CANCELLATION DOES NOT IMPLY STABLE RANK ONE

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Abstract

A unital C*-algebra $A$ is said to have cancellation of projections if the semigroup $D(A)$ of Murray–von Neumann equivalence classes of projections in matrices over $A$ is cancellative. It has long been known that stable rank one implies cancellation for any $A$, and some partial converses have been established. In this paper it is proved that cancellation does not imply stable rank one for simple, stably finite C*-algebras.

1. Introduction

Rieffel introduced the notion of stable rank for C*-algebras in his 1983 paper [4]: a unital C*-algebra $A$ is said to have stable rank $n$ ($\text{sr}(A) = n$) if $n$ is the least natural number such that the set

$$Lg_n(A) \overset{\text{def}}{=} \left\{ (a_1, \ldots, a_n) \in A^n \mid \exists b_i \in A, 1 \leq i \leq n : \sum_{i=1}^{n} b_i a_i = 1 \right\}$$

is dense in $A^n$. If no such $n$ exists, then one says that the stable rank of $A$ is infinite. In the case of a commutative C*-algebra, the stable rank is proportional to the covering dimension of the spectrum; stable rank may be viewed as a kind of non-commutative dimension.

Given a unital C*-algebra $A$, let $D(A)$ be the Abelian semigroup obtained by endowing the set of Murray–von Neumann equivalence classes of projections in matrix algebras over $A$ with the addition operation coming from direct sums. The algebra $A$ is said to have cancellation of projections if $x + y = x + z$ implies that $y = z$ for any $x, y, z \in D(A)$.Shortly after the appearance of Rieffel’s paper, Blackadar showed that stable rank one implies cancellation of projections [1]. He also established a partial converse: if a C*-algebra of real rank zero has cancellation of projections, then it has stable rank one. The relationship between cancellation and stable rank for general simple, stably finite C*-algebras, however, remained unclear. The lack of examples of simple, stably finite C*-algebras with non-minimal stable rank was a serious obstacle. Villadsen provided the first such examples in [7], but determining whether his examples had cancellation of projections was all but impossible, due to their extremely complicated $K$-theory.

Recently, the author has been able to apply Villadsen’s techniques to construct simple, stably finite C*-algebras with non-minimal stable rank and cyclic $K_0$-groups. These algebras constitute the first simple, nuclear and stably finite counterexamples to Elliott’s classification conjecture for nuclear C*-algebras [2, 6]. In this paper we study one such algebra in order to prove our main result.
Theorem. There is a simple, separable, nuclear, and stably finite C*-algebra with non-minimal stable rank, which nevertheless has cancellation of projections.

Thus, Blackadar’s partial converse cannot be extended to cover general simple, stably finite C*-algebras.

2. The proof of the main result

We proceed by a close analysis of the structure of the simple, separable, and stably finite C*-algebra $B_2$ of [6], which has non-minimal stable rank. We prove that $B_2$ nevertheless has cancellation of projections.

Let $C$ and $D$ be C*-algebras, and let $\phi_0$ and $\phi_1$ be *-homomorphisms from $C$ to $D$. The generalised mapping torus of $C$ and $D$ with respect to $\phi_0$ and $\phi_1$ is

$$A := \{(c, d) \mid d \in C([0, 1]; D), \ c \in C, \ d(0) = \phi_0(c), \ d(1) = \phi_1(c)\}.$$ 

We denote $A$ by $A(C, D, \phi_0, \phi_1)$ for clarity when necessary. Let $U(A)$ denote the unitary group of a unital C*-algebra $A$.

The algebra $B_2$ of [6] is constructed as the limit of an inductive sequence $(A_i, \theta_i)$ of generalised mapping torus $A_i = A(C_i, D_i, \phi_i^0, \phi_i^1)$ and unital *-homomorphisms $\theta_i : A_i \longrightarrow A_{i+1}$ where, for each $i \in \mathbb{N}$,

$$C_i \overset{\text{def}}{=} p_i(C(X_i) \otimes K)p_i$$

and

$$D_i \overset{\text{def}}{=} M_{k_i} \otimes C_i$$

for some connected compact Hausdorff space $X_i$, projection $p_i \in C(X_i) \otimes K$ and natural number $k_i$. The maps $\phi_i^0$ and $\phi_i^1$ are unital. The spaces $X_i$, $i \in \mathbb{N}$, have the property that

$$\dim(p_i) = \frac{\dim(X_i)}{2},$$

and the maps $\phi_i^0$ and $\phi_i^1$ are chosen to ensure that

$$(K_0A_i, K_0A_i^+, [1_{A_i}]) = (\mathbb{Z}, \mathbb{Z}^+, 1),$$

where $1_{A_i} \in A_i$ is the unit; $A_i$ is projectionless except for zero and $1_{A_i}$.

To prove our theorem, it will suffice to prove that $A_i$ has cancellation of projections for every $i \in \mathbb{N}$. Let $p, q \in M_n(A_i)$ be projections having the same $K_0$-class. We must show that $p$ and $q$ are Murray–von Neumann equivalent. Since $K_0(A_i) = \mathbb{Z}[1_{A_i}]$, we may assume that $p$ is a multiple of the unit of $A_i$, say $p = U1_{A_i}$. Now $M_n(A_i)$ can be viewed as an algebra of functions from $[0, 1] \times X_i$ into matrices. Given $f \in M_n(A_i)$, we let $f(t), t \in [0, 1]$, denote the restriction of $f$ to $\{t\} \times X_i \subseteq [0, 1] \times X_i$. Both $f(0)$ and $f(1)$ are images of a single element in $M_n(C_i)$, which we denote by $f(\infty)$. If two vector bundles over a compact, connected CW-complex $X$ of covering dimension $m$ with the same $K^0$-class have fibre dimension at least $m/2$, then the bundles are isomorphic (cf. [3, Theorem 1.5, Chapter 8]).

In the language of C*-algebras, the projections in $M_k \otimes C(X)$, for some $k \in \mathbb{N}$, corresponding to these vector bundles are Murray–von Neumann equivalent. Since $p(\infty)$ and $q(\infty)$ can be viewed as vector bundles over $X_i$ having the same $K^0$-class, and since they must both have fibre dimension at least $\dim(X_i)/2$ by the construction of $A_i$, they are Murray–von Neumann equivalent, as are their images.
under \( \phi_0^1 \) and \( \phi_1^1 \). Note that if one considers \( M_n(A_i) \) as a unital sub-C*-algebra of \( C_t \otimes M_{n_k} \otimes C([0,1]) \), then fibre dimension considerations show \( q \) and \( p \) to be Murray–von Neumann equivalent inside \( C_t \otimes M_{n_k} \otimes C([0,1]) \). This does not, however, prove that \( q \) and \( p \) are Murray–von Neumann equivalent inside \( M_n(A_i) \).

We may assume without loss of generality that \( l1_{A_i} \) and \( q \) are constant over some small interval \([1/2-\varepsilon,1/2+\varepsilon]\) in the interval factor of the spectrum of \( M_n(A_i) \), since small perturbations do not disturb the Murray–von Neumann equivalence class. Consider \( l1_{A_i} \) and \( q \) as vector bundles over \([0,1] \times X_i \). Define

\[
q_0 := q|_{[0,1/2-\varepsilon] \times X_i}, \quad q_1 := q|_{[1/2+\varepsilon,1] \times X_i},
\]

and

\[
1_{A_i,0} := 1_{A_i}|_{[0,1/2-\varepsilon] \times X_i}, \quad 1_{A_i,1} := 1_{A_i}|_{[1/2+\varepsilon,1] \times X_i}.
\]

The following statement appears as [3, Chapter 3, Corollary 4.4].

**Lemma.** Let \( \gamma \) be a vector bundle over \( X \times [0,1] \), \( X \) paracompact, and \( \omega \) a vector bundle over \( X \) such that \( \gamma|_{X \times \{0\}} \cong \omega \). Then \( \gamma \) is isomorphic to the induced bundle \( \pi^*(\omega) \), where \( \pi : X \times [0,1] \rightarrow X \times \{0\} \) is given by \( \pi(x,t) = (x,0) \).

Define maps

\[
\pi_0 : [0, \frac{1}{2} - \varepsilon] \times X_i \rightarrow \{0\} \times X_i, \quad \pi_1 : [\frac{1}{2} + \varepsilon, 1] \times X_i \rightarrow \{1\} \times X_i,
\]

by

\[
\pi_0(t,x) = (0,x), \quad \pi_1(t,x) = (1,x).
\]

We have \( l1_{A_i}(j) \cong q(j) \) for \( j \in \{0,1\} \). Moreover, \( l1_{A_i,j} \cong \pi^*_j(l1_{A_i}(j)) \) by construction. We may thus apply our lemma, with \( \gamma = q_j \), \( \omega = l1_{A_i}(j) \), and \( \pi = \pi_j \), to conclude that \( l1_{A_i,j} \cong q_j \). In other words, there is a continuous path of partial isometries \( v(t) \), \( t \in [0,1/2-\varepsilon] \cup [1/2+\varepsilon,1] \), such that \( v(t)^*v(t) = l1_{A_i}(t) \), \( v(t)v(t)^* = q(t) \), and, for each \( j \in \{0,1\} \), the partial isometry \( v(j) \) is the image under \( \phi_j^0 \otimes \text{id}_{M_{n_i}} \) of a single partial isometry \( v \in M_n(C_i) \) such that \( v^*v = l1_{C_i} \) and \( vv^* = q(\infty) \). This last property ensures that if we can find a continuous extension of \( v(t) \) to a partial isometry defined on \([0,1]\), then our proof is complete: \( v(t) \) will lie in \( M_n(A_i) \).

From [5] we have the formula

\[
\text{sr}(p(C(X) \otimes \mathcal{K})p) = \left\lfloor \frac{\text{dim}(X)/2}{\text{rank}(p)} \right\rfloor + 1,
\]

where \( \mathcal{K} \) denotes the compact operators on a separable Hilbert space, \( X \) is a compact connected Hausdorff space, and \( p \) is a projection in \( C(X) \otimes \mathcal{K} \). Straightforward calculation then shows that \( \text{sr}(C_i) = 2 \), for all \( i \in \mathbb{N} \). For a unital C*-algebra \( A \), let \( \mathcal{U}(A) \) denote the unitary group of \( A \), and let \( \mathcal{U}(A)_0 \) denote the connected component of \( \mathcal{U}(A) \) containing the identity. Now, [4, Theorem 10.12] states that one has an isomorphism

\[
\frac{\mathcal{U}(M_r(A))}{\mathcal{U}(M_r(A))_0} \rightarrow K_1(A)
\]

whenever \( r \geq \text{sr}(A) + 2 \). In the construction of \( A_i \), the parameter \( k_i \) in the definition \( D_i := M_{k_i}(C_i) \) is chosen to be much larger than \( \text{sr}(C_i) \). Furthermore, one has (again,
by construction) that $K_1(C_i) = 0$, for all $i \in \mathbb{N}$. Thus, $U(M_l(D_i))$ is connected for every $l \in \mathbb{N}$.

We may view $u := v(1/2 + \varepsilon)^*v(1/2 - \varepsilon)$ as a unitary element in $M_l(D_i)$. By the discussion above, there is a path of unitary elements $u(t)$, $t \in [1/2 - \varepsilon, 1/2 + \varepsilon]$, inside $M_l(D_i)$ such that $u(1/2 + \varepsilon) = l_1 A_i$, and $u(1/2 - \varepsilon) = u$.

For $t \in [1/2 - \varepsilon, 1/2 + \varepsilon]$, define $\tilde{v}(t) = v(1/2 + \varepsilon)u(t)$. Clearly, $\tilde{v}(t)$ is a partial isometry in $M_n(D_i)$ for each $t$ in its domain. One has
\[
\tilde{v}(1/2 + \varepsilon) = v(1/2 + \varepsilon)
\]
and
\[
\tilde{v}(1/2 - \varepsilon) = v(1/2 + \varepsilon)v(1/2 - \varepsilon)^*v(1/2 - \varepsilon) = q(1/2 - \varepsilon)u(1/2 - \varepsilon) = v(1/2 - \varepsilon).
\]
Then
\[
v(t) := \begin{cases} v(t), & t \in [0, 1/2 - \varepsilon] \cup [1/2 + \varepsilon, 1], \\ \tilde{v}(t), & t \in (1/2 - \varepsilon, 1/2 + \varepsilon) \end{cases}
\]
defines a partial isometry in $M_n(A_i)$ such that $v(t)^*v(t) = l_1 A_i(t)$ and $v(t)v(t)^* = q(t)$, for all $t \in [0, 1]$. Thus $q$ and $l_1 A_i$ are Murray–von Neumann equivalent, as desired.

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References


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