A TWO-PHASE PROBLEM WITH A LOWER-DIMENSIONAL FREE BOUNDARY

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Abstract. We study minimizers of the energy functional

$$\int_D |\nabla u|^2 + \int_{D\cap(\mathbb{R}^{n-1}\times\{0\})} \lambda^+\chi_{\{u>0\}} + \lambda^-\chi_{\{u<0\}}\ d\mathcal{H}^{n-1}$$

without any sign restriction on the function u. The main result states that the free boundaries

$$\Gamma^+ = \partial\{u(\,\cdot\,,0) > 0\} \text{ and } \Gamma^- = \partial\{u(\,\cdot\,,0) < 0\}$$

never touch and in dimension three they are C^1 regular on a dense subset.

1. Introduction

For a bounded domain D in $\mathbb{R}^n$ consider the problem of minimizing the energy functional

$$J(u) = \int_D |\nabla u|^2\ dx + \int_{D\cap(\mathbb{R}^{n-1}\times\{0\})} \lambda^+\chi_{\{u>0\}} + \lambda^-\chi_{\{u<0\}}\ d\mathcal{H}^{n-1} \tag{1.1}$$

among all functions $u \in W^{1,2}(D)$ with $u - u_0 \in W_0^{1,2}(D)$ for a prescribed u_0. We assume that λ^+ and λ^- are positive constants. The main objective of this paper is to study the local properties of the free boundaries

$$\Gamma^\pm = \Gamma_u^\pm \overset{\text{def}}{=} \partial\Omega_u^\pm \cap D, \quad \text{where } \Omega_u^\pm \overset{\text{def}}{=} \{\pm u > 0\} \cap (\mathbb{R}^{n-1}\times\{0\}).$$

The boundary here is defined by the topology of $\mathbb{R}^{n-1}\times\{0\}$, so formally it is of codimension two in $\mathbb{R}^n$.

This problem bears resemblance to the one of minimizing the functional

$$J(u) = \int_D |\nabla u|^2 + \lambda^+\chi_{\{u>0\}} + \lambda^-\chi_{\{u<0\}} \tag{1.2}$$

studied in [2], the same paper where the renown Alt-Caffarelli-Friedman monotonicity formula has been introduced. The minimizers of (1.2) are generalized solutions of a classical two-phase free boundary problem

$$\begin{aligned} \Delta u = 0 &\quad \text{in } \{u>0\}\cup\{u<0\} \\ |\nabla u_+|^2 - |\nabla u_-|^2 = M &\quad \text{on } \partial\{u>0\}\cup\partial\{u<0\}, \end{aligned} \tag{1.3}$$

with $M = \lambda^+ - \lambda^-$. In one particular application, the problem (1.3) appears in a simplified model for premixed equidiffusional flames, in the stationary case. More

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specifically, one considers the limits as $\epsilon \to 0+$ of a singular perturbation problem

$$\Delta u = \beta_\epsilon(u) \quad \text{in } D,$$

where the nonlinearities β_ϵ, $\epsilon > 0$, are supported in $[0, \epsilon]$ and have a fixed total energy $\int_0^\epsilon \beta_\epsilon(s) ds = M/2$, see e.g. [4].

When long range interactions are present, it is relevant to replace the Laplacian by nonlocal operators, such as the fractional Laplacian. See survey papers [11] and [3]. If one formally considers the equation

$$(-\Delta_{x'})^\alpha u = -\beta_\epsilon(u) \quad \text{on } \mathbb{R}^{n-1},$$

where $\Delta_{x'}$ is the Laplacian on $\mathbb{R}^{n-1}$ and $0 < \alpha < 1$, then in the case $\alpha = 1/2$ this equation can be rewritten as a boundary reaction problem

$$\Delta u = 0 \quad \text{in } \mathbb{R}^{n-1} \times (0, \infty),$$
$$\partial_{x_n} u = \beta_\epsilon(u) \quad \text{on } \mathbb{R}^{n-1} \times \{0\},$$

solutions of which can be found by minimizing a suitably smoothed version of the energy functional (1.1) on $\mathbb{R}^n$. Letting $\epsilon \to 0+$, we obtain thereby that the minimization problem (1.1) can be viewed as a “localized” version of the free boundary problem (1.3) for the half Laplacian $(-\Delta_{x'})^{1/2}$.

The one-phase version of our problem (i.e. nonnegative minimizers) has been recently considered in [6]. The authors of [6], in fact, consider the analogous problem for all fractional powers of Laplacian, by using the extension of Caffarelli and Silvestre [5]. While our study of the two-phase problem is only for $\alpha = 1/2$, there are more technical tools available at our disposal (such as the Alt-Caffarelli-Friedman monotonicity formula) which allows us to obtain richer results.

Main Results and Outline of Paper. The main results obtained in this paper are as follows.

- *Existence of minimizers.* In Section 2 we show the existence of minimizers (Theorem 2.1), including the maximal and minimal ones for the given boundary data (Theorem 2.3).
- *Optimal regularity.* In Section 3, we show that the bounded minimizers are in fact $C^{1/2}$ regular (Theorem 3.1). This is the best regularity possible since in fact $C \operatorname{Re}(x_{n-1} + i|x_n|)^{1/2}$ is a minimizers for appropriately chosen constant C (Theorem 9.2).
- *Convergence properties.* Having the optimal regularity, in Section 4 we study the convergence properties of sequences of minimizers (Theorem 4.1), including the strong convergence in $W^{1,2}$ (Theorem 4.2).
- *Nondegeneracy.* In Section 5 we show that the minimizers cannot decay faster than the square root of the distance from the free boundaries, in both phases, (Theorem 5.1), even restricted to the thin space D' (Theorem 5.5). As a consequence, we obtain that $\Omega_u^\pm$ satisfy a $\mathcal{H}^{n-1}$-density property (Theorem 5.7), which implies that $\mathcal{H}^{n-1}(\Gamma^\pm) = 0$.
- *Separation of phases.* In Section 6 we prove an unexpected result that that the two phases Ω_u^+ and Ω_u^- are separated in a sense that $\Gamma^+ \cap \Gamma^- = \emptyset$, and that in fact the minimizers don't change sign in solid neighborhoods of points on $\Gamma^\pm$ (Theorem 6.1). This effectively reduces the two-phase problem to an one-phase problem, at least for the study of the local properties of the free boundary.

The proof is obtained by the application of Alt-Caffarelli-Friedman monotonicity formula.

This result is in complete contrast with two-phase free-boundary problem (1.3), where the two-phase points create a major complication even in the proof of the optimal (Lipschitz in that case) regularity of solutions, see [2].

- $\mathcal{H}^{n-3/2}$ *measure of the free boundary* In Section 7 we show that the free boundary has $\mathcal{H}^{n-3/2}$ measure zero (Theorem 7.1). This result is not optimal, but it is a simple corollary for an estimate on $\mathcal{H}^{n-1}$-density of Δu on D' (Lemma 7.2), that is instrumental for the remaining part of the paper.
- *Monotonicity formula and blow-ups.* In Section 8 we prove a Weiss-type monotonicity formula (Theorem 8.1). It has an immediate corollary that the blow-ups are homogeneous of degree 1/2, see Section 9. We then give a characterization of so-called regular free boundary points (i.e. the points where the blow-up has a flat free boundary) in terms of the Weiss energy functional (Theorem 9.4). The proofs are heavily based on the use of Steiner symmetrization.
- *Regularity of the free boundary in dimension $n = 3$.* In Section 10 we prove the main result of the paper. We restrict ourselves to dimension three, and therefore the free boundary will be contained in $\mathbb{R}^2 \times \{0\}$. We use the Alexandrov reflection technique to prove that the set of regular points is relatively open subset of the free boundary, and is locally a C^1 curve (Theorem 10.1).

Notation and Terminology. Throughout the paper we will use the following notation.

- We denote a point $x \in \mathbb{R}^n$ by (x', x_n) where $x' = (x_1, \dots, x_{n-1})$.

- For $\alpha \in \mathbb{R}$, we define $\alpha_\pm = \max(\pm\alpha, 0)$, the positive and negative parts of α, so that we have $\alpha = \alpha_+ - \alpha_-$.

- For any set $\Omega \subset \mathbb{R}^n$, we define

$$\Omega' \stackrel{\text{def}}{=} \Omega \cap (\mathbb{R}^{n-1} \times \{0\})$$

We will refer to $\mathbb{R}^{n-1} \times \{0\}$ as the thin space.

- The balls $B = B_r(x) = \{y \in \mathbb{R}^n \mid |x - y| < r\}$ will be often referred to as *solid balls*; whereas, $B' = B'_r(x') = \{y' \in \mathbb{R}^{n-1} \mid |y' - x'| < r\}$ will be referred to as the *thin balls*.

- The unit sphere in dimension n will be denoted by S^{n-1}.

- The spherical coordinates $(r, \theta_1, \dots, \theta_{n-1}) \in (0, \infty) \times (0, 2\pi) \times (0, \pi) \times \dots \times (0, \pi)$ for a nonzero point $x = (x_1, x_2, \dots, x_n)$ are defined by the relations

$$\begin{aligned} r &= |x| \\ x_n &= r \cos\theta_{n-1} \\ &\dots \\ x_{n-k} &= r \sin\theta_{n-1} \sin\theta_{n-2} \cdots \sin\theta_{n-k} \cos\theta_{n-k-1} \\ &\dots \\ x_1 &= r \sin\theta_{n-1} \sin\theta_{n-2} \cdots \sin\theta_1. \end{aligned}$$

- We will call the sets $\Omega_u^\pm \stackrel{\text{def}}{=} \{\pm u > 0\} \cap (\mathbb{R}^{n-1} \times \{0\})$ positive and negative *phases* of u and $\Lambda_u \stackrel{\text{def}}{=} \{u = 0\} \cap (\mathbb{R}^{n-1} \times \{0\})$ the *coincidence set*. The *free boundary* Γ_u is the union of Γ_u^+ and Γ_u^-, where $\Gamma_u^\pm \stackrel{\text{def}}{=} \partial\Omega_u^\pm \cap D$.

- It is useful when studying local properties of free boundary points to consider the *rescalings* at $x_0 \in \Gamma_u$.

$$u_r \stackrel{\text{def}}{=} \frac{u(rx + x_0)}{r^{1/2}}.$$

It is easy to see that the rescalings are still minimizers of the functional J. When we let $r \to 0$, the process is known as *blow-up*. If for a certain subsequence $r \to 0$, u_r converges to u_0 (in a certain sense) we will also refer to u_0 as a blow-up.

2. Existence

We say that u is a minimizer of the functional J in (1.1) if

$$J(u) \le J(v), \quad \text{for } v \in u + W_0^{1,2}(D). \tag{2.1}$$

Many of the results in this paper can be generalized also for local minimizers, for which (2.1) is satisfied with v such that $\operatorname{supp}(u - v) \Subset D$ and $\operatorname{diam} \operatorname{supp}(u - v) < \delta$ for some $\delta > 0$.

Throughout the paper we will assume that the domain D is bounded and that the subdomains $D^{\pm} = D \cap \{\pm x_n > 0\}$ have Lipschitz boundaries. This guarantees, for instance, that the trace operator $W^{1,2}(D) \to L^2(D')$ is compact.

The next theorem establishes the existence of minimizers with a given Sobolev trace on ∂D.

Theorem 2.1 (Existence)**.** *For any $\phi \in W^{1,2}(D)$ there exists a minimizer u to the functional J in the class $\mathfrak{K} = \phi + W_0^{1,2}(D)$.*

Proof. $J(v) \ge 0$, so there exists a minimizing sequence $\{u_k\}$. Since $\|\nabla u_k\|_{L^2(D)}$ is bounded, and $u_k|_{\partial D} = \phi$, we obtain

$$\|u_k\|_{W^{1,2}(D)} \text{ is bounded.}$$

Hence, we may extract a further subsequence s.t.

$$u_k \rightharpoonup u \text{ in } W^{1,2}(D).$$

It is clear that $u - \phi \in W_0^{1,2}$, so $u \in \mathfrak{K}$. Furthermore, since the trace operator $v \mapsto v|_{D'}$ is compact we may pass to a further subsequence to obtain

$$u_k \to u \text{ in } L^2(D'), \quad \text{and} \quad u_k \to u \quad \mathcal{H}^{n-1}\text{a.e. in } D'.$$

Then there exist two functions $\gamma^{\pm}$ with $0 \le \gamma^{\pm} \le 1$ such that

$$\chi_{\{u_k>0\}} \overset{*}{\rightharpoonup} \gamma^{+} \text{ and } \quad \chi_{\{u_k<0\}} \overset{*}{\rightharpoonup} \gamma^{-}.$$

Since $\int_{D'} u_k^{\pm}(1-\chi_{\{\pm u_k>0\}}) = 0$, passing to the limit, we obtain $\int_{D'} u^{\pm}(1-\gamma^{\pm}) = 0$, which implies that

$$\chi_{\{u>0\}} \le \gamma^{+}, \quad \chi_{\{u<0\}} \le \gamma^{-} \quad \mathcal{H}^{n-1} \text{ a.e. in } D'.$$

We then obtain:

$$\begin{aligned}
\int_D |\nabla u|^2 &+ \int_{D'} \lambda^{+}\chi_{\{u>0\}} + \lambda^{-}\chi_{\{u<0\}} \\
&\le \lim_{k\to\infty} \int_D |\nabla u_k|^2 + \int_{D'} \lambda^{+}\gamma^{+} + \lambda^{-}\gamma^{-} \\
&= \lim_{k\to\infty} \int_D |\nabla u_k|^2 + \lim_{k\to\infty} \int_{D'} \lambda^{+}\chi_{\{u_k>0\}} + \lambda^{-}\chi_{\{u_k<0\}}
\end{aligned}$$

Hence u is a minimizer. ∎

Note that since the functional J is not convex, we may not necessarily have the uniqueness of the solution, and in general we may not necessarily conclude that if u and v are two minimizers with $u \leq v$ on ∂D, then $u \leq v$ in D. Instead we have the following lemma.

Lemma 2.2 (Lattice Property). *Let u, v be two minimizers of the functional J in a domain D with $u|_{\partial D} \leq v|_{\partial D}$. If we define $\overline{w} \equiv \max\{u, v\}$ and $\underline{w} \equiv \min\{u, v\}$, then $\overline{w}$ and $\underline{w}$ are minimizers of the functional J with boundary values v and u respectively.*

Proof. It is fairly straightforward to check that

$$J(w_1) + J(w_2) = J(u) + J(v)$$

Since $\overline{w}|_{\partial D} = v$ and $\underline{w}|_{\partial D} = u$, we conclude that $\overline{w}$ and $\underline{w}$ are minimizers of the functional J. □

Theorem 2.3. *For any $\phi \in W^{1,2}(D)$ there exists a maximal (minimal) minimizer u^* (u_*) of J on D with boundary data ϕ on ∂D such that $v \leq u^*$ ($v \geq u_*$) for all other minimizers v with $v|_{\partial D} = \phi$.*

Moreover, if u_1^* and u_2^* are maximal minimizers corresponding to boundary data ϕ_1 and ϕ_2 on ∂D and such that $\phi_1 \leq \phi_2$ then $u_1^* \leq u_2^*$. Similar statement holds for minimal minimizers.

Proof. The existence of u^* and u_* is obtained by the limiting procedure by using the lattice property, similar to the standard Perron method.

The second part of the theorem is a direct consequence of the lattice property. □

Corollary 2.4. *If D and the boundary data ϕ on ∂D are symmetric about the line $(0, \ldots, 0, x_n)$, then so will be u^* and u_*.*

Proof. By theorem 2.3, if R is any rotation about the line $(0, \ldots, 0, x_n)$, then $u^* \circ R$ is a minimizer with the same boundary data and therefore $u^* \circ R \leq u^*$. This is possible only if u^* is symmetric about $(0, \ldots, 0, x_n)$. The same proof holds for u_*. □

3. Optimal Regularity

In this section we show the Hölder-1/2 regularity of minimizers. This regularity is suggested by the natural scaling of the problem. Namely, if u is a minimizer of J, then $u_r(x) = u(rx + x_0)/r^{1/2}$ is still a minimizer of J with the respective boundary data.

Theorem 3.1 (Hölder-1/2 Regularity). *Let u be a minimizer of J in B_1 with $\|u\|_{L^\infty(\partial B_1)} \leq M$. Then*

$$\|u\|_{C^{1/2}(B_{1/2})} \leq C,$$

where C is a constant depending only on n, M, and ρ.

Remark 3.2. In the above theorem we only need to control $\|u\|_{L^\infty}$ on the boundary of the ball, since it is straight forward to show that if $\|u\|_{L^\infty(\partial B_1)} \leq M$, then $\|u\|_{L^\infty(B_1)} \leq M$. Similarly, we note that if $u|_{\partial B_1} \geq 0$ (≤ 0) then $u \geq 0$ (≤ 0) in all of B_1.

To prove the above theorem, we will need a Caccioppoli inequality. Without assuming apriori that u is continuous, we do not necessarily know that u_+ and u_- are subharmonic. Instead we prove the Caccioppoli inequality directly from the fact that u is a minimizer.

Lemma 3.3. *Let u be a minimizer of the functional J in B_{2r}. Then*

$$\int_{B_r} |\nabla u|^2 \ \leq \frac{C}{r^2} \int_{B_{2r}} u^2,$$

where C is a constant depending only on the dimension n.

Proof. Choose a cut-off function $\eta \in C_0^\infty(B_{2r})$ such that

$$0 \leq \eta \leq 1, \qquad \eta \equiv 1 \text{ on } B_r, \qquad |\nabla \eta| \leq \frac{C_n}{r}$$

and consider a competing function $u^\epsilon = u + \epsilon u_+ \eta^2$ for a small $\epsilon > 0$. Note that $\{u^\epsilon > 0\} = \{u > 0\}$ and $\{u^\epsilon < 0\} = \{u < 0\}$, besides $u^\epsilon = u$ on ∂B_{2r}. Therefore, from the minimality of u, we must have

$$\int_{B_{2r}} |\nabla u|^2 \leq \int_{B_{2r}} |\nabla(u + \epsilon u_+ \eta^2)|^2,$$

which by letting $\epsilon \to 0+$ yields

$$\int_{B_{2r}} \nabla u \nabla(u_+ \eta^2) \leq 0.$$

Proceeding as in the standard proof of the Caccioppoli inequality, we arrive at

$$\int_{B_r} |\nabla u_+|^2 \leq \frac{C}{r^2} \int_{B_{2r}} u_+^2.$$

Similar inequality holds also for u_-. This completes the proof of the lemma. $\square$

We are now able to prove the optimal regularity of minimizers.

Proof of Theorem 3.1. Let $B = B_r(y)$ be a ball contained in B_1 and v the harmonic function such that $v|_{\partial B} = u$. Then since $J(u) \leq J(v)$ and v is harmonic,

$$\begin{aligned}
\int_B |\nabla(u - v)|^2 &= \int_B |\nabla u|^2 - |\nabla v|^2 \\
&\leq \int_{B'} \lambda^+ (\chi_{\{v>0\}} - \chi_{\{u>0\}}) \ + \lambda^- (\chi_{\{v<0\}} - \chi_{\{u<0\}}) \\
&\leq C(n, \lambda^+, \lambda^-) r^{n-1}
\end{aligned}$$

Now let $M = \|u\|_{L^\infty(\partial B_1)}$, and $B = B_r(y)$ with $y \in B_{7/8}$ and $0 < r \leq 1/16$. Then by the Caccioppoli inequality

$$\int_B |\nabla u|^2 \leq \ \frac{C}{r^2} \int_{2B} u^2 \ \leq C M^2 r^{n-2}$$

Since v is harmonic, we have the following well known inequality

$$\sup_{1/2B} |\nabla v| \leq \left(\frac{C}{r^n} \int_B |\nabla v|^2 \right)^{1/2}$$

and since v is harmonic, we also know

$$\int_B |\nabla v|^2 \leq \int_B |\nabla u|^2$$

Then we may conclude that

$$\sup_{1/2B} |\nabla v| \leq \frac{CM}{r}$$

Let $\rho = r^2$. Then

$$\begin{aligned}
\|\nabla u\|_{L^2(B_\rho(y))} &\leq \|\nabla(u-v)\|_{L^2(B_{r^2}(y))} + \|\nabla v\|_{L^2(B_{r^2}(y))} \\
&\leq C\rho^{\frac{n-1}{2}} + Cr^{2n/2}\|\nabla v\|_{L^\infty(B_{r/2}(y))} \\
&\leq C(\rho^{\frac{n-1}{2}} + CMr^{n-1}) \\
&\leq C\rho^{\frac{n-1}{2}}
\end{aligned}$$

Applying Morrey's theorem, see e.g. [9] we conclude that u is Hölder- 1/2. □

Remark 3.4. Now that we know that minimizers are continuous and we may use first variation to conclude that u is harmonic on the set $\{u \neq 0\} \cup D^+ \cup D^-$. In particular, we also obtain that u_+ and u_- are continuous subharmonic functions in entire D.

4. Convergence of Minimizers

In this section we have collected some results on the convergence of sequences of minimizers, that are going to be important in blow-up analysis and compactness type arguments throughout the paper.

Theorem 4.1 (Convergence of minimizers)**.** *Let $\{u_k\}$ be a sequence of minimizers of the functional J in the domain D with $\|u_k\|_{L^\infty(\partial D)} \leq M$. Then there exists a subsequence and a function u_0 such that for every open $U \Subset D$*

(1) $u_0 \in W^{1,2}(U) \cap C^{1/2}(\overline{U})$

(2) $u_k \to u_0$ *in* $C^\alpha(\overline{U})$ *for* $\alpha < 1/2$

(3) $u_k \rightharpoonup u_0$ *in* $W^{1,2}(U)$

(4) u_0 *is a minimizer of* J *in* U.

Proof. Properties (1)–(3) follow immediately from Lemma 3.3 and Theorem 3.1. So we will concentrate on the proof of (4). We must show $J(u_0) \leq J(u_0 + \psi)$ for $\psi \in W_0^{1,2}(U)$. Since minimizers exist and are Hölder-1/2 continuous, we only have to show the inequality for ψ continuous. Choose a cut-off function $\eta \in C_0^\infty(D)$ such that

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on a neighborhood of } \overline{U}.$$

For the minimizer u_k consider the following competing function:

$$v_k^\epsilon = (u_k + \psi - \epsilon\eta)_+ - (u_k + \psi + \epsilon\eta)_-.$$

Then we have

$$\begin{aligned}
J(u_k) &\leq J(v_k^\epsilon) \\
&\leq \int_D |\nabla(u_k+\psi)|^2 + \int_{\operatorname{supp}\eta\setminus\overline{U}} (2\epsilon|\nabla u_k||\nabla\eta| + \epsilon^2|\nabla\eta|^2) \\
&\quad + \int_{D'} \lambda^+\chi_{\{u_k+\psi-\epsilon\eta>0\}} + \lambda^-\chi_{\{u_k+\psi+\epsilon\eta<0\}}.
\end{aligned}$$

Now since $\nabla u_k \rightharpoonup \nabla u$, and $u_k \to u$ uniformly we obtain in the limit that

$$J(u) \le \int_D |\nabla(u+\psi)|^2 + \int_{D'} \lambda^+ \chi_{\{u+\psi>0\}} + \lambda^- \chi_{\{u+\psi<0\}} + C\epsilon$$

Letting $\epsilon \to 0$ we obtain $J(u) \le J(u+\psi)$. □

Our next result strengthens the convergence given in part (4) of Theorem 4.1 from weak convergence in $W^{1,2}$ to strong convergence, so minimizers under the assumptions of Theorem 4.1 will be locally compact in $W^{1,2}$.

Theorem 4.2 (Strong Convergence)**.** *Let $\{u_k\}$ and u_0 be as in Theorem 4.1. Then, over a subsequence, $u_k \to u_0$ strongly in $W^{1,2}(U)$ for any open $U \Subset D$.*

To prove this theorem, we will need the following result on the structure of Δu for the minimizer u.

Lemma 4.3. *Let u be a minimizer of the functional J. Then Δu is a signed Radon measure supported in $\Lambda = \{u = 0\} \cap D'$ with the total variation $|\Delta u|$ satisfying*

$$\langle |\Delta u|, \chi_K \rangle \le C(n,K) \|\nabla u\|_{L^2(D)}$$

for any $K \Subset D$. Moreover, Δu is absolutely continuous with respect to $\mathcal{H}^{n-1}|_\Lambda$ and

$$\Delta u = \omega \mathcal{H}^{n-1}|_{\Lambda\setminus\Gamma}, \quad \omega = \frac{\partial u}{\partial x_n} + \frac{\partial u}{\partial x_{-n}}$$

or equivalently, for $\phi \in C_0^\infty(D)$,

$$\langle \Delta u, \phi \rangle = \int_{\Lambda\setminus\Gamma} \phi\, \omega \; d\mathcal{H}^{n-1}.$$

In the statement of the lemma above we have used the notation

$$\frac{\partial u}{\partial x_n}(x_0,0) = \lim_{h\to 0} \frac{u(x_0,h) - u(x_0,0)}{h}$$
$$\frac{\partial u}{\partial x_{-n}}(x_0,0) = \lim_{h\to 0} \frac{u(x_0,-h) - u(x_0,0)}{-h}$$

where $h > 0$, which exist at every point $x_0 \in D' \setminus \Gamma$.

Proof. The functions $u_\pm$ are nonnegative, continuous, harmonic where positive. Hence $u_\pm$ are subharmonic in B_1, implying that $\Delta u_\pm$ are nonnegative Radon measures, and consequently that $\Delta u = \Delta u_+ - \Delta u_-$ is a signed Radon measure. Besides, we know that u is harmonic in $B_1 \setminus \Lambda$ which implies that Δu lives on Λ. The quantitative estimate on $|\Delta u|$ follows from a standard argument for subharmonic functions. Indeed, let $\eta \in C_0^\infty(B_1)$ be a nonnegative cut-off function such that

$$\eta \equiv 1 \quad \text{on } K, \quad \|\nabla \eta\|_{L^2(D)} \le C(n,K).$$

Then

$$\langle \Delta u_\pm, \chi_K \rangle \le \langle \Delta u_\pm, \eta \rangle = -\int_D \nabla u_+ \nabla \eta \le C(n,K) \|u\|_{L^2(D)}$$

and the claimed estimate follows.

For the second part of the theorem, we will essentially prove the divergence theorem directly. If fact, we will need to jump ahead and use the fact that $\mathcal{H}^{n-1}(\Gamma) = 0$, see Corollary 5.8. (We just note here that the proof of Theorem 5.7 and Corollary 5.8 is independent of Lemma 4.3.)

We first break up the integral over D as follows:

$$\int_D u\Delta\phi = \int_{D_+} u\Delta\phi + \int_{D_-} u\Delta\phi$$

We now break up the Laplacian as:

$$\int_{D_+} u\Delta\phi = \int_{D_+} u\frac{\partial^2\phi}{\partial x_n^2} + \sum_{i=1}^{n-1}\int_{D_+} u\frac{\partial^2\phi}{\partial x_i^2}$$

We now use iterated integrals as follows:

$$\int_{D_+} u\frac{\partial^2\phi}{\partial x_n^2} = \int_{D'\setminus\Gamma}\int_0^\infty u\frac{\partial^2\phi}{\partial x_n^2}$$

(We may integrate over $D'\setminus\Gamma$ since $\mathcal{H}^{n-1}(\Gamma) = 0$). We are now able to use integration by parts on each line to obtain

$$\int_{D'\setminus\Gamma}\int_0^\infty u\frac{\partial^2\phi}{\partial x_n^2} = \int_{D'\setminus\Gamma} \frac{\partial u}{\partial x_n}\phi - \frac{\partial\phi}{\partial x_n}u - \int_{D_+}\frac{\partial^2 u}{\partial x_n^2}\phi$$

Also

$$\sum_{i=1}^{n-1}\int_{D_+} u\frac{\partial^2\phi}{\partial x_i^2} = \sum_{i=1}^{n-1}\int_{D_+}\phi\frac{\partial^2 u}{\partial x_i^2}$$

Then

$$\int_{D_+} u\Delta\phi = \int_{D_+}\phi\Delta u + \int_{D'\setminus\Gamma}\frac{\partial u}{\partial x_n}\phi - \frac{\partial\phi}{\partial x_n}u$$

Now u is harmonic and differentiable off the coincidence set, so when we add the integral over D_- we get

$$\int_D u\Delta\phi = \int_{\Lambda\setminus\Gamma}\left(\frac{\partial u}{\partial x_n} + \frac{\partial u}{\partial x_{-n}}\right)\phi. \qquad \square$$

We can now prove the strong convergence.

Proof of Theorem 4.2. Take a test function $\eta \in C_0^\infty(D)$ such that

$$0 \le \eta \le 1 \quad \eta \equiv 1 \text{ in a neighborhood of } \overline{U}.$$

Next, suppose that k is so large that $|u_k - u| < \epsilon$ on $\operatorname{supp}\eta$. Then we will have

$$|\langle\Delta(u_k - u), (u_k - u)\eta^2\rangle| \le \epsilon(\langle|\Delta u_k|, \eta^2\rangle + \langle|\Delta u|, \eta^2\rangle) \le C\epsilon,$$

where C depends only on L^2 norms of ∇u_k and ∇u on D, by Lemma 4.3. Therefore

$$\int_D |\nabla(u_k - u)|^2\eta^2 \le C\epsilon - 2\int_D \eta(u_k - u)\langle\nabla(u_k - u), \nabla\eta\rangle$$

and applying Young's inequality, we arrive at

$$\int_U |\nabla(u_k - u)|^2 \le C\epsilon + C\int_{\operatorname{supp}\eta}(u_k - u)^2.$$

$\square$

5. Nondegeneracy

As mentioned in the introduction, in order to study local properties of the free boundary it is useful to study so called blow-ups. If u is a minimizer in $B_1(x_0)$, then

$$u_r(x) \stackrel{\text{def}}{=} \frac{u(x_0 + rx)}{r^{1/2}}$$

is a minimizer in $B_{1/r}$. Theorems 3.1 and 4.1 guarantee that if x_0 is a free boundary point and we let $r \to 0$, we may extract a subsequence such that $u_r \to u_0$. This blow-up process is in essence an infinite zoom. The basic idea is to transfer properties of u_0 to the free boundary of u close to x_0. It is not immediately obvious if u_0 could be degenerate, that is $u_0 \equiv 0$. If u_0 were degenerate then we would not be able to gain much insight for the free boundary of u near x_0. Theorem 5.1 will guarantee that u_0 will not be degenerate.

We now state the main theorem of this section.

Theorem 5.1 (Nondegeneracy)**.** *Fix $0 < t < 1$, and let u be a minimizer of J in B_r. There exists $\epsilon > 0$ with ϵ depending only on $\{\lambda^+, \lambda^-, t\}$ such that if $u|_{\partial B_r} \leq \epsilon r^{1/2}$ $(u|_{\partial B_r} \geq -\epsilon r^{1/2})$ then*

$$u(x) \leq 0 \quad (u(x) \geq 0) \qquad \text{for } x \in B'_{tr}$$

To prove this theorem, we will need the following estimate.

Lemma 5.2. *Fix $0 < \kappa < 1$. If u is a minimizer on B_1 with $u \leq M$ on ∂B_1, then*

$$\int_{B'_\kappa} \lambda^+ \chi_{\{u>0\}} \leq \; M^2 C_{n,\kappa},$$

where $C_{n,\kappa}$ is a constant depending only on dimension n and κ.

Proof. For $0 < \kappa < 1$, let ϕ_κ be the solution to

$$\phi_\kappa|_{\partial B} = 1, \quad \phi_\kappa|_{B'_\kappa} = 0, \quad \Delta\phi_\kappa(x) = 0 \text{ for } x \notin B'_\kappa.$$

We first note that the solution ϕ_κ exists since $B \setminus B'_\kappa$ is a regular domain for the Dirichlet problem by the Wiener criterion. Let $v \equiv M\phi_\kappa$ and $w \equiv \min\{u, v\}$. Then $J(u) \leq J(w)$, and by grouping similar terms we find that

$$\int_{D' \cap \{v=0\}} \lambda^+ \chi_{\{u>0\}} \leq \int_{D \cap \{u>v\}} |\nabla v|^2 - |\nabla u|^2. \tag{5.1}$$

Note that we have used that $u^- = w^-$. Now we use that u is harmonic in the open set $\{u > v\}$ so that

$$\int_{\{u>v\}} |\nabla u|^2 = \int_{\{u>v\}} \langle \nabla u, \nabla v \rangle$$

Substituting this into inequality (5.1) gives

$$\int_{D' \cap \{v=0\}} \lambda^+ \chi_{\{u>0\}} \leq \int_{\{u>v\}} \langle \nabla v, \nabla(v-u) \rangle = \int_{B'_\kappa} (\Delta v) u = \langle \Delta v, u \rangle$$

Here, Δv is a nonnegative Radon measure in B_1 whose support is $\{v = 0\} = B'_\kappa$ (see e.g. proof of Lemma 4.3). Now

$$\langle \Delta v, u \rangle \leq \; M^2 \langle \Delta\phi_\kappa, \chi_{B'_\kappa} \rangle \leq \; M^2 C_{n,\kappa}. \qquad \square$$

Next lemma improves the statement of Corollary 2.4 when the boundary data is constant.

Lemma 5.3. *Let u be a minimizer of J on B_1 such that the values of $u|_{\partial B_1} = M > 0$. Then u is symmetric about the line $(0, \dots, 0, x_n)$, and the coincidence set $\Lambda = \{u = 0\} \cap B_1'$ is connected and centered at the origin.*

Proof. Extend u to be a function on the cube Q_1 with side length 2, by defining $u(x) = M$ for $x \notin B_1$. We now apply Steiner symmetrization (as defined in [8, page 82]) to the function $w = M - u$ on lines parallel to $\mathbb{R}^{n-1} \times \{0\}$. If we only consider $\{x \mid |x_n| > \epsilon\}$, then w is Lipschitz. Then by [8, page 82], if we Steiner symmetrize w to obtain v, we get:

$$\int_{B_1 \cap \{|x_n| > \epsilon\}} |\nabla u|^2 = \int_{B_1 \cap \{|x_n| > \epsilon\}} |\nabla w|^2 \geq \int_{B_1 \cap \{|x_n| > \epsilon\}} |\nabla v|^2$$

Equality is only achieved if w (and hence u) is already Steiner symmetric along the lines we symmetrize. Furthermore, v will have the same boundary values as w on ∂B. Then by letting $\epsilon \to 0$ we obtain

$$\int_{B_1} |\nabla u|^2 = \int_{B_1} |\nabla w|^2 \geq \int_{B_1} |\nabla v|^2$$

Finally, we see that $\mathcal{H}^{n-1}(\{u = 0\})$ is invariant under Steiner symmetrization. Then by a limiting process, we see that u is a minimizer if and only if u is symmetric about the line $(0, \dots, 0, x_n)$ and $\{u = 0\}$ is connected and centered at the origin. □

We are now able to prove the nondegeneracy result.

Proof of Theorem 5.1. First we note that by rescaling we only need to prove Theorem 5.1 on the unit ball B_1. Also, Theorem 2.3 reduces Theorem 5.1 to proving the theorem for the maximal minimizer u_ϵ^* where $u_\epsilon^*|_{\partial B} = \epsilon$. Lemma 5.3 proves that $\{u_\epsilon^* = 0\} = B_\rho'$ for some $\rho < 1$. Lemma 5.2 shows

$$\int_{B_\kappa'} \lambda^+ \chi_{\{u_\epsilon^* > 0\}} \to 0 \text{ as } \epsilon \to 0$$

Then there exists ϵ depending only on $\{t, \lambda^+\}$ such that if $u|_{\partial B_1} = \epsilon$ then

$$u|_{B_t'} = 0$$

The case for which $u \geq -\epsilon$ is proven similarly. □

Corollary 5.4. *If u is a minimizer and $0 \in \Gamma^+$ $(0 \in \Gamma^-)$, then*

$$\sup_{\partial B_r} u \geq C r^{1/2} \qquad \left(\inf_{\partial B_r} u \leq -C r^{1/2} \right), \tag{5.2}$$

where C depends only on λ^+, λ^- and n.

We have nondegeneracy for the solid ball. This next theorem will give us nondegeneracy in the thin space. We will need this result to prove the positive density of the free boundary. We note that Theorems 5.5 and 5.7 have already been proven in [6] for the one-phase problem.

Theorem 5.5 (Nondegneracy in the thin space)**.** *Let u be a minimizer of J in B_r with $\|u\|_{C^{1/2}(B_r)} \leq M$. Then there exists $C > 0$ and $0 < \rho < 1$, depending only on n, M, and λ^+ (λ^-), such that if $0 \in \overline{\Omega_u^+}$ $(0 \in \overline{\Omega_u^-})$ then*

$$\sup_{B_r' \setminus B_{\rho r}'} u \geq C r^{1/2} \qquad \left(\inf_{B_r' \setminus B_{\rho r}'} u \leq -C r^{1/2} \right).$$

Proof. By rescaling, it is enough to prove the lemma for the case when $r = 1$. Suppose first that $0 \in \overline{\Omega_u^+}$ and $u \le 0$ on $B_1' \setminus B_\rho'$ with $\rho < 1/4$ to be chosen later. Let $\phi_{1/2}$ be as defined in Lemma 5.2. Then as before:

$$\lambda^+ \int_{B'_{1/2}} \chi_{\{u>0\}} \le M \int_{B'_{1/2}} (\Delta\phi_{1/2})u \tag{5.3}$$

Now, $\Delta\phi_{1/2}$ is a nonnegative Radon measure, with a support in $\overline{B}'_{1/2}$ and in fact

$$\Delta\phi_{1/2} = 2\frac{\partial\phi_{1/2}}{\partial x_n}\,\mathcal{H}^{n-1}|_{B'_{1/2}}.$$

Moreover, it is easy to see that $\partial\phi_{1/2}/\partial x_n \le C_n$ on $B'_{1/4} \supset B'_\rho$. Using also that $u \le M\rho^{1/2}$ on B'_ρ, we can write

$$\lambda^+ \int_{B'_{1/2}} \chi_{\{u>0\}} \le C_n M\rho^{1/2} \int_{B'_{1/2}} \chi_{\{u>0\}}.$$

So by making ρ small enough we would obtain that $u \le 0$ on B'_ρ. This would be a contradiction since $0 \in \Gamma^+$.

Suppose by way of contradiction that there exists a sequence $\{u_k\}$ of minimizers in B_1 with $\|u_k\|_{C^{1/2}(B_1)} \le M$ and $0 \in \overline{\Omega^+_{u_k}}$ such that

$$u_k(x',0) < \frac{1}{k}|x'|^{1/2}, \text{ for all } (x',0) \text{ with } \rho \le |x'| \le 1.$$

Then $u_k \to u_0$ in C^α for $\alpha < 1/2$. Furthermore, by Theorem 4.1, u_0 is a minimizer of J in any ball B_r, $0 < r < 1$. Since $0 \in \overline{\Omega^+_{u_k}}$, then u_0 inherits the same nondegeneracy properties of Theorem 5.1 that each u_k has. Then $0 \in \overline{\Omega^+_{u_0}}$. But in the limit $u_0 \le 0$ on $B_1' \setminus B_\rho'$. This is a contradiction. □

The nondegeneracy in the thin space has one immediate corollary. We omit the simple proof.

Corollary 5.6. *Let $\{u_k\}$ and u_0 be as in Theorem 4.1. If $x_k \in \Gamma^\pm_{u_k}$ and $x_k \to x_0 \in D$, then $x_0 \in \Gamma^\pm_{u_0}$.*

We next show the positive density of the free boundary.

Theorem 5.7. *Let u be a minimizer of J in B_1 with $\|u\|_{C^{1/2}(B_1)} \le M$ and $0 \in \Gamma^+$. Then there exists $c = c(n, M, \lambda^+) > 0$ such that*

$$c < \frac{\mathcal{H}^{n-1}(\Omega_u^+ \cap B_r')}{\mathcal{H}^{n-1}(B_r')} < 1 - c \tag{5.4}$$

for every $0 < r < 1$. Similar estimate holds also for Ω_u^- if $0 \in \Gamma^-$.

Proof. Since $0 \in \Gamma^+$, by Theorem 5.5 there exists $(x',0)$ such that $|x'| > \rho r$ and $u(x',0) \ge C|x'|^{1/2}$. By uniform Hölder-regularity, u will be positive in a small ball around $(x',0)$. This proves the estimate from below.

To prove the estimate from above we argue as follows. If $B'_{r/2} \cap \Omega_u^-$ is nonempty, then by the argument above show that the set $\Omega_u^- \cap B_r' \subset B_r' \setminus \Omega_u^+$ is large enough and completes the proof. So it remains to consider the case when $u \ge 0$ in $B'_{r/2}$. Besides, scaling if necessary, we may assume that $r = 1/2$. Now, if the estimate from above fails in this case, we can find a sequence of minimizers u_k as in the statement of the theorem such that $u_k \geq 0$ in $B'_{1/4}$ and

$$\mathcal{H}^{n-1}(\{u_k = 0\} \cap B'_{1/4}) \to 0.$$

Let now v_k be a harmonic function in $B_{1/4}$, with the same boundary values as u_k on $\partial B_{1/4}$. Then arguing as in the beginning of the proof of Theorem 3.1, we will have

$$\int_{B_{1/4}} |\nabla(v_k - u_k)|^2 \leq \lambda^+ \int_{B'_{1/4}} \chi_{\{u_k=0\}} \to 0.$$

Next, passing to a subsequence, we may assume that $u_k \to u_0$ and $v_k \to v_0$ uniformly in $B_{1/4}$. Clearly, v_0 is harmonic in $B_{1/4}$ and u_0 is a minimizer of J in B_ρ for $\rho < 1/4$, with $0 \in \Gamma^+_{u_0}$. Besides, by Fatou's lemma $\nabla u_0 = \nabla v_0$ in $B_{1/4}$, which implies that $u_0 \equiv v_0 + c$. Consequently, u_0 is also harmonic in $B_{1/4}$. Now observe that $u_0 \geq 0$ in $B_{1/4}$ and $u_0(0) = 0$. By the strong maximum principle this implies that $u_0 \equiv 0$ in $B_{1/4}$. However, this contradicts the fact that $0 \in \Gamma^+_{u_0}$. □

Corollary 5.8. *The free boundaries $\Gamma^\pm$ have $\mathcal{H}^{n-1}$ measure zero.*

Proof. Apply Lebesgue's density theorem and use the property (5.4). □

The zero $\mathcal{H}^{n-1}$ measure of the free boundaries allows to conclude the following fact about the convergence of the positivity and negativity sets of minimizers.

Theorem 5.9. *Let $\{u_k\}$ and u_0 be as in Theorem 4.1. Then, over a subsequence,*

$$\chi_{\{u_k>0\}} \to \chi_{\{u_0>0\}}, \quad \chi_{\{u_k<0\}} \to \chi_{\{u_0<0\}} \quad \mathcal{H}^{n-1}\text{-a.e. on } D'.$$

Proof. Without loss of generality we may assume that D is the unit ball B_1 and prove only that $\chi_{\{u_k>0\}} \to \chi_{\{u_0>0\}}$ $\mathcal{H}^{n-1}$-a.e. on $B'_{1/2}$. We will also assume that the properties (1)–(4) in Theorem 4.1 hold.

First, if $x \in \{u_0 > 0\} \cap B'_{1/2}$, then from C^α convergence, $u_k(x) > 0$ for k sufficiently large.

Next, if $x \in B'_{1/2}$ and $u_0 \leq 0$ on $B'_\delta(x)$, then we claim that $u_k \leq 0$ on $B'_{\delta/2}(x)$ for k sufficiently large. Indeed, let $y \in B'_{\delta/2}(x)$. By C^α convergence we obtain that $\limsup_{B'_\delta(x)} u_k \leq 0$ as $k \to \infty$. By Theorem 5.5 there exists C such that if for all $z \in B'_{\delta/4}(y)$

$$u_k(z) < C(\rho\delta/4)^{1/2}$$

then y cannot be in $\overline{\Omega^+_{u_k}}$. Thus, $u_k \leq 0$ on $B'_{\delta/2}(x)$ if k is large enough, as claimed.

Hence, we have established that $\chi_{\{u_k>0\}} \to \chi_{\{u_0>0\}}$ everywhere on $B'_{1/2} \setminus \Gamma^+_{u_0}$. Since $\mathcal{H}^{n-1}(\Gamma^+_{u_0}) = 0$, this completes the proof of the theorem. □

6. Separation of phases

In this section we prove that the free boundaries Γ^+, Γ^- cannot meet and that effectively near the free boundary we deal only with a one phase problem.

Theorem 6.1 (Separation of phases)**.** *Let u be a minimizer of J. Then $\Gamma^+ \cap \Gamma^- = \emptyset$, i.e. the free boundaries Γ^+ and Γ^- cannot touch. Moreover, for any $x_0 \in \Gamma^+$ ($x_0 \in \Gamma^-$) there exists an open ball $B_t(x_0) \subset D$ such that $u \geq 0$ ($u \leq 0$) in $B_t(x_0)$.*

As we will see, this follows from the combination of the nondegeneracy in Theorem 5.1 and the following ACF monotonicity formula.

Lemma 6.2 (Alt-Caffarelli-Friedman monotonicity formula)**.** *Let $\{u_+, u_-\}$ be a pair of nonnegative continuous subharmonic functions in B_1 such that $u_+ \cdot u_- = 0$ in B_1. Then the functional*

$$r \mapsto \Phi(r, u_+, u_-) = \frac{1}{r^4} \int_{B_r} \frac{|\nabla u_+|^2}{|x|^{n-2}} \int_{B_r} \frac{|\nabla u_-|^2}{|x|^{n-2}}$$

is finite and nondecreasing for $0 < r < 1$.

Lemma 6.2 was proven in [2].

Remark 6.3. For $u_r = u(rx)/r^{1/2}$, the functional Φ enjoys the following rescaling property:

$$\Phi(R, (u_r)_+, (u_r)_-) = r^2 \Phi(rR, u_+, u_-). \tag{6.1}$$

If we have a blow-up sequence $\{u_r\}$, then for a subsequence $u_r \to u_0$ with the convergence and properties of u_0 as described in Theorem 4.1. The next statement immediately follows from Lemma 6.2 and Remark 6.3.

Lemma 6.4. *If u_r is a blow-up sequence, and $u_r \to v$, then*

$$\int_{B_R} \frac{|\nabla v_+|^2}{|x|^{n-2}} \int_{B_R} \frac{|\nabla v_-|^2}{|x|^{n-2}} = 0,$$

for any $R > 0$. So either $v_+ \equiv 0$ or $v_- \equiv 0$ in entire $\mathbb{R}^n$.

Proof. Using the scaling and monotonicity properties of Φ, we have

$$\Phi(R, (u_r)_+, (u_r)_-) = r^2 \Phi(rR, u_+, u_-) \le r^2 \Phi(1, u_+, u_-) \to 0 \quad \text{as } r \to 0,$$

and applying Fatou's lemma, we obtain that $\Phi(R, v_+, v_-) = 0$ for any $R > 0$. This may happen only if v_+ or v_- is identically constant in $\mathbb{R}^n$. In fact, this implies that one of the functions $v_\pm$ must vanish identically in $\mathbb{R}^n$. □

Proof of Theorem 6.1. We split the proof into two steps.

1) *Thin separation*: $\Gamma^+ \cap \Gamma^- = \emptyset$.

Suppose by way of contradiction, that $0 \in \Gamma^+ \cap \Gamma^-$. Let the blow-up $u_{r_k} \to u_0$. Then by Theorem 5.1 for every k there exist points x_k and y_k such that $u_{r_k}(x_k) \ge C_n$ and $u_{r_k}(y_k) \le -C_n$. Since the convergence is in C^α for $\alpha < 1/2$, it means there exist two points x and y such that $u_0(x) \ge C$ and $u_0(y) \le -C$. Since u_0 will be continuous, we notice that $(u_0)_+$ and $(u_0)_-$ cannot be identically zero. This is a contradiction to Lemma 6.4. Hence, the free boundaries cannot meet.

2) *Solid separation*: If $0 \in \Gamma^+$, then there exists $t > 0$ such that $u \ge 0$ for all $x \in B_t$.

In the previous step we have showed that u has a sign in the thin ball B'_t for a small t. Here we show that u has a sign in the solid ball B_t.

Let $u_{r_k} \to u_0$ be a blow-up of u at the origin. By Lemma 6.4, $u_0 \ge 0$. Since each $u_{r_k}(x', x_n)$ is harmonic in the open set $\{x \in B_{1/r_k} \mid x_n \ne 0\}$, then u_0 will be harmonic in the open set $\{x \in \mathbb{R}^n \mid x_n \ne 0\}$. We define

$$\delta = \inf u_0 \text{ over the set } B_1 \cap \{|x_n| \ge 1/2\}.$$

We claim that $\delta > 0$. Indeed, otherwise by the strong minimum principle $u_0 \equiv 0$ in $\mathbb{R}^n_+$ or $\mathbb{R}^n_-$, and therefore $u_0 \equiv 0$ on $\mathbb{R}^{n-1} \times \{0\}$, which contradicts the fact that $0 \in \Gamma^+_{u_0}$ (see Corollary 5.6). Then, by C^α convergence, for large enough k,

$u_{r_k}(x', x_n) \geq \delta/2$ for $|x_n| \geq 1/2$ in B_1. Also by C^α convergence, $\inf_{B_1} u_{r_k} \to 0$. Now by thin separation, for large enough k,

$$u_{r_k}(x', 0) \geq 0 \text{ in } B'_1$$

Without loss of generality it suffices to show that $u_{r_k} \geq 0$ in $B^+_{1/2}$. Let v_k be the harmonic function such that

$$v_k|_{B'_1} = 0, \text{ and } v_k|_{\partial B^+_1} = u_{r_k}$$

Then $v_k \leq u_{r_k}$ in all of B^+_1. We show for k large enough that $v_k \geq 0$ in $B^+_{1/2}$. To this end, consider two subsets E_1 and E_2 of $\partial(B^+_1)$:

$$E_1 = \partial(B^+_1) \cap \{x_n \geq 1/2\}, \quad E_2 = \partial(B^+_1) \cap \{0 < x_n < 1/2\},$$

and there harmonic measures ω_1 and ω_2 with respect to the domain B^+_1. The latter means that ω_i are harmonic functions in B^+_1 satisfying

$$\omega_i|_{\partial(B^+_1)} = \chi_{E_i}, \quad i = 1, 2.$$

By using an explicit representation with the Poisson kernel or the boundary Harnack inequality, one then has that

$$c_n x_n \leq \omega_i(x) \leq C_n x_n \quad \text{in } B^+_{1/2}.$$

for some positive dimensional constants c_n and C_n. Now, by using the maximum principle we then can write that in $B^+_{1/2}$

$$\begin{aligned} v_k(x) &\geq (\delta/2)\omega_1(x) + \omega_2(x) \inf_{B^+_1} v_k \\ &\geq x_n[(\delta/2)c_n - C_n \sup_{(\partial B_1)^+} u^-_{r_k}]. \end{aligned}$$

Since $u^-_{r_k} \to 0$ uniformly on compact subsets of $\mathbb{R}^n$, we obtain that $v_k(x) \geq 0$ in $B^+_{1/2}$ for large k. This completes the proof. □

7. $\mathcal{H}^{n-3/2}$ MEASURE OF THE FREE BOUNDARY

For the remainder of the paper we will consider local properties of the free boundary. Using the results of the previous section, it will suffice to assume unless otherwise stated that we now consider minimizers of the functional

$$J(v) = \int_D |\nabla v|^2 + \int_{D'} \chi_{\{v>0\}},$$

where $v \in \mathfrak{K} = \{w \in W^{1,2} \mid w|_{\partial D} = \phi\}$ and $\phi \geq 0$. Since there is no negative phase, it will be natural to denote the free boundary Γ^+_v simply by Γ_v.

We start with an improvement on Corollary 5.8.

Theorem 7.1. *Let u be a minimizer of J. Then*

$$\mathcal{H}^{n-3/2}(\Gamma) = 0.$$

This result is far from being optimal, indeed, one would expect the free boundary to have locally finite $\mathcal{H}^{n-2}$ measure. We state it partially because its proof uses and estimate on the measure Δu that we will apply more than once throughout this paper.

Lemma 7.2. *Let u be a minimizer with $\|u\|_{C^{1/2}(D)} \leq M$. For any compact $K \Subset D$, there exist two positive constants c, C depending only on n, M and K such that if $u(x', 0) = 0$ and $(x', 0) \in K \setminus \Gamma$, then*

$$\frac{c}{\sqrt{\operatorname{dist}(x', \Gamma)}} < \frac{\partial u}{\partial x_n}(x', 0) < \frac{C}{\sqrt{\operatorname{dist}(x', \Gamma)}}.$$

Proof. Suppose by way of contradiction that there exists a sequence of minimizers $\{u_k\}$ and points $x_k \in \{u_k = 0\} \setminus \Gamma_{u_k}$ with $\|u_k\|_{C^\alpha(\overline{B_1})} \leq M$ such that

$$\frac{\partial u_k}{\partial x_n}(x_k) \leq \frac{1}{k\sqrt{\operatorname{dist}(x_k, \Gamma_k)}}$$

By rescaling with

$$\tilde{u}_k(x) = \frac{u_k(2d_k x + y_k)}{\sqrt{2d_k}},$$

where $y_k \in \Gamma_{u_k}$ and $d_k = \operatorname{dist}(x_k, \Gamma) = |x_k - y_k|$, we obtain that $\{\tilde{u}_k\}$ is a uniformly bounded family of minimizers in B_1 with $0 \in \tilde{u}_k$. Then by Theorem 4.1, we may extract a subsequence $\tilde{u}_k \to u$, where u is a minimizer on every ball B_r, $r < 1$. Also, if we denote $\xi_k = (x_k - y_k)/2d_k$ then $|\xi_k| = 1/2$ and we may also assume that we get that $\xi_k \to \xi_0$ with $|\xi_0| = 1/2$. Furthermore, since $\tilde{u}_k$ is harmonic in $B^+_{1/4}(\xi_k)$ and $\tilde{u}_k(x) = 0$ for $x \in B'_{1/4}(\xi_k)$, we get that u is harmonic in $B^+_{1/4}(\xi_0)$ and $u(x) = 0$ for $x \in B'_{1/4}(\xi_0)$. By C^1 convergence up to the boundary for harmonic functions we obtain that

$$\frac{\partial u}{\partial x_n}(\xi_0) = 0$$

This is a violation of the Hopf principle, and therefore we obtain a contradiction. The proof of the estimate from above is similar. Only this time we suppose that

$$\frac{\partial u_k}{\partial x_n}(x_k) > k\frac{1}{\sqrt{\operatorname{dist}(x_k, \Gamma)}}$$

This time, in the limit we obtain that

$$\frac{\partial u}{\partial x_n}(\xi_0) = \infty$$

which is of course a contradiction. This completes the proof. □

We are now able to prove Theorem 7.1.

Proof. Without loss of generality we will assume that u is a minimizer on B_1 and show that $\mathcal{H}^{n-3/2}(\Gamma \cap B_{1/2}) = 0$. By lemmas 4.3 and 7.2, we know that as a measure for $x' \in \{u = 0\}$

$$\frac{c}{\sqrt{\operatorname{dist}(x, \Gamma)}} < \frac{\partial u}{\partial x_n}(x', 0) + \frac{\partial u}{\partial x_{-n}}(x', 0) < \frac{C}{\sqrt{\operatorname{dist}(x, \Gamma)}}$$

Let $A_\epsilon = \{x' \in \{u = 0\} \mid 0 < \operatorname{dist}(x', \Gamma) < \epsilon\}$. Then

$$\mathcal{H}^{n-1}(A_\epsilon) \leq c\epsilon^{1/2}\langle \Delta u, A_\epsilon \rangle \tag{7.1}$$

Let $B = B_{1/2}$, and take a cover of $\Gamma \cap B$ by a collection thin balls B'_i of radius ϵ centered on the free boundary with the property that at most N balls intersect. N

is dependent only on dimension. We get

$$\begin{aligned}\sum_i |B_i'| &\le \frac{1}{\beta}\sum_i |B_i' \cap A_\epsilon| \qquad \text{(from positive density)}\\ &\le \frac{N}{\beta}|B\cap A_\epsilon|\\ &\le \frac{\langle \Delta u, B\cap A_\epsilon\rangle N c\epsilon^{1/2}}{\beta}\\ &\le \frac{\langle \Delta u, B\rangle N c\epsilon^{1/2}}{\beta}\end{aligned}$$

This shows that the $\mathcal{H}^{n-3/2}$ measure of $\Gamma\cap B$ is finite. To show that actually $\mathcal{H}^{n-3/2}(\Gamma\cap B)=0$, we just notice that

$$\langle \Delta u, B\cap A_\epsilon\rangle \to 0 \quad \text{as } \epsilon\to 0$$

due to the fact that Δu is a Radon measure and $B\cap A_\epsilon \searrow \emptyset$. □

8. Monotonicity Formula

In this section we establish a Weiss type monotonicity formula. This formula will prove that all blow-ups are homogeneous of degree 1/2. We prove this formula for minimizers without a sign restriction; although, it's application will be to prove local properties of the free boundary.

Theorem 8.1. *Let u be a minimizer of J as in* (1.1) *in $B_R(x_0)$, $x_0\in\mathbb{R}^{n-1}\times\{0\}$. Define the* Weiss energy

$$W(r,u,x_0)=\frac{1}{r^{n-1}}\left(\int_{B_r(x_0)}|\nabla u|^2+\int_{B_r'(x_0)}\lambda^+\chi_{\{u>0\}}+\lambda^-\chi_{\{u<0\}}\right) - \frac{1/2}{r^n}\int_{\partial B_r(x_0)}u^2 \tag{8.1}$$

for $0<r<R$. Then $W(r,u,x_0)$ is monotone increasing in r. Furthermore, if $r_1<r_2$, then $W(r_1,u,x_0)=W(r_2,u,x_0)$ if and only if u is homogeneous of degree $1/2$ with respect to x_0 on the ring $r_1<|x-x_0|<r_2$.

Remark 8.2. Although, we may choose x_0 to be any point, it will be most useful to choose $x_0\in\Gamma$. We will also use a short-cut notation $W(r,u)$ when $x_0=0$.

Remark 8.3. The Weiss energy is essentially a normalized and boundary adjusted version of J. The normalization is chosen so that it is preserved under the scaling $u_r(x)=u(rx)/r^{1/2}$ in a sense that $W(r,u)=W(1,u_r)$.

Proof. The proof is along the lines of that given by G. Weiss in [10]. Without loss of generality we may assume $x_0=0$. Let $\tau_\epsilon(x)=x+\epsilon\eta_k x$, where

$$\eta_k(x)=\max\left(0,\min(1,\frac{r-|x|}{k})\right).$$

Then $\eta_k(x)=0$ outside of $B_r(0)$, and

$$\eta_k(x)\to\chi_{B_r(0)} \text{ as } k\to 0.$$

Notice that $\tau_\epsilon(x) = x(1+\epsilon\eta_k(x))$ leaves $\mathbb{R}^{n-1}\times\{0\}$ invariant. We will also denote by τ'_ϵ the restriction of τ_ϵ to $\mathbb{R}^{n-1}\times\{0\}$. Now

$$\nabla\eta_k(x) = \frac{-x}{|x|k}\chi_{B_r\setminus B_{r-k}}$$

and

$$D\tau_\epsilon(x) = I + \epsilon\left(\eta_k(x)I + x\nabla\eta_k(x)\right) + o(\epsilon)$$

Now let $u_\epsilon\left(\tau_\epsilon(x)\right) = u(x)$ and $y = \tau_\epsilon(x)$. Then

$$\frac{1}{\epsilon}\left(J(u_\epsilon) - J(u)\right) \geq 0$$

and

$$\begin{aligned} J(u_\epsilon) - J(u) = &\int_D |\nabla u_\epsilon(y)|^2 + \int_{D'} \lambda^+\chi_{\{u_\epsilon>0\}} + \lambda^-\chi_{\{u_\epsilon<0\}} \\ &- \int_D |\nabla u(x)|^2 - \int_{D'} \lambda^+\chi_{\{u>0\}} + \lambda^-\chi_{\{u<0\}}. \end{aligned}$$

Now

$$\begin{aligned} \det D\tau_\epsilon(x) &= 1 + \epsilon\operatorname{trace} D(\eta_k(x)x) + o(\epsilon) \\ \operatorname{trace} D(\eta_k(x)x) &= \operatorname{div}(\eta_k(x)x) \\ D\tau_\epsilon^{-1} &= I - \epsilon D(\eta_k(x)x) + o(\epsilon) \end{aligned}$$

Then substituting these into the equality above we get:

$$\begin{aligned} J(u_\epsilon) - J(u) &= \int_D |\nabla u(x)(D\tau_\epsilon(x))^{-1}|^2 \det D\tau_\epsilon \\ &\quad + \int_{D'} \left[\lambda^+\chi_{\{u>0\}} + \lambda^-\chi_{\{u<0\}}\right] \det D\tau'_\epsilon(x) \\ &\quad - \int_D |\nabla u|^2 - \int_{D'} \lambda^+\chi_{\{u>0\}} + \lambda^-\chi_{\{u<0\}} \\ &= \int_D \left[|\nabla u|^2 - 2\epsilon\nabla u D(\eta_k(x)x)\nabla u\right]\left[1 + \epsilon\operatorname{div}(\eta_k(x)x)\right] + o(\epsilon) \\ &\quad + \int_{D'} \left[\lambda^+\chi_{\{u>0\}} + \lambda^-\chi_{\{u<0\}}\right]\left[1 + \epsilon\operatorname{div}(\eta'_k(x',0)x')\right] + o(\epsilon) \\ &\quad - \int_D |\nabla u|^2 - \int_{D'} \lambda^+\chi_{\{u>0\}} + \lambda^-\chi_{\{u<0\}} \\ &= \int_D \epsilon|\nabla u|^2 \operatorname{div}\eta_k(x)x - 2\epsilon\nabla u D(\eta_k(x)x)\nabla u + o(\epsilon) \\ &\quad + \int_{D'} \left[\lambda^+\chi_{\{u>0\}} + \lambda^-\chi_{\{u<0\}}\right]\left[\epsilon\operatorname{div}(\eta'_k(x',0)x')\right] + o(\epsilon). \end{aligned}$$

Now we may let ϵ be both positive and negative, so

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[J(u_\epsilon) - J(u)\right] = 0.$$

Then we obtain the following equality:

$$0 = \int_D |\nabla u|^2 \operatorname{div} \eta_k(x)x - 2\nabla u D(\eta_k(x)x)\nabla u + \int_{D'} \left[\lambda^+ \chi_{\{u>0\}} + \lambda^- \chi_{\{u<0\}}\right] \operatorname{div}(\eta_k'(x',0)x').$$

Now

$$\operatorname{div}\ (\eta_k(x)x) = n\eta_k(x) - \frac{|x|}{k}\chi_{B_r \setminus B_{r-k}}$$

and

$$\operatorname{div}\ (\eta_k(x',0)x') = (n-1)\eta_k' - \frac{|x'|}{k}\chi_{B_r' \setminus B_{r-k}'}.$$

Then

$$\begin{aligned} 0 &= (n-2)\int_{B_r} |\nabla u|^2 \eta_k - \frac{1}{k}\int_{B_r\setminus B_{r-k}} |x| \left[|\nabla u|^2 - 2\left\langle \nabla u \frac{x}{|x|}\right\rangle^2 \right] \\ &+ \quad (n-1)\int_{B_r'} \left[\lambda^+ \chi_{\{u>0\}} + \lambda^- \chi_{\{u<0\}}\right] \eta_k' \\ &- \frac{1}{k}\int_{B_r'\setminus B_{r-k}'} |x'| \left[\lambda^+ \chi_{\{u>0\}} + \lambda^- \chi_{\{u<0\}}\right], \end{aligned}$$

so as $k \to 0$,

$$\begin{aligned} 0 &= (n-2)\int_{B_r} |\nabla u|^2 - r\int_{\partial B_r} |\nabla u|^2 - 2(u_\nu)^2 \\ &\quad + (n-1)\int_{B_r'} \lambda^+ \chi_{\{u>0\}} + \lambda^- \chi_{\{u<0\}} - r\int_{\partial B_r'} \lambda^+ \chi_{\{u>0\}} + \lambda^- \chi_{\{u<0\}} \\ &= (n-1)\int_{B_r} |\nabla u|^2 - r\int_{\partial B_r} |\nabla u|^2 \\ &\quad + (n-1)\int_{B_r'} \lambda^+ \chi_{\{u>0\}} + \lambda^- \chi_{\{u<0\}} - r\int_{\partial B_r'} \lambda^+ \chi_{\{u>0\}} + \lambda^- \chi_{\{u<0\}} \\ &\quad - \int_{B_r} |\nabla u|^2 + 2r\int_{\partial B_r} u_\nu^2. \end{aligned}$$

Now for smooth ψ

$$\int_{B_r} u\Delta\psi + \int_{B_r} \langle \nabla\psi, \nabla u\rangle = \int_{\partial B_r} u\psi_\nu.$$

We let ψ to be a standard mollification of u, i.e. $\psi = \eta_\epsilon * u$, and let $\epsilon \to 0$ to obtain

$$\int_{B_r} u\Delta u + \int_{B_r} |\nabla u|^2 = \int_{\partial B_r} uu_\nu.$$

Now, as a measure, $u\Delta u = 0$, so

$$\begin{aligned} 0 &= (n-1)\int_{B_r} |\nabla u|^2 - r\int_{\partial B_r} |\nabla u|^2 \\ &\quad + (n-1)\int_{B_r'} \lambda^+ \chi_{\{u>0\}} + \lambda^- \chi_{\{u<0\}} - r\int_{\partial B_r'} \lambda^+ \chi_{\{u>0\}} + \lambda^- \chi_{\{u<0\}} \\ &\quad - \int_{\partial B_r} uu_\nu + 2r\int_{\partial B_r} u_\nu^2. \end{aligned}$$

Now multiply both sides of the equation by $-r^{-n}$ to obtain

$$0 = \left[\frac{1}{r^{n-1}}\int_{B_r}|\nabla u|^2\right]' + \left[\frac{1}{r^{n-1}}\int_{B_r'}\lambda^+\chi_{\{u>0\}} + \lambda^-\chi_{\{u<0\}}\right]'$$
$$- \left[\frac{1/2}{r^n}\int_{\partial B_r}u^2\right]' - \frac{1}{r^{n-1}}\int_{\partial B_r}\left(\frac{u}{\sqrt{2}r} - \sqrt{2}u_\nu\right)^2.$$

Thus, $W' \geq 0$, and $W' = 0$ on the interval $r_1 < r < r_2$ if and only if u is homogeneous of degree $1/2$ on the ring $r_1 < |x| < r_2$. □

9. Blow-ups and Regular Points

To study the local properties of the free boundary, it becomes useful to use the so called blow-up process. This is accomplished by rescaling. Recall that if u is a minimizer in B_1, then $u_r(x) = u(rx)/r^{1/2}$ is a minimizer in $B_{1/r}$. If we pick a sequence $r_k \to 0$, then by Theorem 4.1 we know that there exists a subsequence such that $u_{r_k} \to u_0$ and u_0 is a minimizer in every open $U \Subset \mathbb{R}^n$. We call limiting solutions *blow-ups*. When we let $r_k \to 0$ we are in effect doing an infinite zoom. If in the limiting solution the free boundary is flat, then intuitively we would like to say that the origin is a regular (or differentiable) point of the free boundary of u. (By translation we may assume the free boundary point we are considering to be the origin.) The main difficulty in studying blow-ups is that apriori it is not clear if different subsequences converge to different blow-up solutions. i.e. if r_k and r_j are two subsequences going to zero and $u_{r_k} \to u_1$ and $u_{r_j} \to u_2$, is it true that $u_1 \equiv u_2$? A partial answer will be given in this section by classifying points on the free boundary in terms of what we will define as the Weiss energy. A more complete answer will then be furnished in Section 10.

When studying blow-ups, one of the first questions that one may ask is, "what type of solutions may arise as blow-ups?" It is immediate that blow-ups are defined and minimizers in all of $\mathbb{R}^n$; that is they are minimizers in all compact subsets of $\mathbb{R}^n$. One of the most important uses of Weiss-type monotonicity formulas is the ability to prove the following result about all blow-ups.

Corollary 9.1. *If u_0 is any blow-up, then u_0 is homogeneous of degree $1/2$.*

Proof. By Theorem 4.2, if $u_r \to u_0$, over a subsequence $r = r_k \to 0$ then

$$\int_{B_1}|\nabla u_0|^2 = \lim_{r=r_k\to 0}\int_{B_1}|\nabla u_r|^2$$

Also, by Theorem 5.9

$$\chi_{\{u_{r_k}>0\}} \to \chi_{\{u_0>0\}} \quad \mathcal{H}^{n-1} - \text{ a.e. on } \mathbb{R}^{n-1}$$

so

$$\int_{B_1'}\chi_{\{u_0>0\}} = \lim_{r=r_k\to 0}\int_{B_1'}\chi_{\{u_r>0\}}$$

Then for any $\rho > 0$,

$$W(\rho, u_0) = \lim_{r=r_k\to 0}W(\rho, u_r) = \lim_{r=r_k\to 0}W(\rho r, u) = W(0+, u),$$

where we have used the strong convergence of u_{r_k} to u_0 in B_ρ, see Theorem 4.2. Thus, $W(r, u_0)$ is constant; consequently, u_0 is homogeneous of degree $1/2$. □

Now that we know all blow-ups are homogeneous of degree 1/2, we may find what we will term our half plane solution. Also, we will be able to define what we will call the regular points of the free boundary.

Theorem 9.2. *The function*

$$u(x,y) = \sqrt{2/\pi}\,\mathrm{Re}\,\sqrt{x+iy} = \sqrt{\frac{x+\sqrt{x^2+y^2}}{\pi}} = \sqrt{2/\pi}r^{1/2}\cos(\theta/2)$$

is a minimizer on any $U \Subset \mathbb{R}^2$.

Proof. Blow-ups are minimizers and homogeneous of degree 1/2. In the plane the only harmonic function (up to rotation) that is homogeneous of degree 1/2 is $c\,\mathrm{Re}\sqrt{z}$. So we only need to find the constant out in front. This will be accomplished by using variational techniques similar to the proof of Theorem 8.1. This time we choose $\tau_\epsilon(x,y) = (x,y) + \epsilon\eta(x,y)$ where $\eta(x,y) = (\gamma(x)\psi(y), 0)$. Now following the same ideas as in [1] and the proof of Theorem 8.1, we obtain

$$\int_{\{u>0\}} \mathrm{div}\left(|\nabla u|^2\eta - 2(\eta\cdot\nabla u)\nabla u\right) = -\int_{(\mathbb{R}\times\{y=0\})\cap\{u>0\}} \psi(0)\,\mathrm{div}\,\gamma \tag{9.1}$$

Now let $u = c\,\mathrm{Re}\sqrt{z}$ and $\psi \equiv 1$ and $\gamma \equiv 1$ on B_δ for some small δ. Then the right hand side is equal to 1 by the fundamental theorem of calculus. For the left hand side we approximate by doing the following:

$$\int_{\{u>0\}} \mathrm{div}\left(|\nabla u|^2\eta - 2(\eta\cdot\nabla u)\nabla u\right) = \lim_{\epsilon\to 0}\int_{\{u>0\}\setminus B_\epsilon} \mathrm{div}\left(|\nabla u|^2\eta - 2(\eta\cdot\nabla u)\nabla u\right)$$

Now we use the divergence theorem and knowledge of how we have chosen u to obtain that on the coincidence set $\{u=0\}$:

$$\langle \nabla u, \eta\rangle = \langle (0,u_y), (\gamma\psi, 0)\rangle = 0$$

so

$$\lim_{\epsilon\to 0}\int_{\{u>0\}\setminus B_\epsilon} \mathrm{div}\left(|\nabla u|^2\eta - 2(\eta\cdot\nabla u)\nabla u\right) = \lim_{\epsilon\to 0}\int_{\partial B_\epsilon} -|\nabla u|^2\frac{x}{\epsilon} - 2u_x u_\nu$$

Explicit computations of $u = c\,\mathrm{Re}\sqrt{z}$ show that

$$|\nabla u|^2 = \frac{c^2}{4r} \quad \text{and } u_x = u_\nu = c\sqrt{\frac{\cos(\theta)+1}{8r}}$$

Then

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\partial B_\epsilon} -|\nabla u|^2\frac{x}{\epsilon} - 2u_x u_\nu &= c^2\int_{\partial B_\epsilon} \frac{-\cos(\theta)}{4\epsilon} + 2\frac{\cos(\theta)+1}{8r} \\ &= \frac{c^2}{4}\int_{\partial B_1} 1 \\ &= c^2\frac{\pi}{2}\end{aligned}$$

Thus $c^2\pi/2 = 1$, so $c = \sqrt{2/\pi}$. □

It is fairly straight forward to show that any minimizer in dimension n is a minimizer in dimension $n+k$ by defining $\tilde{u}(x_1,\dots,x_n,x_{n+1},\dots,x_{n+k}) = u(x_1,\dots,x_n)$. We now define our *half plane solution* to be

$$\sqrt{2/\pi}\,\mathrm{Re}\sqrt{x_{n-1}+ix_n}. \tag{9.2}$$

Definition 9.3. Let u be a minimizer and $x \in \Gamma$. We call x a regular point if there exists a blow-up v of u at x_0 such that v is a rotation (in the first $n-1$ variables) of the half-plane solution.

This next theorem will give a classification of regular points in terms of the Weiss energy. If $x_0 \in \Gamma$, then the Weiss energy of u at x_0 is defined by $W(0+, u, x_0)$.

Theorem 9.4. *Let u be a minimizer and $x_0 \in \Gamma$, then*

$$W(0+, u, x_0) \geq |B'|/2$$

and equality holds if and only if x_0 is a regular point.

Henceforth we will use a shorthand notation $|B'| = \mathcal{H}^{n-1}(B \cap \mathbb{R}^{n-1})$.

To prove Theorem 9.4 we will use the following two lemmas.

Lemma 9.5. *Let $\{f_1, g_1, f_2, g_2\}$ be functions defined on the interval $[0, \pi]$ satisfying the following*

$$(1) \qquad f_1(t) + g_1(t) \leq f_2(t) + g_2(t) \qquad \textit{for a.e. } t \in [0, \pi]$$

$$(2) \qquad f_1(t) \leq f_2(t) \qquad \textit{for a.e. } t \in [0, \pi]$$

Then for any $a \in \mathbb{R}$

$$\int_0^\pi f_1(t) \sin^{a-2}(t) + g_1(t) \sin^a(t)\ dt \leq \int_0^\pi f_2(t) \sin^{a-2}(t) + g_2(t) \sin^a(t)\ dt$$

Proof. By property (1) we know that

$$(f_1(t) + g_1(t)) \sin^a(t) \leq (f_2(t) + g_2(t)) \sin^a(t) \text{ for a.e. } t$$

By property (2) we may add $(\sin^{a-2}(t) - \sin^a(t))f_1$ to the left hand side of the inequality above and add $(\sin^{a-2}(t) - \sin^a(t))f_2$ to the right hand side and the inequality will be preserved. We then integrate. ☐

Lemma 9.6. *Let $f : S^{n-1} \to \mathbb{R}$ be Lipschitz. By using spherical coordinates we may consider $f : R \to \mathbb{R}$ where $R \overset{def}{=} (0, 2\pi) \times (0, \pi) \times \cdots \times (0, \pi)$. If we Steiner symmetrize f in the θ_1 variable to obtain f^*, then*

$$\int_{S^{n-1}} |\nabla_\theta f^*|^2 \leq \int_{S^{n-1}} |\nabla_\theta f|^2 \tag{9.3}$$

$$\int_{S^{n-1}} |f^*|^2 = \int_{S^{n-1}} |f|^2 \tag{9.4}$$

Equality holds in (9.3) if and only if f is already Steiner symmetric in the θ_1 variable.

Proof. The spherical gradient of f on S^{n-1} in local coordinates is given by

$$\nabla_\theta f = \left(\frac{f_{\theta_i}}{\sin\theta_{n-1} \cdots \sin\theta_{i+1}} \right) \tag{9.5}$$

so that

$$|\nabla_\theta f|^2 = \sum_{i=1}^{n-1} \frac{f_{\theta_i}^2}{\sin^2\theta_{n-1} \cdots \sin^2\theta_{i+1}}. \tag{9.6}$$

We now use Steiner symmetrization on f in the same way as in described in Section 5. If we assume that f in local coordinates is Lipschitz on R, and that f is periodic in the θ_1 variable, i.e. $f(0,\theta_2,\dots,\theta_{n-1}) = f(2\pi,\theta_2,\dots,\theta_{n-1})$ (which is the case since $f : S^{n-1} \to \mathbb{R}$), then by [8], if we Steiner symmetrize in the θ_1 variable to obtain f^*, then

$$\int_R (f^*)^2 = \int_R f^2$$

and

$$\int_R |\nabla f^*|^2 \le \int_R |\nabla f|^2 \tag{9.7}$$

and equality holds in (9.7) if and only if f is already Steiner symmetric in the θ_1 variable. (The gradient in inequality (9.7) is the regular Euclidean gradient.) We may actually deduce more than just inequality (9.7). By fixing the variables $(\theta_k,\dots,\theta_{n-1})$ for $k>1$ we have reduced ourselves to a lower dimension, and so we obtain

$$\sum_{i=1}^{k-1}\int\cdots\int (f^*_{\theta_i})^2\ d\theta_1\dots d\theta_{k-1} \le \sum_{i=1}^{k-1}\int\cdots\int f^2_{\theta_i}\ d\theta_1\dots d\theta_{k-1} \tag{9.8}$$

Since in (9.8) $k>1$, we may restrict θ_{k-1} to any interval contained in $[0,\pi]$, then we may further deduce that for almost every fixed $(\theta_{k-1},\dots,\theta_{n-1})$

$$\sum_{i=1}^{k-1}\int\cdots\int (f^*_{\theta_i})^2\ d\theta_1\dots d\theta_{k-2} \le \sum_{i=1}^{k-1}\int\cdots\int f^2_{\theta_i}\ d\theta_1\dots d\theta_{k-2} \tag{9.9}$$

We now define

$$\begin{aligned}
f_1(\theta_{n-1}) &= \sum_{i=1}^{n-2}\int\cdots\int (f^*_{\theta_i})^2\ d\theta_1\dots d\theta_{n-2}\\
g_1(\theta_{n-1}) &= \int\cdots\int (f^*_{\theta_{n-1}})^2\ d\theta_1\dots d\theta_{n-2}\\
f_2(\theta_{n-1}) &= \sum_{i=1}^{n-2}\int\cdots\int f^2_{\theta_i}\ d\theta_1\dots d\theta_{n-2}\\
g_2(\theta_{n-1}) &= \int\cdots\int f^2_{\theta_{n-1}}\ d\theta_1\dots d\theta_{n-2}
\end{aligned}$$

By inequalities (9.8) and (9.9) we notice that $\{f_1, g_1, f_2, g_2\}$ satisfy the hypotheses of Lemma 9.5. We may then pick $a = n-2$ and integrate over θ_{n-1} to obtain

$$\begin{aligned}
&\int_R \left(\sum_{i=1}^{n-2}\frac{(f^*_{\theta_i})^2}{\sin^2\theta_{n-1}} + (f^*_{\theta_{n-1}})^2\right)\sin^{n-2}\theta_{n-1}\\
&\qquad\le \int_R \left(\sum_{i=1}^{n-2}\frac{f^2_{\theta_i}}{\sin^2\theta_{n-1}} + f^2_{\theta_{n-1}}\right)\sin^{n-2}\theta_{n-1}
\end{aligned} \tag{9.10}$$

We continue in the inductive process by noting that as before, we may let the interval of integration for θ_{n-2} in (9.10) vary, so we obtain the inequality

$$\begin{aligned}(9.11)\quad &\int\cdots\int\left(\sum_{i=1}^{n-2}\frac{(f^*_{\theta_i})^2}{\sin^2\theta_{n-1}}+(f^*_{\theta_{n-1}})^2\right)\sin^{n-2}\theta_{n-1}\ d\theta_1\dots d\theta_{n-3}d\theta_{n-1}\\ &\le\int\cdots\int\left(\sum_{i=1}^{n-2}\frac{f^2_{\theta_i}}{\sin^2\theta_{n-1}}+f^2_{\theta_{n-1}}\right)\sin^{n-2}\theta_{n-1}\ d\theta_1\dots d\theta_{n-3}d\theta_{n-1}\end{aligned}$$

We now define

$$\begin{aligned}f_1(\theta_{n-2})&=\sum_{i=1}^{n-3}\int\cdots\int(f^*_{\theta_i})^2\sin^{n-4}\theta_{n-1}\ d\theta_1\dots d\theta_{n-3}d\theta_{n-1}\\ g_1(\theta_{n-2})&=\int\cdots\int(f^*_{\theta_{n-2}})^2\sin^{n-4}\theta_{n-1}+(f^*_{\theta_{n-1}})^2\sin^{n-2}\theta\ d\theta_1\dots d\theta_{n-3}d\theta_{n-1}\\ f_2(\theta_{n-2})&=\sum_{i=1}^{n-3}\int\cdots\int f^2_{\theta_i}\sin^{n-4}\theta_{n-1}\ d\theta_1\dots d\theta_{n-3}d\theta_{n-1}\\ g_2(\theta_{n-2})&=\int\cdots\int f^2_{\theta_{n-2}}\sin^{n-4}\theta_{n-1}+f^2_{\theta_{n-1}}\sin^{n-2}\theta\ d\theta_1\dots d\theta_{n-3}d\theta_{n-1}\end{aligned}$$

Multiply both sides of inequality (9.8) (with $k=n-2$) by $\sin^{n-4}\theta_{n-1}$ and integrate over θ_{n-1} to find that $f_1(\theta_{n-2})\le f_2(\theta_{n-2})$. Now we may apply inequality (9.11) to notice that again $\{f_1,g_1,f_2,g_2\}$ satisfy the hypotheses of Lemma 9.5. We may then pick $a=n-3$ and integrate over θ_{n-2} to obtain

(9.12)

$$\begin{aligned}&\int_R\left(\sum_{i=1}^{n-3}\frac{(f^*_{\theta_i})^2}{\sin^2\theta_{n-2}\sin^2\theta_{n-1}}+\frac{(f^*_{\theta_{n-2}})^2}{\sin^2\theta_{n-1}}+(f^*_{\theta_{n-1}})^2\right)\sin^{n-3}\theta_{n-2}\sin^{n-2}\theta_{n-1}\\ &\le\int_R\left(\sum_{i=1}^{n-3}\frac{f^2_{\theta_i}}{\sin^2\theta_{n-2}\sin^2\theta_{n-1}}+\frac{f^2_{\theta_{n-2}}}{\sin^2\theta_{n-1}}+f^2_{\theta_{n-1}}\right)\sin^{n-3}\theta_{n-2}\sin^{n-2}\theta_{n-1}\end{aligned}$$

We then proceed in the same inductive manner to obtain

$$\begin{aligned}(9.13)\quad &\sum_{i=1}^{n-1}\int_R\frac{(f^*_{\theta_i})^2}{\sin^2\theta_{n-1}\dots\sin^2\theta_{i+1}}\sin^{n-2}\theta_{n-1}\dots\sin\theta_2\\ &\le\sum_{i=1}^{n-1}\int_R\frac{f^2_{\theta_i}}{\sin^2\theta_{n-1}\dots\sin^2\theta_{i+1}}\sin^{n-2}\theta_{n-1}\dots\sin\theta_2\end{aligned}$$

We now recognize that inequality (9.13) is the local coordinate version of inequality (9.3). From the manner of our proof we see that equality holds in (9.3) if and only if equality holds in (9.7) if and only if f is already Steiner symmetric in the θ_1 variable. □

We may now prove Theorem 9.4.

Proof. Let u be a minimizer and $x_0\in\Gamma$. Let v be a blow-up of u at x_0. Since v is homogeneous of degree $1/2$ and harmonic when positive, restricted to the unit sphere S^{n-1}, v satisfies the differential equation

$$\Delta_{S^{n-1}} v = -\lambda_n v \qquad \text{whenever } v > 0,$$

where

$$\lambda_n = \frac{1}{2}\big(n - 2 + \frac{1}{2}\big) = \frac{2n-3}{4}.$$

Since v is nonnegative, we conclude that v is the principle eigenfunction associated with its coincidence set. We now consider v in the place of f in Lemma 9.6. If for $\epsilon > 0$, we restrict ourselves to R_ϵ where

$$R_\epsilon \overset{\text{def}}{=} \{\theta \in R \mid \theta_{n-1} \notin (\pi/2 - \epsilon, \pi/2 + \epsilon)\}$$

Then v will be both periodic in θ_1 and Lipschitz on R_ϵ, so we obtain every inequality for v^* over R_ϵ that we had for f^* over R. In particular, we obtain the analog of inequality (9.13) for v^* over R_ϵ. We now let $\epsilon \to 0$, to find that v^* satisfies inequality (9.13) and hence v^* also satisfies inequality (9.3) and equality is achieved in (9.3) if and only if v is already symmetric in the θ_1 variable. It is clear that

$$\int_{S^{n-1}} (v^*)^2 = \int_{S^{n-1}} v^2$$

And

$$\int_{(S^{n-1})'} \chi_{\{v^*>0\}} \, d\mathcal{H}^{n-2} = \int_{(S^{n-1})'} \chi_{\{v>0\}} \, d\mathcal{H}^{n-2}$$

Then

$$\frac{\int_{S^{n-1}} |\nabla v^*|^2}{\int_{S^{n-1}} (v^*)^2} \le \frac{\int_{S^{n-1}} |\nabla v|^2}{\int_{S^{n-1}} v^2} \tag{9.14}$$

Equality is achieved if and only if v is already Steiner symmetric in the θ_1 variable. By making orthogonal transformations on the first $n-2$ angles, and using a limiting procedure with Steiner symmetrization, we obtain the principle eigenfunction v^* whose coincidence set is a spherical cap on $(S^{n-1})'$ i.e. $\{v^* = 0\} \equiv \{(x', 0) \mid x_{n-1} \ge t\}$ for some t. The coincidence set of v^* on S^{n-1} has the same $\mathcal{H}^{n-2}$ measure as the coincidence set of v. v^* is the principle eigenfunction associated with its coincidence set, and also v is the principle eigenfunction associated with its own coincidence set. We recall that the eigenvalue λ_n of v is a fixed constant depending on dimension n. Inequality (9.14) shows that the eigenvalue λ^* of v is such that

$$\lambda^* \le \lambda_n$$

Increasing the radius of the coincidence set of v^* (which is a spherical cap) will increase the eigenvalue of v^*; consequently, the upper bound on the eigenvalue $\lambda^* \le \lambda_n$ gives an upper bound on the measure of the coincidence set of v^*. By considering the halfspace solution, we are able to conclude

$$\mathcal{H}^{n-1}(\{v^* > 0\}) \ge |B'|/2$$

If equality above is achieved above, then

(1) v^* must be a rotation of the half plane solution

(2) $\lambda^* = \lambda_n$

(1) is obtained by recalling that principle eigenfunction is unique. (1) immediately implies (2) since the eigenvalues of v and of the half plane solution are a fixed constant depending on n.

If $\lambda^* = \lambda_n$, then equality is achieved in inequality (9.14), and so v is invariant under our Steiner symmetrization. Then necessarily the coincidence sets of v and v^* coincide, and since principle eigenfunctions are unique, we conclude that $v = v^*$. Then v is a rotation of the half plane solution, and so x_0 is a regular point. This concludes the proof. □

Corollary 9.7. *If $x \in \Gamma$ and x is a regular point, then all blow-ups at x are a rotation of the half-plane solution.*

Corollary 9.7 states that all blow-ups of a regular point are some rotation of the half-plane solution. Corollary 9.7 is the partial answer to the question posed in the beginning of this section. Although each blow-up at a regular point is a half-plane solution, it is not immediately clear that the blow-ups have to coincide. They could, in principle, be rotations of each other. In Section 10 we show that this is not the case for minimizers in three dimensions.

This next theorem will be useful in Section 10. We present the proof here because the ideas are similar to those presented in the proof of Theorem 9.4.

Theorem 9.8. *Let $n = 3$, so that the free boundary is contained in $\mathbb{R}^2 \times \{0\}$. Let u be a minimizer with $0 \in \Gamma$, and suppose that in a small enough ball B_ρ, the free boundary of u can be represented as a graph of the x_1 variable. Then every free boundary point of u in B_ρ is a regular point.*

We will need the following lemma for the proof of Theorem 9.8.

Lemma 9.9. *Assume the same hypotheses as given in Theorem 9.8. If $x_0 \in \Gamma \cap B_\rho$, and v is a blow-up of u at x_0, then the coincidence set of v consists of a single connected cone.*

Remark 9.10. The coincidence set of a blow-up is always connected topologically. When we say that the coincidence set consists of a single connected cone, we mean that $\{v = 0\} \setminus \{0\}$ is connected.

Proof. Without loss of generality assume that $\{v > 0\}$ lies above Γ. Since blow-ups are homogeneous it is clear that the coincidence set of v is a cone. Let $y \in \{v = 0\}^\circ$ (the interior here is given by the topology of $\mathbb{R}^{n-1} \times \{0\}$). We claim that

$$v(t, y_2, 0) = 0 \text{ for all } t < y_1$$

Suppose by way of contradiction that $v(t, y_2, 0) > 0$ for some $t < y_1$. Let u_{r_k} be the blow-up sequence converging to v. By C^α convergence we know that for large enough k, u_{r_k} is strictly positive in a small neighborhood of $(t, y_2, 0)$. From Theorem 5.9 we know that for large enough k, $u_{r_k} = 0$ in a small thin ball around $(y_1, y_2, 0)$. Since Γ can be represented as a graph with the positivity set lying above Γ this is a contradiction and thus the claim is proven. The claim along with the fact that the coincidence set is a cone proves the lemma. □

Proof of Theorem 9.8. Let $x_0 \in \Gamma \cap B_\rho$. Let v be a blow-up of u at x_0. Then v is homogeneous of degree $1/2$ and harmonic off the coincidence set. We also obtain on S^2

$$\Delta_{S^2}\, v = -3/4 v.$$

Here $3/4 = \lambda_3$ is the dimensional constant that is a result of v being homogeneous of degree $1/2$. Since the free boundary of u can be represented as a graph, by Lemma 9.9 we know that the coincidence set of v consists of a single connected cone. v, which is nonnegative, must then be the principle eigenfunction on S^2 associated with its coincidence set. We now utilize the fact that our minimizer is in dimension three. Specifically, $\Gamma \cap (S^2)'$ must be a spherical cap (since it is a 1-dimensional curve). Since the eigenvalue of v is fixed at 3/4, the aperture of the coincidence set which is a cone must be fixed at π. Then

$$\mathcal{H}^2(\{v > 0\}) = |B'|/2$$

By Theorem 9.4, we conclude that x_0 is a regular point. □

10. Regularity of the free boundary in dimension three

In this section we prove that in three dimensions the free boundary is locally a C^1 graph near regular points.

Theorem 10.1. *Let $n = 3$. The set of regular points is dense and relatively open in Γ. Furthermore, the set of regular points is locally a C^1 graph in $\mathbb{R}^2 \times \{0\}$.*

We start with a technical lemma on the growth of minimizers away from the plane $\mathbb{R}^{n-1} \times \{0\}$.

Lemma 10.2 (Linear growth)**.** *Let u be a minimizer with $\|u\|_{C^{1/2}(B_1)} \leq M$, and let $\epsilon > 0$. There exists a constant $c = c_{n,M} > 0$ depending only on n and M such that if $x' \in B'_{1/2}$, $\operatorname{dist}(x', \Gamma) \leq \epsilon$ and $|x_n| \leq \epsilon$, then*

$$u(x', x_n) \geq \frac{c|x_n|}{\sqrt{\epsilon}}$$

Proof. Throughout this proof c and C will be constants depending only on dimension and on M. Also, without loss of generality we may consider only the case $x_n \geq 0$. We consider the three cases:

$$x' \in \{u = 0\} \setminus \Gamma, \qquad x' \in \Gamma, \qquad u(x', 0) > 0.$$

If $x' \in \Gamma$, then we may use a compactness argument similar to the one used in Lemma 7.2, to conclude:

$$u(x', x_n) \geq c\sqrt{x_n} = \frac{cx_n}{\sqrt{x_n}} \qquad \text{in } B_1(x', 0)$$

So, in particular,

$$u(x', x_n) \geq \frac{cx_n}{\sqrt{\epsilon}} \text{ for } x_n < \epsilon$$

If $x' \notin \Gamma$, then let $\operatorname{dist}(x', \Gamma) = \delta \leq \epsilon$. If $x' \in \{u = 0\}$, then by Lemma 7.2

$$\frac{c}{\sqrt{\operatorname{dist}(x', \Gamma)}} \leq \frac{\partial u}{\partial x_n}(x', 0) \leq \frac{C}{\sqrt{\operatorname{dist}(x', \Gamma)}}$$

Then we may compare u to the harmonic function $f(x) = x_n$, and by the boundary Harnack principle [7],

$$\frac{cx_n}{\sqrt{2\epsilon}} \leq u(x', x_n) \;\; \text{in } B_{\delta/2}(x', 0) \tag{10.1}$$

If $x_n > \delta/2$, then let y' be the closest point on the free boundary to x'. As stated earlier,

$$u(y', y_n) \geq \frac{cy_n}{\sqrt{y_n}}.$$

For $t > \delta$, we can connect (y', t) to (x', t) by five balls of radius $\delta/4$. Then by using the Harnack inequality on this Harnack chain, we obtain that for $\delta/2 < t \le \epsilon$:

$$u(x',t) \ge \frac{c^5 t}{\sqrt{t}} \ge \frac{c^5 t}{\sqrt{\epsilon}}. \tag{10.2}$$

We now consider the last case when $u(x',0) > 0$. For $\delta < t \le \epsilon$ we may again use a Harnack chain and obtain as in Equation (10.2)

$$u(x',t) \ge \frac{ct}{\sqrt{\delta}} \ge \frac{ct}{\sqrt{\epsilon}}$$

But now since u is harmonic in the tube $B'_\delta \times (-\epsilon, \epsilon)$, we may continue a chain of three more Harnack balls of radius $\delta/2$, and we obtain for $0 \le t \le \delta$

$$u(x',t) \ge \frac{c^3 \delta}{\sqrt{\delta}} \ge \frac{c^3 t}{\sqrt{\epsilon}}$$

Thus the estimate is shown. □

For the remainder of this section we will be working in dimension three, so the free boundary will be contained in $\mathbb{R}^2 \times \{0\}$. Accordingly our functional J will be

$$J(v) = \int_{\mathbb{R}^3} |\nabla v|^2 + \int_{\mathbb{R}^2 \times \{0\}} \chi_{\{v(x_1,x_2,0)>0\}} \, d\mathcal{H}^2$$

and our half plane solution will be

$$\sqrt{2/\pi}\, \operatorname{Re} \sqrt{x_2 + i x_3}. \tag{10.3}$$

To prove that the free boundary is differentiable we will use the Alexandrov Reflection technique. Accordingly, we will need the following Lemma.

Lemma 10.3 (Reflection principle)**.** *Define the tube $T_r(x_0) = B'_r(x_0) \times (-1,1)$. Let u be a minimizer in $T_r = T_r(0)$ such that for every $y \in \partial T_r$ with $y_2 \ge 0$,*

$$u(y_1, y_2, y_3) \ge u(y_1, -y_2, y_3).$$

Also, suppose that there exists at least one point $z \in \partial T_r$ such that $z_2 > 0$ and

$$u(z_1, z_2, z_3) > u(z_1, -z_2, z_3).$$

Then for every $x \in T_r$ with $x_2 \ge 0$

$$u(x_1, x_2, x_3) \ge u(x_1, -x_2, x_3).$$

Proof. Define $\tilde u(y_1, y_2, y_3) \equiv u(y_1, -y_2, y_3)$. Then u and $\tilde u$ are both minimizers in the set $UT_r = \{x \in T_r \mid x_2 \ge 0\}$. (The upper half tube with respect to the variable x_2). By Theorem 2.2 we obtain that $w = \max\{u, \tilde u\}$ is a minimizer. Since $u, \tilde u, w$ are all harmonic when nonzero in UT_r and $w = \max\{u, \tilde u\}$, we obtain that if $\tilde u(z) < u(z)$ for some point $z \in \partial(UT_r)$, then $\tilde u \le u$ in all of UT_r. □

Theorem 10.4. *Let $\{u_k\}$ be a sequence of minimizers in the ball B_4 converging to the minimizer u_0, where u_0 is the half-plane solution as defined in* (10.3)*. Then for every $\delta > 0$, there exists $\epsilon = \epsilon_\delta$ and k_δ, such that if $k > k_\delta$, $y \in (-1,1) \times (-\epsilon, \epsilon)$, $|x_3| \le 1$, $0 < r \le 1$, $\theta \in (0, \pi)$ then*

$$\phi(y, r, \omega, \theta, x_3, u_k) \begin{cases} \ge 0, & \omega \in (\delta, \frac{\pi}{2} - \delta) \cup (\frac{3\pi}{2} + \delta, 2\pi - \delta) \\ \le 0, & \omega \in (\frac{\pi}{2} + \delta, \pi - \delta) \cup (\pi + \delta, \frac{3\pi}{2} - \delta), \end{cases} \tag{10.4}$$

where

$$\phi(y, r, \omega, \theta, x_3, u) = u(y_1 + r\cos(\omega+\theta), y_2 + r\sin(\omega+\theta), x_3) \\ - u(y_1 + r\cos(\omega-\theta), y_2 + r\sin(\omega-\theta), x_3).$$

Proof. We only prove the theorem for the case $\omega \in (\delta, \pi/2-\delta) \cup (3\pi/2+\delta, \pi-\delta)$, the other case being similar.

Let $\epsilon = \epsilon_\delta > 0$ be a small constant to be specified due course. Then split B_2 into the following four regions

$$\begin{aligned} U_1 &= \{x \in B_2 \mid |x_3| \le \epsilon, \text{ and } x_2 > 2\epsilon\}, \\ U_2 &= \{x \in B_2 \mid |x_3| \le \epsilon, \text{ and } |x_2| \le 2\epsilon\}, \\ U_3 &= \{x \in B_2 \mid |x_3| \le \epsilon, \text{ and } x_2 < -2\epsilon\}, \\ U_4 &= \{x \in B_2 \mid |x_3| \ge \epsilon\}. \end{aligned}$$

We also define the following two regions

$$\begin{aligned} \tilde U_1 &= \{x \in B_2 \mid |x_3| \le \epsilon, \text{ and } x_2 \ge \delta/2\}, \\ \tilde U_3 &= \{x \in B_2 \mid |x_3| \le \epsilon, \text{ and } x_2 \le -\delta/2\}. \end{aligned}$$

We note that $\tilde U_1 \subset U_1$ and $\tilde U_3 \subset U_3$.

1) Consider first the case $|x_3| \ge 1$. Take a unit vector $e = (e_1, e_2, 0)$ such that$(0,1,0)\cdot e \ge \sin\delta$. Then a direct computation show that

$$\partial_e u_0 \ge c_\delta > 0 \quad \text{in } U_4.$$

If k is large enough so that $u_k - u_0$ is small in C^1 norm in U_4, then

$$\partial_e u_k > 0 \quad \text{in } U_4, \tag{10.5}$$

and it is easy to see that this implies estimate (10.4) if $|x_3| \ge \epsilon$.

2) Let now $|x_3| \le \epsilon$. We claim that

$$\phi(y, 1, \omega, \theta, t, u_k) \ge 0 \quad \text{for } |t| \le \epsilon. \tag{10.6}$$

By the same arguments as in the proof of Theorem 4.2, we may choose k large enough such that

$$\Gamma \cap B_2(0) \subset \{|x_2| < \epsilon\} \cap B_2'(0).$$

We then consider several subcases, depending whether $|y_2 + \sin(\omega \pm \theta)| \le \epsilon$ or not. In this regard, we note that if $\omega \in (\delta, \pi/2-\delta) \cup (3\pi/2+\delta, 2\pi-\delta)$ and $\theta \in (0,\pi)$, then

$$|\sin(\omega \pm \theta)| \le |\sin\delta| \quad \Rightarrow \quad |\sin(\omega \mp \theta)| > |\sin\delta|.$$

Hence, if we choose $\epsilon < (\sin\delta)/4$, and take $y \in (-1,1)\times(-\epsilon,\epsilon)$, then

$$|y_2 + \sin(\omega \pm \theta)| \le 2\epsilon \quad \Rightarrow \quad |y_2 + \sin(\omega \mp \theta)| > \delta/2.$$

Also notice that $y_2 + \sin(\omega+\theta) \ge y_2 + \sin(\omega-\theta)$ in the considered ranges of ω and θ.

Then we have the following subcases.

2a) $|y_2 + \sin(\omega-\theta)| \le \epsilon$ and consequently $y_2 + \sin(\omega+\theta) > \delta/2$. Let ϵ be so small that

$$\sup_{U_2} u_k \le C\sqrt{\epsilon} \le \sqrt{\delta/4\pi} = \frac{1}{2}\inf_{\tilde U_1} u_0.$$

Then, using L^∞ convergence of u_k to u_0 on $\tilde{U}_1$, we can guarantee that

$$\sup_{U_2} u_k \le \inf_{\tilde{U}_1} u_k,$$

if k is large enough. This implies (10.6).

2b) $|y_2 + \sin(\omega+\theta)| \le \epsilon$ and consequently $y_2 + \sin(\omega-\theta) < -\delta/2$. We can make ϵ even smaller to have

$$\frac{c}{\sqrt{\epsilon}}|x_3| \ge 2u_0(x_1,x_2,x_3) \quad \text{ for } x_2 \le -\delta/2, \text{ and } |x_3| \le \epsilon,$$

for the constant c as given in Lemma 10.2. Then by Lemma 10.2 we obtain that for $z \in U_2$ and $x \in \tilde{U}_3$ with $z_3 = x_3$

$$u_k(z) \ge 2u_0(x).$$

For $x = (x', 0) \in U_3$, we know that for large k, $u_k(x', 0) = 0$, so we may use C^1 convergence of harmonic functions up to the set $\{x_3 = 0\}$ and we obtain that

$$u_k(z) \ge u_k(x), \quad \text{ for } z \in U_2 \text{ and } x \in \tilde{U}_3 \text{ with } z_3 = x_3. \tag{10.7}$$

Therefore inequality (10.6) also follows in this case.

2c) $y_2 + \sin(\omega+\theta) \ge 2\epsilon$ and $y_2 + \sin(\omega-\theta) \le -2\epsilon$. In this case, we may use L^∞ convergence of $u_k \to u_0$ again to obtain

$$\sup_{U_3} u_k \le \inf_{U_1} u_k,$$

which implies (10.6).

2d) Both $y_2 + \sin(\omega \pm \theta) \ge 2\epsilon$. Similar to the case 1), if $x \in U_1$, we may use C^1 convergence to obtain that for unit vectors $e = (e_1, e_2, 0)$ with $(0,1,0)\cdot e \ge \sin\delta$ we have

$$\partial_e u_k > 0 \quad \text{in } U_1. \tag{10.8}$$

Hence inequality (10.6) is true also in this case.

2e) Both $y_2 + \sin(\omega \pm \theta) \le -2\epsilon$. This is our only remaining concern. Without loss of generality we may further assume $x_3 \ge 0$. On U_3 we know that $\partial_{x_3}\partial_e u_0 > c$. (This constant c will depend on δ and ϵ). On U_3 we may use C^2 convergence up to the boundary $\{x_3 = 0\}$ to obtain that for large enough k

$$\partial_{x_3}\partial_e u_k > 0 \quad \text{in } U_3$$

Since $\partial_e u_k(x_1, x_2, 0) = 0$ for $x_2 < -\epsilon$ we obtain

$$\partial_e u_k > 0 \quad \text{ in } U_3 \tag{10.9}$$

This implies (10.6).

Thus, we have considered all cases, so we may conclude that inequality (10.6) is true for large enough k.

Now that we have inequalities (10.6) and (10.5) , we know that for $y \in (-1,1) \times (\epsilon, \epsilon)$, u and its mirror reflection with respect to the plane $-\sin\omega(x_1 - y_1) + \cos\omega(x_2 - y_2) = 0$ satisfy the hypotheses of Lemma 10.3 on the cylindrical tube $T_1(y)$. Therefore, we may utilize Lemma 10.3 to conclude that inequality (10.4) is true for all $0 \le r \le 1$. □

The technical details of the previous proof would have been simpler if instead of letting $y \in (-1,1) \times (-\epsilon, \epsilon)$, we had only considered $y = 0$. The usefulness of allowing y to vary in the rectangular region $(-1,1) \times (-\epsilon, \epsilon)$ is seen in the following corollary.

Corollary 10.5. *Let $\{u_k\}$, u_0, δ, ϵ, k_δ be as in Theorem 10.4. If $k \geq k_\delta$ and $\omega \in (\delta, \pi/2-\delta) \cup (\pi/2+\delta, \pi-\delta)$, then in the rectangular region $(-1,1)\times(-\epsilon,\epsilon)\times\{0\}$, u is monotone in the direction $(\cos\omega, \sin\omega)$.*

Proof. We are allowed to vary the point $y \in (-1,1)\times(-\epsilon,\epsilon)$. Suppose by way of contradiction that $u(x_1,x_2,0) > u(x_1 + r\cos\omega, x_2 + r\sin\omega, 0)$, then pick y to be the midpoint, so that $y = (x_1 + r/2\cos\omega, x_2 + r/2\sin\omega, 0)$. By Theorem 10.4 we obtain:

$$\begin{aligned}
&u(x_1 + r\cos\omega, x_2 + r\sin\omega, 0) \\
&= u(y_1 + r/2\cos(3\pi/2 + \omega + \pi/2), y_2 + r/2\sin(3\pi/2 + \omega + \pi/2), 0) \\
&\geq u(y_1 + r/2\cos(3\pi/2 + \omega - \pi/2), y_2 + r/2\sin(3\pi/2 + \omega - \pi/2), 0) \\
&= u(x_1, x_2, 0).
\end{aligned}$$

This is a contradiction, and hence the corollary is proven. □

Corollary 10.6. *Let $\{u_k\}$, u_0, δ, $\epsilon = \epsilon_\delta$, k_δ be as in Theorem 10.4. If $k > k_\delta$, then the free boundary in B_1' is a graph.*

Proof. This is a direct consequence of the directional monotonicity proven in Corollary 10.5 and the fact that $\Gamma \cap B_1' \subset (-1,1)\times(-\epsilon,\epsilon)$. □

Corollary 10.7. *Let $\{u_k\}$, u_0, δ, ϵ, k_δ be as in Theorem 10.4. If $k > k_0$ and $y \in \Gamma \cap B_1'$, then y is a regular point.*

Proof. By Corollary 10.7 we know the free boundary can be represented by a graph. Recalling that we are in dimension three, by Theorem 9.8 we conclude that every free boundary point is regular. □

Remark 10.8. The previous corollary shows that the set of regular points is relatively open in the free boundary Γ. This is seen by letting $u_r \to u_0$ be a blow up of u a minimizer at x_0 a regular point. We may apply Corollary 10.7 to u_r for r small enough and then rescale backwards.

We also note that the set of regular points is relatively dense in Γ. This is seen by noting that free boundary points that have tangent balls from one side (as defined in [6]) are dense in Γ. It has been shown in [6] that free boundary points that have a tangent ball are regular.

Corollary 10.9 (Differentiability)**.** *If x_0 is a regular point, then the blow-up at x_0 is unique.*

Proof. Let u_0 and v_0 be two blow-ups of u at x_0. By Corollary 9.7, we know that u_0 and v_0 are both half plane solutions. If we use polar coordinates for variables x_1 and x_2, we may write $u_0(x_1,x_2,t) = u_0(r,\theta,t)$. If $u_0 \neq v_0$, then $v(r,\theta,t) = u(r,\theta+\alpha,t)$. By Corollary 10.5, u will obtain directional monotonicity from both u_0 and v_0. Specifically, in a small enough ball, u will be monotone in in the angular direction

$$\omega \in (\delta+\alpha, \pi/2-\delta+\alpha) \cup (\pi/2+\delta+\alpha, \pi-\delta+\alpha)$$

Then u_0 will inherit monotonicity in the same angular direction, and this will be a contradiction for u_0. □

We now prove the main result of the paper Theorem 10.1. We only state what remains to be proven.

Theorem 10.10 (C^1 Regularity)**.** *Let u be a nonnegative bounded minimizer of J in B_1 in dimension $n = 3$ and let the origin be a regular point. Then there exists $\rho > 0$ small enough such that $\Gamma \cap B_\rho$ is a C^1 graph.*

Proof. Let $u_r \to u_0$ be a blow-up, and fix $\delta > 0$. By using Corollary 10.6 and rescaling backward we obtain that there is a small enough ρ such that $\Gamma \cap B_\rho$ is a graph. Furthermore, $\Gamma \cap B'_\rho \subset S_{\rho\epsilon}$, and by Corollary 10.5 u is monotone in $S_{\rho\epsilon}$ in the ω direction for $\omega \in (\delta, \pi/2 - \delta) \cup (\pi/2 + \delta, \pi - \delta)$. By Corollary 10.7, we know that for all $y \in \Gamma \cap B_\rho$, y is a regular point. By Corollary 10.9 we know that if v_0 is a blow-up at y, then v_0 is unique. Also, v_0 will inherit monotonicity in the ω direction for $\omega \in (\delta, \pi/2 - \delta) \cup (\pi/2 + \delta, \pi - \delta)$. Then $v_0(r, \theta, t) = u_0(r, \theta + \alpha, t)$ for $|\alpha| < \delta$. This proves the C^1 regularity. □

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