tangential way. More exactly one has the following: Take $s = 0$ for simplicity. Let Z, on the fixed boundary $x_1 = 0$, be also a free boundary, i.e. there exists a sequence $Z^j \to Z$ such that $u(Z^j) > g(Z^j)$, and $u(Z) = g(Z)$. Let us once again simplify things by assuming $Z = (0, 0)$. Then there exists r_0 such that

$$\mathcal{E}(u) \cap Q_{r_0} \subset \{(x, t) : \; 0 \leq x_1 \leq C|t|\} \cap Q_{r_0},$$

for some constant C.

The proof of this results is very much in the spirit of that of Theorem 5.4. We refer the reader to the papers [AUS03].

A similar behavior can be established for the American up-and-out call option. We refrain ourselves discussing it here.

Appendix A. Numerical Solutions of American Put Options by Teitur Arnarson

Below are presented pictures concerning the global solutions of the problem

$$\begin{aligned} -\partial_t u + \Delta u \le 0, \quad (-\partial_t u + \Delta u)(\psi - u) = 0, \\ u \ge \psi(x), \quad u(x,0) = \psi(x) \end{aligned} \tag{A.1}$$

in $\mathbf{R}^2 \times (0,T]$, where $\psi(x) = \min(x_1, x_2)^+$, and the two-dimensional Black-Scholes equation

$$\begin{aligned} \mathcal{L}V + \partial_t V \le 0, \quad (\mathcal{L}V + \partial_t V)(\psi - V) = 0, \\ V \ge \psi, \quad V(S,T) = \psi(S) \end{aligned} \tag{A.2}$$

in $(\mathbf{R}^+)^2 \times [0,T)$, where $\psi(S) = \min\{(E - S^1)_+, (E - S^2)_+\}$. The equations are solved numerically using finite difference schemes. For stability reasons we use implicit methods as follows. Let A be the finite difference coefficient matrix and denote the time index n. We then use Gaussian elimination to calculate u^{n+1} from $u^n = Au^{n+1}$ for the forward equation and V^n from $V^{n+1} = AV^n$ for the backwards equation.

Since numerical computations can only be carried out in finite domains we approximate $\mathbf{R}^2$ in (A.1) with $[-10,4] \times [-10,4]$ and $(\mathbf{R}^+)^2$ in (A.2) by $[0,10.5] \times [0,10.5]$. We then impose as boundary conditions the solution to the corresponding one-dimensional problem on the boundaries where the obstacle is greater then or equal to zero, and zero boundary conditions on the boundaries where the obstacle is zero. The boundary conditions are verified to be reasonable by solving the problem in a larger domain and using the values of this solution as boundary conditions for the smaller domain. The difference between the two approaches is insignificant.

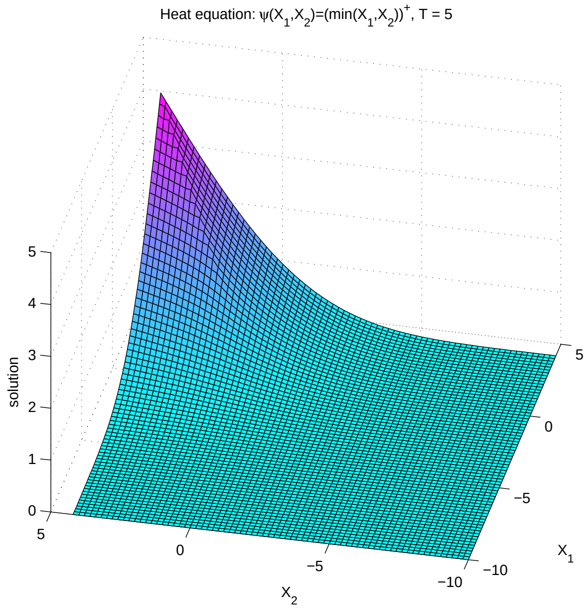

FIGURE 1. Approximation of global solution, $-\partial_t u + \Delta u \leq 0,\ u \geq \psi$

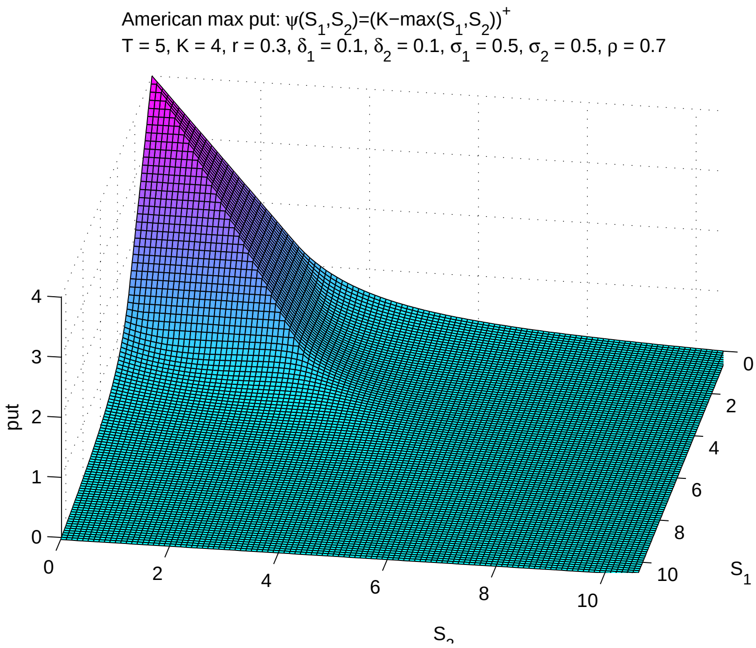

FIGURE 2. Price of the American max put option for given parameters, $\partial_t V + \mathcal{L}V \leq 0,\ V \geq \psi$.

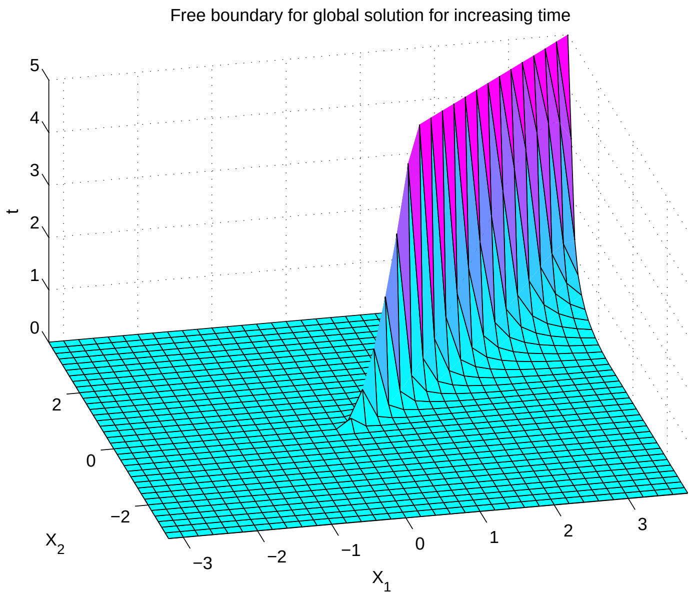

FIGURE 3. The contact set of the global solution for increasing time. It exists on the diagonal $x_1 = x_2$.

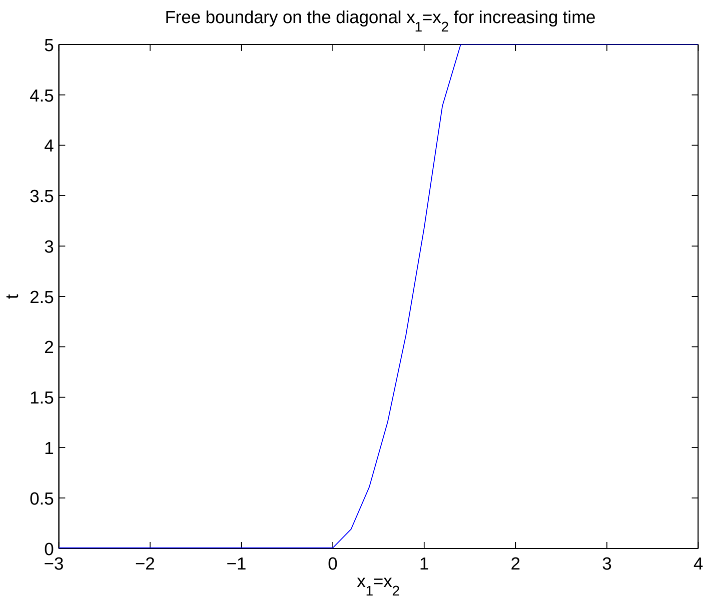

FIGURE 4. The contact set on the diagonal $x_1 = x_2$ for the global solution. It decreases as time increases. The picture is similar for the American max put on the diagonal.

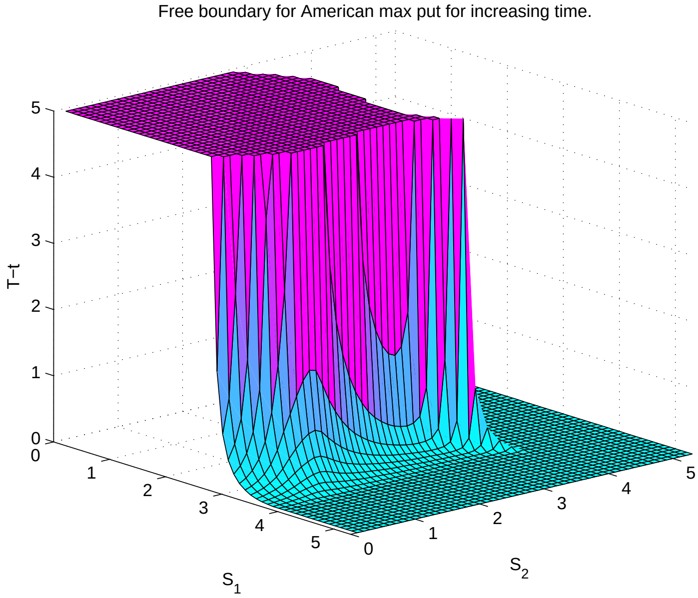

FIGURE 5. The exercise region (i.e. contact set) of the American max put for increasing time.

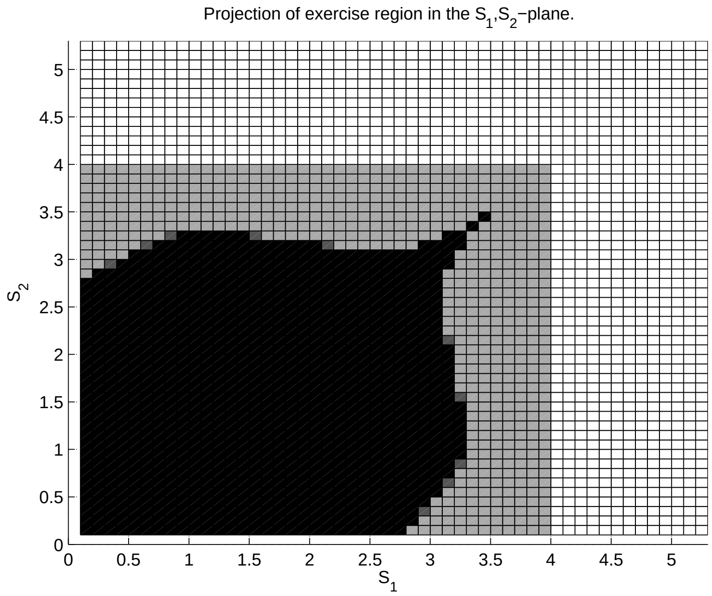

FIGURE 6. Projection of the exercise region of the American max put at time 0. The payoff function (i.e. the obstacle) is zero outside the gray region.

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