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# Dimension growth for $C^*$ -algebras

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## Abstract

We introduce the growth rank of a  $C^*$ -algebra—a  $(\mathbb{N} \cup \{\infty\})$ -valued invariant whose minimal instance is equivalent to the condition that an algebra absorbs the Jiang–Su algebra  $\mathcal{Z}$  tensorially—and prove that its range is exhausted by simple, nuclear  $C^*$ -algebras. As consequences we obtain a well developed theory of dimension growth for approximately homogeneous (AH)  $C^*$ -algebras, establish the existence of simple, nuclear, and non- $\mathcal{Z}$ -stable  $C^*$ -algebras which are not tensorially prime, and show the assumption of  $\mathcal{Z}$ -stability to be particularly natural when seeking classification results for nuclear  $C^*$ -algebras via K-theory.

The properties of the growth rank lead us to propose a universal property which can be considered inside any class of unital and nuclear  $C^*$ -algebras. We prove that  $\mathcal{Z}$  satisfies this universal property inside a large class of locally subhomogeneous algebras, representing the first uniqueness theorem for  $\mathcal{Z}$  which does not depend on the classification theory of nuclear  $C^*$ -algebras.

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## 1. Introduction

In the late 1980s, Elliott conjectured that separable nuclear  $C^*$ -algebras would be classified by K-theoretic invariants. He bolstered his claim by proving that certain inductive limit  $C^*$ -algebras

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(the  $\text{AT}$  algebras of real rank zero, [8]) were so classified, generalising broadly his seminal classification approximately of finite-dimensional (AF) algebras by their scaled, ordered  $K_0$ -groups ([7], 1976). His conjecture was confirmed in the case of simple algebras throughout the 1990s and early 2000s. Highlights include the Kirchberg–Phillips classification of purely infinite simple  $C^*$ -algebras satisfying the Universal Coefficients Theorem (UCT), the Elliott–Gong–Li classification of simple unital approximately homogeneous (AH) algebras of bounded topological dimension, and Lin’s classification of certain tracially AF algebras. The classifying invariant, consisting of topological  $K$ -theory, traces (in the stably finite case), and a connection between them is known as the Elliott invariant. (See [28] for a thorough introduction to this invariant and the classification program for separable, nuclear  $C^*$ -algebras.)

Counterexamples to Elliott’s conjecture appeared first in 2002: Rørdam’s construction of a simple, nuclear, and separable  $C^*$ -algebra containing a finite and an infinite projection [29] was followed by two stably finite counterexamples, due to the author [31,32]. (The second of these [32] shows that Elliott’s conjecture will not hold even after adding to the Elliott invariant every continuous (with respect to direct limits) homotopy invariant functor from the category of  $C^*$ -algebras.) The salient common feature of these counterexamples is their failure to absorb the Jiang–Su algebra  $\mathcal{Z}$  tensorially. The relevance of this property to Elliott’s classification program derives from the following fact: taking the tensor product of a simple unital  $C^*$ -algebra  $A$  with  $\mathcal{Z}$  is trivial at the level of the Elliott invariant when  $A$  has weakly unperforated ordered  $K$ -theory [15], and so the Elliott conjecture predicts that any simple, separable, unital, and nuclear  $A$  satisfying this  $K$ -theoretic condition will also satisfy  $A \otimes \mathcal{Z} \cong A$ . This last condition is known as  $\mathcal{Z}$ -stability, and any  $A$  satisfying it is said to be  $\mathcal{Z}$ -stable. In recent work with Wilhelm Winter, the author has proved that every class of unital, simple, and infinite-dimensional  $C^*$ -algebras for which the Elliott conjecture is so far confirmed consists of  $\mathcal{Z}$ -stable algebras [34]. The emerging consensus, suggested first by Rørdam and well supported by these results, is that the Elliott conjecture should hold for simple, nuclear, separable, and  $\mathcal{Z}$ -stable  $C^*$ -algebras.

A recurring theme in theorems confirming the Elliott conjecture is that of minimal rank. There are various notions of rank for  $C^*$ -algebras—the real rank, the stable rank, the tracial topological rank, and the decomposition rank—which attempt to capture a non-commutative version of dimension. A natural and successful approach to proving classification theorems for separable and nuclear  $C^*$ -algebras has been to assume that one or more of these ranks is minimal (see [9,20], for instance). But there are examples which show these minimal rank conditions to be variously too strong or too weak to characterise those algebras for which the Elliott conjecture will be confirmed. One wants to assume  $\mathcal{Z}$ -stability instead, but a fair objection has been that this assumption seems unnatural.

In the sequel, we situate  $\mathcal{Z}$ -stability as the minimal instance of a well-behaved rank for  $C^*$ -algebras, which we term the *growth rank*. The growth rank measures “how far” a given algebra is from being  $\mathcal{Z}$ -stable, and inherits excellent behaviour with respect to common operations from the robustness of  $\mathcal{Z}$ -stability. Our terminology is motivated by the fact that the growth rank may be viewed as a theory of dimension growth for AH algebras, and, more generally, locally type-I  $C^*$ -algebras. We prove that for every  $n \in \mathbb{N} \cup \{\infty\}$ , there is a simple, separable, and nuclear  $C^*$ -algebra  $A_n$  with growth rank equal to  $n$ . The algebras constructed in the proof of this theorem are entirely new, and rather exotic; for all but two of them, the other ranks for  $C^*$ -algebras above are simultaneously infinite. We use these algebras to obtain the unexpected (see [29]): a simple, nuclear, and non- $\mathcal{Z}$ -stable  $C^*$ -algebra which is not tensorially prime.

Motivated by the properties of the growth rank, we propose a pair of conditions on a unital and nuclear  $C^*$ -algebra  $A$  which constitute a universal property. The first of these conditions is known

to hold for  $\mathcal{Z}$ . We verify the second condition for  $\mathcal{Z}$  inside a large class of separable, unital, nuclear, and locally subhomogeneous  $C^*$ -algebras which, significantly, contains projectionless algebras. This represents the first uniqueness theorem for  $\mathcal{Z}$  among projectionless algebras which does not depend on the classification of such algebras via the Elliott invariant.

Our paper is organised as follows: in Section 2 we introduce the growth rank and establish its basic properties; in Section 3 we show that the growth rank may be viewed as an abstract version of dimension growth for AH algebras; in Section 4 we establish the range of the growth rank, and consider the growth rank of some examples; tensor factorisation and the existence of a simple, nuclear, and non- $\mathcal{Z}$ -stable  $C^*$ -algebra which is not tensorially prime are contained in Section 5; two universal properties for a simple, separable, unital, and nuclear  $C^*$ -algebra are discussed in Section 6, and the second of these is shown to be satisfied by  $\mathcal{Z}$  inside certain classes of algebras; connections between the growth rank and other ranks for  $C^*$ -algebras are drawn in Section 7, and it is argued that the growth rank is connected naturally to Elliott's conjecture.

## 2. The growth rank of a $C^*$ -algebra

Recall that the Jiang–Su algebra  $\mathcal{Z}$  is a simple, unital, nuclear, and infinite-dimensional  $C^*$ -algebra which is KK-equivalent to the complex numbers (cf. [16]). We say that a  $C^*$ -algebra  $A$  is  $\mathcal{Z}$ -stable if  $A \otimes \mathcal{Z} \cong A$ . The existence of simple, nuclear, separable, and non-elementary  $C^*$ -algebras which are not  $\mathcal{Z}$ -stable was established by Villadsen in [35].

**Definition 2.1.** Let  $A$  be a  $C^*$ -algebra. The *growth rank*  $\text{gr}(A)$  is the least non-negative integer  $n$  such that

$$A^{\otimes n} \stackrel{\text{def}}{=} \underbrace{A \otimes \cdots \otimes A}_{n \text{ times}}$$

is  $\mathcal{Z}$ -stable, assuming the minimal tensor product. If no such integer exists, then say  $\text{gr}(A) = \infty$ .

The growth rank is most interesting for  $C^*$ -algebras without finite-dimensional representations, as these are the only algebras whose finite tensor powers may be  $\mathcal{Z}$ -stable. Thus, the growth rank differs significantly from other notions of rank for nuclear  $C^*$ -algebras—the stable rank, the real rank, the tracial topological rank, and the decomposition rank—see [4,19,26] and [18], respectively, for definitions and basic properties—in that it is not proportional to the covering dimension of the spectrum in the commutative case. Rather, it is designed to recover information about  $C^*$ -algebras which are pathological with respect to the Elliott conjecture.

The permanence properties of  $\mathcal{Z}$ -stability, most of them established in [33], show the growth rank to be remarkably well behaved with respect to common operations.

**Theorem 2.1.** Let  $A, B$  be separable, nuclear  $C^*$ -algebras,  $I$  a closed two-sided ideal of  $A$ ,  $H$  a hereditary subalgebra of  $A$ , and  $k \in \mathbb{N}$ . Then,

- (i)  $\text{gr}(H) \leq \text{gr}(A)$ ;
- (ii)  $\text{gr}(A/I) \leq \text{gr}(A)$ ;
- (iii)  $\text{gr}(A) = \text{gr}(A \otimes M_k) = \text{gr}(A \otimes K)$ ;
- (iv)  $\text{gr}(A \otimes B) \leq \min\{\text{gr}(A), \text{gr}(B)\}$ ;
- (v)  $\text{gr}(A \oplus B) \leq \text{gr}(A) + \text{gr}(B)$ ;

(vi) if  $A_1, \dots, A_k$  are hereditary subalgebras of  $A$ , then

$$\text{gr}\left(\bigoplus_{i=1}^k A_i\right) \leq \text{gr}(A);$$

(vii) if  $A$  is the limit of an inductive sequence  $(A_i, \phi_i)$ , where  $A_i$  is separable, nuclear and satisfies  $\text{gr}(A_i) \leq n$  for each  $i \in \mathbb{N}$ , then  $\text{gr}(A) \leq n$ ;

(viii) if  $\text{gr}(I) = \text{gr}(A/I) = 1$ , then  $\text{gr}(A) = 1$ .

**Proof.** (i) and (ii) are clearly true if  $\text{gr}(A) = \infty$ . Suppose that  $\text{gr}(A) = n \in \mathbb{N}$ , so that  $A^{\otimes n}$  is  $\mathcal{Z}$ -stable. Since  $H^{\otimes n}$  is a hereditary subalgebra of  $A^{\otimes n}$  we conclude that it is  $\mathcal{Z}$ -stable by Corollary 3.3 of [33]— $\mathcal{Z}$ -stability passes to hereditary subalgebras. (ii) follows from Corollary 3.1 of [33] after noticing that  $(A/I)^{\otimes n}$  is a quotient of  $A^{\otimes n}$ .

(iii) is Corollary 3.2 of [33].

For (iv), suppose that  $\text{gr}(A) \leq \text{gr}(B)$ . Then,

$$(A \otimes B)^{\otimes \text{gr}(A)} \cong A^{\otimes \text{gr}(A)} \otimes B^{\otimes \text{gr}(A)}$$

is  $\mathcal{Z}$ -stable since is the tensor product of two algebras, one of which— $A^{\otimes \text{gr}(A)}$ —is  $\mathcal{Z}$ -stable.

For (v), one can use the binomial theorem to write

$$(A \oplus B)^{\otimes \text{gr}(A)+\text{gr}(B)} \cong \bigoplus_{i=0}^{\text{gr}(A)+\text{gr}(B)} A^{\otimes i} \otimes B^{\otimes \text{gr}(A)+\text{gr}(B)-i}.$$

For each  $0 \leq i \leq \text{gr}(A) + \text{gr}(B)$  one has that either  $i \geq \text{gr}(A)$  or  $\text{gr}(A) + \text{gr}(B) - i \geq \text{gr}(B)$ , whence  $A^{\otimes i} \otimes B^{\otimes \text{gr}(A)+\text{gr}(B)-i}$  is  $\mathcal{Z}$ -stable. It follows that  $(A \oplus B)^{\text{gr}(A)+\text{gr}(B)}$  is  $\mathcal{Z}$ -stable, as required.

For (vi) we use the fact that

$$\left(\bigoplus_{i=1}^k A_i\right)^{\otimes \text{gr}(A)}$$

is a direct sum of algebras of the form

$$A_1^{\otimes n_1} \otimes A_2^{\otimes n_2} \otimes \cdots \otimes A_k^{\otimes n_k}, \quad \sum_{i=1}^k n_i = \text{gr}(A),$$

and each such algebra is a hereditary subalgebra of  $A^{\otimes \text{gr}(A)}$ . The desired conclusion now follows from (2).

(vii) is Corollary 3.4 of [33], while (viii) is Theorem 4.3 of the same paper.  $\square$

We defer our calculation of the range of the growth rank until Section 4.

### 3. The growth rank as abstract dimension growth

In this section we couch the growth rank as a measure of dimension growth in the setting of AH algebras. Recall that an unital AH algebra is an inductive limit

$$A \cong \lim_{i \rightarrow \infty} (A_i, \phi_i) \quad (1)$$

where  $\phi_i : A_i \rightarrow A_{i+1}$  is an unital  $*$ -homomorphism and

$$A_i := \bigoplus_{l=1}^{m_i} p_{i,l} (\mathcal{C}(X_{i,l}) \otimes \mathcal{K}) p_{i,l} \quad (2)$$

for compact connected Hausdorff spaces  $X_{i,l}$  of finite covering dimension, projections  $p_{i,l} \in \mathcal{C}(X_{i,l}) \otimes \mathcal{K}$  ( $\mathcal{K}$  is the algebra of compact operators on a separable Hilbert space  $\mathcal{H}$ ), and natural numbers  $m_i$ . Put

$$\phi_{ij} := \phi_{j-1} \circ \phi_{j-2} \circ \cdots \circ \phi_i.$$

We refer to this collection of objects and maps as a *decomposition* for  $A$ . Decompositions for  $A$  are highly non-unique. The proof of Theorem 2.5 in [13] shows that one may assume the  $X_{i,l}$  above to be finite CW-complexes. We make this assumption throughout the sequel.

When we speak of dimension growth for an AH algebra we are referring, roughly, to the asymptotic behaviour of the ratios

$$\frac{\dim(X_{i,l})}{\text{rank}(p_{i,l})},$$

assuming, due to the non-unique nature of decompositions for  $A$ , that we are looking at a decomposition for which these ratios grow at a rate close to some lower limit. If there exists a decomposition for  $A$  such that

$$\lim_{i \rightarrow \infty} \max_{1 \leq l \leq m_i} \left\{ \frac{\dim(X_{i,l})}{\text{rank}(p_{i,l})} \right\} = 0, \quad (3)$$

then  $A$  is said to have *slow dimension growth*. This definition appeared first in [2]. As it turns out, this definition is not suitable for non-simple algebras, at least from the point of view that slow dimension growth should entail good behaviour in ordered K-theory. This is pointed out by Goodearl in [13], and a second, more technical definition of slow dimension growth is introduced. We are interested in a demonstration of principle—that the growth rank yields a theory of dimension growth for AH algebras—and so will limit technicalities by restricting our attention to direct sums of simple and unital AH algebras. In this setting, Goodearl's definition and the one above coincide.

(Slow dimension growth or, occasionally, a slightly stronger version thereof, is an essential hypothesis in classification theorems for simple unital AH algebras.)

Observe that taking the tensor product of an unital AH algebra with itself reduces dimension growth. Indeed, for compact Hausdorff spaces  $X$  and  $Y$  and natural numbers  $m$  and  $n$  one has

$$\mathbf{M}_n(\mathcal{C}(X)) \otimes \mathbf{M}_m(\mathcal{C}(Y)) \cong \mathbf{M}_{nm}(\mathcal{C}(X \times Y));$$

the dimension of the unit in a tensor product of two homogeneous  $C^*$ -algebras is the product of the dimensions of the units, while the dimension of the spectrum of the tensor product is the sum of the dimensions of the spectra. If, for instance, one has a sequence of natural numbers  $n_i \xrightarrow{i \rightarrow \infty} \infty$  and an unital inductive limit algebra  $A = \lim_{i \rightarrow \infty} (A_i, \phi_i)$  where  $A_i = M_{n_i}(C(X_i))$  and  $\dim(X_i) = n_i^m$ , then  $A^{\otimes m+1}$  has slow dimension growth, despite the fact that  $A$  may not;  $A^{\otimes m+1}$  is an inductive limit of the building blocks

$$A_i^{\otimes m+1} \cong M_{n_i^{m+1}}(C((X_i)^{m+1})),$$

and

$$\frac{\dim((X_i)^{m+1})}{n_i^{m+1}} = \frac{(m+1)n_i^m}{n_i^{m+1}} \xrightarrow{i \rightarrow \infty} 0.$$

We use this observation to define a concrete measure of dimension growth for unital AH algebras.

**Definition 3.1.** Let  $A$  be an unital AH algebra. Define the *topological dimension growth*  $\text{tdg}(A)$  to be the least non-negative integer  $n$  such that  $A^{\otimes n}$  has slow dimension growth, if it exists. If no such integer exists, then say  $\text{tdg}(A) = \infty$ .

Roughly, an unital AH algebra with finite topological dimension growth  $n$  has a decomposition for which

$$\dim(X_{i,l}) \propto \text{rank}(p_{i,l})^{n-1}, \quad (4)$$

and no decomposition for which (4) holds with  $n$  replaced by  $m < n$ . One might say that such an algebra has “polynomial dimension growth of order  $n - 1$ .” Similarly, an algebra for which  $\text{tdg} = \infty$  has “exponential dimension growth.”

We now compare the properties of the topological dimension growth to those of the growth rank.

**Lemma 3.1.** *Let  $A$  and  $B$  be unital AH algebras with slow dimension growth. Then,  $A \oplus B$  has slow dimension growth.*

**Proof.** Straightforward.  $\square$

**Lemma 3.2.** *Let  $A$  and  $B$  be simple and unital AH algebras, and suppose that  $A$  has slow dimension growth. Then,  $A \otimes B$  has slow dimension growth.*

**Proof.** We exploit the fact that there is considerable freedom in choosing an inductive limit decomposition for  $A \otimes B$ , even after fixing decompositions for  $A$  and  $B$ . Let  $A$  be decomposed as in (1) and (2), and let

$$B \cong \lim_{j \rightarrow \infty} \left( \bigoplus_{s=1}^{n_j} q_{j,s} M_{t_{j,s}}(C(Y_{j,s})) q_{j,s}, \psi_j \right)$$

be a decomposition of  $B$ , where the  $Y_{j,s}$  are connected compact Hausdorff spaces and the  $q_{j,s} \in M_{t_{j,s}}(C(Y_{j,s}))$  are projections. Put

$$B_j := \bigoplus_{s=1}^{n_j} q_{j,s} M_{t_{j,s}}(C(Y_{j,s})) q_{j,s}.$$

For any strictly increasing sequence  $(r_i)$  of natural numbers one has

$$A \otimes B \cong \lim_{i \rightarrow \infty} (A_{r_i} \otimes B_i, \phi_{r_i, r_{i+1}} \otimes \psi_i).$$

Put

$$M_i \stackrel{\text{def}}{=} \max_{1 \leq s \leq n_i} \{\dim(Y_{i,s})\}.$$

The simplicity of  $A$  implies that for any  $N \in \mathbb{N}$ , there exists  $i_N \in \mathbb{N}$  such that  $\text{rank}(p_{i,l}) \geq N$ ,  $\forall i \geq i_N$ . Choose the sequence  $(r_i)$  so that

$$\min_{1 \leq l \leq m_{r_i}} \{\dim(p_{r_i,l})\} \geq 2^i M_i.$$

A typical direct summand of  $A_{r_i} \otimes B_i$  with connected spectrum has the form

$$(p_{r_i,l} \otimes q_{i,s})(M_{k_{r_i,l} t_{i,s}}(C(X_{r_i,l} \times Y_{i,s}))(p_{r_i,l} \otimes q_{i,s}),$$

whence the condition that  $A \otimes B$  has slow dimension growth amounts to the condition that

$$\liminf_{i \rightarrow \infty} \max_{l,s} \left\{ \frac{\dim(X_{r_i,l}) + \dim(Y_{i,s})}{\text{rank}(p_{r_i,l}) \text{rank}(q_{i,s})} \right\} = 0.$$

We have that

$$\max_{l,s} \left\{ \frac{\dim(X_{r_i,l}) + \dim(Y_{i,s})}{\text{rank}(p_{r_i,l}) \text{rank}(q_{i,s})} \right\}$$

is dominated by

$$\max_l \left\{ \frac{\dim(X_{r_i,l})}{\text{rank}(p_{r_i,l}) \text{rank}(q_{i,s})} \right\} + \max_s \left\{ \frac{\dim(Y_{i,s})}{\text{rank}(p_{r_i,l}) \text{rank}(q_{i,s})} \right\}.$$

In the above sum the first term tends to zero by virtue of  $A$  having slow dimension growth, while the second tends to zero by our choice of  $(r_i)$ . We conclude that  $A \otimes B$  has slow dimension growth, as desired.  $\square$

**Theorem 3.1.** *Let  $A, B$  be simple and unital AH algebras. Then,*

- (i)  $\text{tdg}(A \otimes B) \leq \min\{\text{tdg}(A), \text{tdg}(B)\}$ ;
- (ii)  $\text{tdg}(A \oplus B) \leq \text{tdg}(A) + \text{tdg}(B)$ .

**Proof.** For (i), suppose that  $\text{tdg}(A) \leq \text{tdg}(B)$ . Then,

$$(A \otimes B)^{\otimes \text{tdg}(A)} \cong (A^{\otimes \text{tdg}(A)}) \otimes (B^{\otimes \text{tdg}(A)}).$$

Since  $A^{\otimes \text{tdg}(A)}$  has slow dimension growth by definition, the right-hand side of the equation above has slow dimension growth by Lemma 3.2.

For (ii), use the binomial theorem to write

$$(A \oplus B)^{\otimes \text{tdg}(A)+\text{tdg}(B)} \cong \bigoplus_{i=0}^{\text{tdg}(A)+\text{tdg}(B)} A^{\otimes i} \otimes B^{\otimes \text{tdg}(A)+\text{tdg}(B)-i}.$$

Notice that each direct summand of the right-hand side above has slow dimension growth by part (2) of this proposition, whence the entire direct sum has slow dimension growth by Lemma 3.1.  $\square$

As far as direct sums of simple unital AH algebras are concerned, the properties of the growth rank agree with those of the topological dimension growth, despite the fact that  $\mathcal{Z}$ -stability and slow dimension growth are not yet known to be equivalent for simple, unital and infinite-dimensional AH algebras.

Next, we prove that the topological dimension growth and the growth rank often agree. Recall that a Bauer simplex is a compact metrizable Choquet simplex  $S$  whose extreme boundary  $\partial_e S$  is compact. The set  $\text{Aff}(S)$  of continuous affine real-valued functions on  $S$  are in bijective correspondence with continuous real-valued functions on  $\partial_e S$ . The bijection is given by the map which assigns to a continuous affine function  $f$  on  $S$ , the continuous function  $\hat{f} : \partial_e S \rightarrow \mathbb{R}$  given by

$$\hat{f}(\tau) = f(\tau), \quad \forall \tau \in \partial_e S.$$

**Proposition 3.1.** *Let  $A$  be a simple, unital and infinite-dimensional AH algebra. Suppose that the simplex of tracial states  $\text{TA}$  is a Bauer simplex, and that the image of  $\text{K}_0(A)$  in  $C_{\mathbb{R}}(\partial_e \text{TA})$  is uniformly dense. Then,*

$$\text{tdg}(A) = \text{gr}(A).$$

**Proof.** It is well known that

$$\partial_e \text{TA}^{\otimes n} \cong \bigtimes_{i=1}^n \partial_e \text{TA},$$

whence,

$$C_{\mathbb{R}}(\partial_e \text{TA}^{\otimes n}) \cong C_{\mathbb{R}}(\partial_e \text{TA})^{\otimes n}.$$

Let  $f_1, \dots, f_n \in C_{\mathbb{R}}(\partial_e TA)$  be the images of elements  $x_1, \dots, x_n \in K_0(A)$ , respectively. Write  $x_i = [p_i] - [q_i]$  for projections  $p_i, q_i \in M_\infty(A)$ ,  $1 \leq i \leq n$ . Let  $g_i, h_i \in C_{\mathbb{R}}(\partial_e TA)$  be the images of  $p_i, q_i$ , respectively. Now

$$f_1 \otimes \cdots \otimes f_n = \bigotimes_{i=1}^n (g_i - h_i), \quad (5)$$

and the right-hand side of the equation is a sum of elementary tensor  $\pm(r_1 \otimes \cdots \otimes r_n)$ , where  $r_i \in \{h_i, g_i\}$ . There are thus projections  $t_i \in \{p_i, q_i\}$  such that the image of

$$[t_1 \otimes \cdots \otimes t_n] \in K_0(A^{\otimes n})$$

is

$$r_1 \otimes \cdots \otimes r_n \in C_{\mathbb{R}}(\partial_e TA)^{\otimes n}.$$

Thus, the right-hand side of (5) can be obtained as the image of some  $x \in K_0(A^{\otimes n})$ . Given  $\epsilon > 0$  and an elementary tensor

$$m_1 \otimes \cdots \otimes m_n \in C_{\mathbb{R}}(\partial_e TA)^{\otimes n} \cong C_{\mathbb{R}}(\partial_e TA^{\otimes n}),$$

we may, by the density of the image of  $K_0(A)$  in  $C_{\mathbb{R}}(\partial_e TA)$ , choose  $f_1, \dots, f_n \in C_{\mathbb{R}}(\partial_e TA)$  to satisfy

$$|(m_1 \otimes \cdots \otimes m_n) - (f_1 \otimes \cdots \otimes f_n)| < \epsilon.$$

It follows that the image of  $K_0(A^{\otimes n})$  is dense in  $C_{\mathbb{R}}(\partial_e TA^{\otimes n})$ . Theorem 3.13 of [34] shows that  $\mathcal{Z}$ -stability and slow dimension growth are equivalent for such an algebra, and the proposition follows.  $\square$

Following [34], we may drop the condition that the image of  $K_0$  in  $\text{Aff}(T(A))$  be dense whenever  $A$  has a unique tracial state. Note that an algebra satisfying the hypotheses of Proposition 3.1 need not have real rank zero, even in the case of a unique tracial state (cf. [36]).

As the growth rank and the topological dimension growth often (probably always, in the simple and unital case) agree, we suggest simply using the growth rank as a theory of unbounded dimension growth for AH algebras. It has the advantage of avoiding highly technical definitions involving arbitrary decompositions for a given AH algebra, and works equally well for non-simple and non-unital algebras.

There are definitions of slow dimension growth for more general locally type-I  $C^*$ -algebras—direct limits of recursive subhomogeneous algebras [25], for instance—but these are even more technical than the definition for non-simple AH algebras. The growth rank seems the logical choice for defining dimension growth in these situations, too.

#### 4. A range result

**Theorem 4.1.** *Let  $n \in \mathbb{N} \cup \{\infty\}$ . There exists a simple, nuclear, and non-type-I  $C^*$ -algebra  $A$  such that*

$$\text{tdg}(A) = \text{gr}(A) = n.$$

To prepare the proof of Theorem 4.1, we collect some basic facts about the Euler and Chern classes of a complex vector bundle, and recall results of Rørdam and Villadsen.

Let  $X$  be a connected topological space, and let  $\omega$  and  $\gamma$  be (complex) vector bundles over  $X$  of fibre dimensions  $k$  and  $m$ , respectively. Recall that the Euler class  $e(\omega)$  is an element of  $H^{2k}(X; \mathbb{Z})$  with the following properties:

- (i)  $e(\omega \oplus \gamma) = e(\omega) \cdot e(\gamma)$ , where “.” denotes the cup product in  $H^*(X; \mathbb{Z})$ ;
- (ii)  $e(\theta_l) = 0$ , where  $\theta_l$  denotes the trivial vector bundle over  $X$  of (complex) fibre dimension  $l$ .

The Chern class  $c(\omega) \in H^*(X; \mathbb{Z})$  is a sum

$$c(\omega) = 1 + c_1(\omega) + c_2(\omega) + \cdots + c_k(\omega),$$

where  $c_i(\omega) \in H^{2i}(X; \mathbb{Z})$ . Its properties are similar to those of the Euler class:

- (i)  $c(\omega \oplus \gamma) = c(\omega) \cdot c(\gamma)$ ;
- (ii)  $c(\theta_l) = 1$ .

The key connection between these characteristic classes is this:  $e(\eta) = c_1(\eta)$  for every line bundle.

The next lemma is due essentially to Villadsen (cf. [35, Lemma 1]), but our version is more general.

**Lemma 4.1.** *Let  $X$  be a finite CW-complex, and let  $\eta_1, \eta_2, \dots, \eta_k$  be complex line bundles over  $X$ . If  $l < k$  and  $\prod_{i=1}^k e(\eta_i) \neq 0$ , then*

$$[\eta_1 \oplus \eta_2 \oplus \cdots \oplus \eta_k] - [\theta_l] \notin K^0(X)^+.$$

**Proof.** If  $[\eta_1 \oplus \eta_2 \oplus \cdots \oplus \eta_k] - [\theta_l] \in K^0(X)^+$ , then there is a vector bundle  $\gamma$  of dimension  $k - l$  and  $d \in \mathbb{N}$  such that

$$\eta_1 \oplus \eta_2 \oplus \cdots \oplus \eta_k \oplus \theta_d \cong \gamma \oplus \theta_{d+k-l}.$$

Applying the Chern class to both sides of this equation we obtain

$$\prod_{i=1}^k c(\eta_i) = c(\gamma).$$

Expanding the left-hand side yields

$$\prod_{i=1}^k (1 + c_1(\eta_i)) = \prod_{i=1}^k (1 + e(\eta_i)).$$

The last product has only one term in  $H^{2k}(X; \mathbb{Z})$ , namely,  $\prod_{i=1}^k e(\eta_i)$ , and this, in turn, is non-zero. On the other hand,  $c(\gamma)$  has no non-zero term in  $H^{2i}(X; \mathbb{Z})$  for  $i > k - l$ . Thus, we have a contradiction, and must conclude that

$$[\eta_1 \oplus \eta_2 \oplus \cdots \oplus \eta_k] - [\theta_l] \notin K^0(X)^+. \quad \square$$

Let  $\xi$  be any line bundle over  $S^2$  with non-zero Euler class—the Hopf line bundle, for instance. We recall some notation and a proposition from [29]. Let  $Y$  and  $Z$  be topological spaces, and let  $\eta$  and  $\beta$  be vector bundles over  $Y$  and  $Z$ , respectively. Let

$$\pi_Y : Y \times Z \rightarrow Y, \quad \pi_Z : Y \times Z \rightarrow Z$$

be the co-ordinate projections, and  $\pi_Y^*(\eta)$  and  $\pi_Z^*(\beta)$  the induced bundles over  $Y \times Z$ . The external tensor product of  $\eta$  and  $\beta$  is defined to be the internal (fibre-wise) tensor product  $\pi_Y^*(\eta) \otimes \pi_Z^*(\beta)$ ; we let  $\eta \otimes \beta$  denote this tensor product in the sequel, to avoid cumbersome notation. For each natural number  $s$  and for each non-empty finite set

$$I = \{s_1, \dots, s_k\} \subseteq \mathbb{N}$$

define bundles  $\xi_s$  and  $\xi_I$  over  $S^{2m}$  (for all  $m \geq s$  or  $m \geq \max\{s_1, \dots, s_k\}$ , as appropriate) by

$$\xi_s = \pi_s^*(\xi), \quad \xi_I = \xi_{s_1} \otimes \cdots \otimes \xi_{s_k},$$

where  $\pi_s : S^{2 \times m} \rightarrow S^2$  is the  $s$ th co-ordinate projection.

**Proposition 4.1.** (Rørdam [29, Proposition 3.2].) *Let  $I_1, \dots, I_m \subseteq \mathbb{N}$  be finite sets. The following are equivalent:*

- (i)  $e(\xi_{I_1} \oplus \xi_{I_2} \oplus \cdots \oplus \xi_{I_m}) \neq 0$ .
- (ii) *For all subsets  $F$  of  $\{1, 2, \dots, m\}$  we have  $|\bigcup_{j \in F} I_j| \geq |F|$ .*

(In fact, there is a third equivalence in Proposition 3.2 of [29]. We do not require it, and so omit it.)

**Proof.** (Theorem 4.1) The case where  $n = 1$  is straightforward: any UHF algebra  $\mathcal{U}$  is  $\mathcal{Z}$ -stable by the classification theorem of [16] (or, alternatively, by Theorem 2.3 of [33]), and has  $\text{tdg}(A) = 1$  (as does any AF algebra).

Let  $1 \leq n \in \mathbb{N} \cup \{\infty\}$  be given. We will construct an simple, unital, and infinite-dimensional AH algebra

$$A = \lim_{i \rightarrow \infty} (A_i, \phi_i)$$

along the lines of the construction of [36], and prove that  $\text{gr}(A) = \text{tdg}(A) = n + 1$ . Our strategy is to prove that  $A^{\otimes n}$  has a perforated ordered  $K_0$ -group and is hence not  $\mathcal{Z}$ -stable by Theorem 1 of [15], while  $A^{\otimes n+1}$  is tracially AF and hence approximately divisible by [10] and  $\mathcal{Z}$ -stable by Theorem 2.3 of [34].  $A$  will be constructed so as to have a unique trace, whence  $\text{tdg}(A) = \text{gr}(A)$  by Proposition 3.1.

Let  $X_1 = (S^2)^{n_1}$ , and, for each  $i \in \mathbb{N}$ , let  $X_i = (X_{i-1})^{n_i}$ , where the  $n_i$  are natural numbers to be specified. Set

$$N_i := \prod_{j=1}^i n_j$$

and

$$I_l^i := \{(l-1)N_i + N_{i-1} + 1, \dots, lN_i\} \subseteq \{1, \dots, nN_i\}, \quad l \in \{1, \dots, n\}.$$

We will take

$$A_i = p_i(C(X_i) \otimes \mathcal{K})p_i$$

for some projection  $p_i \in C(X_i) \otimes \mathcal{K}$  to be specified. The maps

$$\phi_i : A_i \rightarrow A_{i+1}$$

are constructed inductively as follows: suppose that  $p_1, \dots, p_i$  have been chosen, and define a map

$$\tilde{\phi}_i : A_i \rightarrow C(X_{i+1}) \otimes \mathcal{K}$$

by taking the direct sum of the map  $\gamma_i : A_i \rightarrow C(X_{i+1}) \otimes \mathcal{K}$  given by

$$\gamma_i(f)(x) = f(\omega_i(x))$$

( $\omega_i : X_{i+1} \rightarrow X_i$  is projection onto the first factor of  $X_{i+1} = (X_i)^{n_{i+1}}$ ) and  $m_{i+1}$  copies of the map  $\eta_i : A_i \rightarrow C(X_{i+1}) \otimes \mathcal{K}$  given by

$$\eta_i(f)(x) = f(x_i) \cdot \xi_{I_i^{i+1}}$$

( $x_i \in X_i$  is a point to be specified, and  $m_{i+1}$  is a natural number to be specified); set

$$A_{i+1} := \tilde{\phi}_i(p_i)C(X_{i+1}) \otimes \mathcal{K}\tilde{\phi}_i(p_i),$$

and let  $\phi_i$  be the restriction of  $\tilde{\phi}_i$  to  $A_{i+1}$ . In [36] it is shown that by replacing the  $x_i$  with various other points from  $X_{i+1}$  in a suitable manner, one can ensure a simple limit  $A := \lim_{i \rightarrow \infty} (A_i, \phi_i)$ .  $A$  is unital by construction.

Let  $p_1$  be the projection over  $X_1$  corresponding to the Whitney sum

$$\theta_1 \oplus \xi_{I_1^1} \oplus \xi_{I_1^1}.$$

By Proposition 3.2 of [29] (Proposition 4.1 of this section) we have that the Euler class of  $\bigoplus_{j=1}^k \xi_{I_j^1}$  is non-zero whenever  $k \leq n_1$ . By Lemma 4.1, we have

$$2[\xi_{I_1^1}] - [\theta_1] \notin K_0(A_1)^+.$$

For each  $i \in \mathbb{N}$  one has

$$K_0(A_i^{\otimes n}) \cong K^0(X_i^n) \cong K^0(X_i)^{\otimes n}$$

—the last isomorphism follows from the Künneth formula and the fact that  $K^0(X_i)$  is torsion free. We will prove that

$$(2[\phi_{1i}(\xi_{I_1^1})] - [\phi_{1i}(\theta_1)]) \otimes [p_i] \otimes [p_i] \otimes \cdots \otimes [p_i] \notin K_0(A_i^{\otimes n})^+, \quad \forall i \in \mathbb{N}, \quad (6)$$

whence  $K_0(A^{\otimes n})$  is a perforated ordered group, and  $A^{\otimes n}$  is not  $\mathcal{Z}$ -stable.

Let  $\pi_l : (X_i)^l \rightarrow X_i$  be the  $l$ th co-ordinate projection, and set  $p_i^l := \pi_l^*(p_i)$ . The tensor product of group elements in (6) corresponds to the external tensor product of the corresponding (formal difference of) vector bundles. In other words, proving that (6) holds amounts to proving that

$$2[\phi_{1i}(\xi_{I_1^1}) \otimes p_i^2 \otimes \cdots \otimes p_i^n] - [\phi_{1i}(\theta_1) \otimes p_i^2 \otimes \cdots \otimes p_i^n] \notin K^0(X_i^n)^+.$$

Straightforward calculation shows that  $\theta_1$  is a direct summand of  $\phi_{1i}(\theta_1) \otimes p_i^2 \otimes \cdots \otimes p_i^n$ , for all  $i \in \mathbb{N}$ . Thus,

$$\begin{aligned} & 2[\phi_{1i}(\xi_{I_1^1}) \otimes p_i^2 \otimes \cdots \otimes p_i^n] - [\phi_{1i}(\theta_1) \otimes p_i^2 \otimes \cdots \otimes p_i^n] \\ & \leq 2[\phi_{1i}(\xi_{I_1^1}) \otimes p_i^2 \otimes \cdots \otimes p_i^n] - [\theta_1], \end{aligned}$$

and (6) will hold if

$$2[\phi_{1i}(\xi_{I_1^1}) \otimes p_i^2 \otimes \cdots \otimes p_i^n] - [\theta_1] \notin K^0(X_i^n)^+. \quad (7)$$

We prove that (7) holds by induction. Assume that  $i = 1$ . The projection  $p_1^l$  corresponds to the vector bundle  $\xi_{I_1^1} \oplus \xi_{I_1^1} \oplus \theta_1 = 2\xi_{I_1^1} \oplus \theta_1$  over  $X_1^n \cong (S^2)^{n_1 n}$ . Now

$$\left[ \bigotimes_{l=2}^n p_1^l \right] = \left[ \bigotimes_{l=2}^n (2\xi_{I_1^1} \oplus \theta_1) \right] = \left[ \bigoplus_{\emptyset \neq J \subseteq \{2, \dots, n\}} \left( \bigotimes_{l \in J} 2\xi_{I_1^1} \right) \oplus \theta_1 \right],$$

so that

$$2[\phi_{1i}(\xi_{I_1^1}) \otimes p_i^2 \otimes \cdots \otimes p_i^n] = \left[ 2\xi_{I_1^1} \otimes \left( \bigoplus_{\emptyset \neq J \subseteq \{2, \dots, n\}} \left( \bigotimes_{l \in J} 2\xi_{I_1^1} \right) \right) \right] + [2\xi_{I_1^1}].$$

By Lemma 4.1 it will suffice to show that

$$e\left(\bigoplus_{\emptyset \neq J \subseteq \{2, \dots, n\}} 2\xi_{I_1^1} \otimes \left(\bigotimes_{l \in J} 2\xi_{I_l^1}\right) \oplus 2\xi_{I_1^1}\right) \neq 0.$$

Letting  $I_J$  denote the union  $\bigcup_{l \in J} I_l^1$  we have

$$\bigoplus_{\emptyset \neq J \subseteq \{2, \dots, n\}} 2\xi_{I_1^1} \otimes \left(\bigotimes_{l \in J} 2\xi_{I_l^1}\right) \oplus 2\xi_{I_1^1} = 2\xi_{I_1^1} \oplus \bigoplus_{\emptyset \neq J \subseteq \{2, \dots, n\}} 2^{|J|+1} \xi_{I_1^1 \cup I_J}. \quad (8)$$

We wish to apply Proposition 3.2 of [29] to conclude that the Euler class of the bundle above is non-zero. This will, of course, require that  $n_1$  be sufficiently large. One easily sees that the dimension of the bundle above is  $2 \cdot 3^{n-1}$ . Let  $R_1 = I_1^1$ , and define a list of subsets  $R_2, \dots, R_{2 \cdot 3^{n-1}}$  of  $\mathbb{N}$  by including, for each  $J \subseteq \{2, \dots, n\}$ ,  $2^{|J|+1}$  copies of  $I_1^1 \cup I_J$  among the  $R_j$ . The  $R_j$ s are the index sets of the tensor products of Hopf line bundles appearing as direct summands in (8). We must choose  $n_1$  large enough so that, for any finite subset  $F$  of  $\{1, \dots, 2 \cdot 3^{n-1}\}$ , we have  $|\bigcup_{j \in F} R_j| \geq |F|$ . Clearly, setting  $n_1 = 3^n$  will suffice. This establishes the base case of our induction argument.

We proceed to the induction step. By Lemma 4.1 it will suffice to prove that

$$e\left(2\phi_{1i}(\xi_{I_1^1}) \otimes \bigotimes_{l=2}^n p_i^l\right) \neq 0.$$

Suppose that for all  $k < i$ ,  $n_k$  has been chosen large enough that

$$e\left(2\phi_{1k}(\xi_{I_1^1}) \otimes \bigotimes_{l=2}^n p_k^l\right) \neq 0.$$

Put  $\omega_{i,l} := \omega_i \circ \pi_l$ . By construction we have

$$p_i^l = \omega_{i,l}^*(p_{i-1}^l) \oplus m_i \cdot \dim(p_{i-1}^l) \otimes \xi_{I_l^i}$$

and

$$\phi_{1i}(\xi_{I_1^1}) = \omega_{i,1}^*(\phi_{1,i-1}(\xi_{I_1^1})) \oplus m_i \cdot \dim(\phi_{1,i-1}(\xi_{I_1^1})) \otimes \xi_{I_1^i}.$$

It follows that

$$\begin{aligned} 2\phi_{1i}(\xi_{I_1^1}) \otimes \bigotimes_{l=2}^n p_i^l &= 2\left(\omega_i^*(\phi_{1,i-1}(\xi_{I_1^1})) \oplus m_i \cdot \dim(\phi_{1,i-1}(\xi_{I_1^1})) \otimes \xi_{I_1^i}\right) \\ &\quad \otimes \left(\bigotimes_{l=2}^n (\omega_{i,l}^*(p_{i-1}^l) \oplus m_i \cdot \dim(p_{i-1}^l) \otimes \xi_{I_l^i})\right) \end{aligned}$$

$$\begin{aligned}
&= \left( \omega_{i,1}^*(2\phi_{1,i-1}(\xi_{I_1^l})) \otimes \bigotimes_{l=2}^n \omega_{i,l}^*(p_{i-1}^l) \right) \oplus B \\
&= \Gamma_i^* \left( 2\phi_{1,i-1}(\xi_{I_1^l}) \otimes \bigotimes_{l=2}^n p_{i-1}^l \right) \oplus B,
\end{aligned}$$

where

$$\Gamma_i := \omega_{i,1} \times \omega_{i,2} \times \cdots \times \omega_{i,n} : (X_i)^n \rightarrow (X_{i-1})^n$$

and  $B$  is a sum of line bundles, each of which has  $\xi_{I_l^i}$  as a tensor factor for some  $l \in \{1, \dots, n\}$ . The index sets of the line bundles making up

$$\Gamma_i^* \left( 2\phi_{1,i-1}(\xi_{I_1^l}) \otimes \bigotimes_{l=2}^n p_{i-1}^l \right)$$

are disjoint from each  $I_l^i$  by construction, so by Proposition 3.2 of [29] we have

$$e \left( \Gamma_i^* \left( 2\phi_{1,i-1}(\xi_{I_1^l}) \otimes \bigotimes_{l=2}^n p_{i-1}^l \right) \oplus B \right) \neq 0$$

if

$$e \left( \Gamma_i^* \left( 2\phi_{1,i-1}(\xi_{I_1^l}) \otimes \bigotimes_{l=2}^n p_{i-1}^l \right) \right) \neq 0, \quad e(B) \neq 0.$$

The first inequality follows from our induction hypothesis. For the second inequality, we have

$$\dim(B) < \dim \left( 2\phi_{1,i}(\xi_{I_1^l}) \otimes \bigotimes_{l=2}^n p_i^l \right) < \dim(p_i^l)^n.$$

Choosing  $n_i$  just large enough (for reasons to be made clear shortly) to ensure that  $|I_1^i| \geq \dim(p_i^l)^n$ , we may conclude by Proposition 3.2 of [29] that  $e(B) \neq 0$ , as desired.

The fact that  $A^{\otimes n}$  has a perforated ordered  $K_0$ -group implies that  $\text{gr}(A) > n$ . We now show that  $\text{gr}(A) \leq n+1$ . First, we compute an upper bound on the dimension of  $X_i$ . We have chosen  $n_i$  to be just large enough to ensure that  $|I_1^i| \geq \dim(p_i)^n$ . Using the fact that  $|I_1^i| = N_i - N_{i-1}$  we have

$$N_i \leq \dim(p_i)^n + 2N_{i-1}, \quad i \in \mathbb{N},$$

since one could otherwise reduce the size of  $n_i$  by one or more. Set  $d_i := \dim(p_i)$  for brevity.  $A^{\otimes n+1}$  will have slow dimension growth if

$$\frac{(n+1)N_i}{d_i^{n+1}} \xrightarrow{i \rightarrow \infty} 0.$$

We have

$$\frac{(n+1)N_i}{d_i^{n+1}} \leq \frac{(n+1)(d_i^n + 2N_{i-1})}{d_i^{n+1}} = \frac{n+1}{d_i} + \frac{2N_{i-1}}{d_i^{n+1}},$$

so that we need only show

$$\frac{2N_{i-1}}{d_i^{n+1}} \xrightarrow{i \rightarrow \infty} 0. \quad (9)$$

But  $N_{i-1}$  does not depend on  $m_i$ , so we may make  $d_i$  large enough for (9) to hold (remember that  $m_i$  may be chosen before  $n_i$ ). Thus,  $A^{\otimes n+1}$  has slow dimension growth.  $A$  has a unique tracial state by the arguments of [36], whence so does  $A^{\otimes n+1}$ . It follows that  $A^{\otimes n+1}$  is of real rank zero by the main theorem of [1]. The reduction theorem of [6] together with the classification theorem of [9] then show that  $A^{\otimes n+1}$  is approximately divisible, whence  $A^{\otimes n+1}$  is  $\mathcal{Z}$ -stable by Theorem 2.3 of [34] and  $\text{gr}(A) = n+1$ . Since  $A$  has a unique trace, it satisfies the hypotheses of Proposition 3.1, whence  $\text{tdg}(A) = \text{gr}(A) = n+1$ . This proves Theorem 4.1 for  $n$  finite.

To produce an algebra with infinite growth rank, we follow the construction above, but choose the  $n_i$  larger at each stage. Begin as above with the same choice of  $A_1$ . Notice that the arguments above not only show that one can choose  $n_i$  large enough so that

$$2[\phi_{1i}(\xi_{I_1^1}) \otimes p_i^2 \otimes \cdots \otimes p_i^n] - [\theta_1] \notin K^0(X_i^n)^+,$$

but also large enough so that

$$2[\phi_{1i}(\xi_{I_1^1}) \otimes p_i^2 \otimes \cdots \otimes p_i^{in}] - [\theta_1] \notin K^0(X_i^{in})^+.$$

With the latter choice of  $n_i$ , one has that  $K_0(A^{\otimes in})$  is a perforated ordered group for every natural number  $i$ . It follows that  $\text{gr}(A) = \infty$ . Now Proposition 3.1 shows that  $\text{tdg}(A) = \infty$ , proving the theorem in full.

Finally, in the case where  $\text{gr}(A) \geq 3$ , we modify the base spaces  $X_i$  to facilitate stable and real rank calculations in the sequel. Replace  $X_i$  with  $X'_i := X_i \times \mathbb{D}^{id_i^2}$ , where  $\mathbb{D}$  denotes the closed unit disc in  $\mathbb{C}$ , and replace the eigenvalue map  $\omega_i$  with the map with a map  $\omega'_i : X'_{i+1} \rightarrow X'_i$  given by the Cartesian product of  $\omega_i$  and any co-ordinate projection  $\lambda_i : \mathbb{D}^{(i+1)d_{i+1}^2} \rightarrow \mathbb{D}^{id_i^2}$ . On the one hand, these modifications are trivial at the level of  $K_0$ , whence the proof of the lower bound on the growth rank of  $A$  carries over to our new algebra. On the other hand, our new algebra has

$$\frac{\dim(X'_i)}{d_i^{n+1}} = \frac{\dim(X_i) + id_i^2}{d_i^{n+1}} \xrightarrow{i \rightarrow \infty} 0,$$

since  $n \geq 2$ . Thus, the specified adjustment to the construction of  $A$  does not increase the topological dimension growth.  $\square$

We now consider the growth rank of some examples.

**gr = 1.** Let  $A$  be a separable, unital, and approximately divisible  $C^*$ -algebra. Then  $\text{gr}(A) = 1$  by Theorem 2.3 of [34].

Let  $A$  be a simple nuclear  $C^*$ -algebra which is neither finite-dimensional nor isomorphic to the compact operators. Suppose further that  $A$  belongs to a class of  $C^*$ -algebras for which Elliott's classification conjecture is currently confirmed (cf. [28]). It follows from various results in [34] that  $A$  is  $\mathcal{Z}$ -stable, whence  $\text{gr}(A) = 1$ .

**gr = 2.** Let  $A$  be a simple, unital AH algebra given as the limit of an inductive system

$$(p_i(C(X_i) \otimes \mathcal{K})p_i, \phi_i),$$

where the  $X_i$  are compact connected Hausdorff spaces,  $p_i \in C(X_i) \otimes \mathcal{K}$  is a projection, and

$$\phi_i : p_i(C(X_i) \otimes \mathcal{K})p_i \rightarrow p_{i+1}(C(X_{i+1}) \otimes \mathcal{K})p_{i+1}$$

is an unital  $*$ -homomorphism. Suppose that

$$\frac{\dim(X_i)}{\dim(p_i)} \xrightarrow{i \rightarrow \infty} c \in \mathbb{R}, \quad c \neq 0.$$

Since  $\dim(p_i) \rightarrow \infty$  as  $i \rightarrow \infty$  by simplicity, we have that  $2 \geq \text{tdg}(A) \geq \text{gr}(A)$ . If  $A$  is not  $\mathcal{Z}$ -stable, then  $\text{gr}(A) = 2$ . Many of the known examples of non- $\mathcal{Z}$ -stable simple, nuclear  $C^*$ -algebras have this form, including the AH algebras of [35] having perforated ordered  $K_0$ -groups, those of [36] having finite non-minimal stable rank, the algebra  $B$  of [27] which is not stable but for which  $M_2(B)$  is stable, and the counterexample to Elliott's classification conjecture in [32].

Let  $A$  be a simple, nuclear  $C^*$ -algebra containing a finite and an infinite projection and satisfying the UCT (the existence of such algebras is established in [29]). Kirchberg proves in [17] that the tensor product of any two simple, unital and infinite-dimensional  $C^*$ -algebras is either stably finite or purely infinite. It follows that  $A \otimes A$  is purely infinite and hence  $\mathcal{Z}$ -stable, so  $\text{gr}(A) = 2$ .

**gr > 2.** The algebras in Theorem 4.1 are the first examples of simple nuclear algebras with finite growth rank strictly greater than 2. The algebra of [36] having infinite stable rank probably also has infinite growth rank—it bears a more than passing resemblance to the algebra of infinite growth rank in Theorem 4.1.

## 5. Tensor factorisation

A simple  $C^*$ -algebra is said to be *tensorially prime* if it cannot be written as a tensor product  $C \otimes D$ , where both  $C$  and  $D$  are simple and non-type-I. It has been surprising to find that the majority of our stock-in-trade simple, separable, and nuclear  $C^*$ -algebras are not tensorially prime—every class of simple, separable, and nuclear  $C^*$ -algebras for which the Elliott invariant is currently confirmed consists of  $\mathcal{Z}$ -stable members [16,34]. Kirchberg [17] has shown that every simple exact  $C^*$ -algebra which is not tensorially prime is either stably finite or purely infinite. Rørdam has produced an example of a simple nuclear  $C^*$ -algebra containing both a finite and an infinite projection which, in light of Kirchberg's result, is tensorially prime [29].

The question of whether every simple, nuclear, and non- $\mathcal{Z}$ -stable  $C^*$ -algebra is tensorially prime has remained open. Theorem 4.1 settles this question, negatively.

**Corollary 5.1.** *There exists a simple, nuclear, and non- $\mathcal{Z}$ -stable  $C^*$ -algebra which is not tensorially prime.*

**Proof.** Let  $A$  be the algebra of growth rank three in Theorem 4.1.  $A \otimes A$  satisfies the hypotheses of the corollary, yet is evidently not tensorially prime.  $\square$

It is interesting to ask whether simple, unital, and nuclear  $C^*$ -algebras which fail to be tensorially prime must in fact have an infinite factorisation, i.e., can be written as  $\bigotimes_{i=1}^{\infty} C_i$ , where each  $C_i$  is simple, unital, nuclear, and non-type-I. This is trivially true for  $\mathcal{Z}$ -stable algebras, since  $\mathcal{Z} \cong \mathcal{Z}^{\otimes \infty}$  (cf. [16]). Rørdam has asked whether every simple, unital, nuclear, and non-type-I  $C^*$ -algebra admits an unital embedding of  $\mathcal{Z}$ . If this turns out to be true for separable algebras, then Theorem 7.2.2 of [28] implies that infinite tensor products of such algebras are always  $\mathcal{Z}$ -stable. This, in turn, will imply that simple, unital, separable, and nuclear  $C^*$ -algebras which do not absorb  $\mathcal{Z}$  tensorially cannot have an infinite tensor factorisation.

## 6. Universal properties and infinite tensor products

Little is known about the extent to which  $\mathcal{Z}$  is unique, save that it is determined by its K-theory inside a small class of  $\mathcal{Z}$ -stable inductive limit algebras [21,34]. The Elliott conjecture, which may well hold for the class of simple, separable, nuclear, infinite-dimensional and  $\mathcal{Z}$ -stable  $C^*$ -algebras, predicts that  $\mathcal{Z}$  will be the unique such algebra which is furthermore unital, projectionless, unique trace, and KK-equivalent to  $\mathbb{C}$ .

Rørdam has suggested the following universal property, which could conceivably be verified for  $\mathcal{Z}$  within the class of separable, unital, and nuclear  $C^*$ -algebras having no finite-dimensional representations.

**Universal Property 6.1.** *Let  $\mathcal{C}$  be a class of separable, unital, and nuclear  $C^*$ -algebras. If  $A$  in  $\mathcal{C}$  is such that*

- (i) *every unital endomorphism of  $A$  is approximately inner, and*
- (ii) *every  $B$  in  $\mathcal{C}$  admits an unital embedding  $\iota: A \rightarrow B$ ,*

*then  $A$  is unique up to  $*$ -isomorphism.*

**Proof.** Elliott's Intertwining Argument (cf. [9]).  $\square$

We propose a second property.

**Universal Property 6.2.** *Let  $\mathcal{C}$  be a class of unital and nuclear  $C^*$ -algebras. If  $A$  in  $\mathcal{C}$  is such that*

- (i)  *$A^{\otimes \infty} \cong A$ , and*
- (ii)  *$B^{\otimes \infty} \otimes A \cong B^{\otimes \infty}$  for every  $B$  in  $\mathcal{C}$ ,*

*then  $A$  is unique up to  $*$ -isomorphism.*

**Proof.** Suppose that  $A, B$  in  $\mathcal{C}$  satisfy (i) and (ii) above. Then,

$$A \xrightarrow{(i)} A^{\otimes\infty} \xrightarrow{(ii)} A^{\otimes\infty} \otimes B \xrightarrow{(i)} A \otimes B^{\otimes\infty} \xrightarrow{(ii)} B^{\otimes\infty} \xrightarrow{(i)} B. \quad \square$$

Universal Properties 6.1 and 6.2 have the same basic structure. In each case, condition (i) is intrinsic and known to hold for the Jiang–Su algebra  $\mathcal{Z}$ , while condition (ii) is extrinsic and potentially verifiable for  $\mathcal{Z}$ . Conditions 6.1(i) and 6.2(i) are skew, but not completely so: the separable, unital  $C^*$ -algebras satisfying both conditions are precisely the strongly self-absorbing  $C^*$ -algebras studied in [33]. Any separable, unital, and nuclear  $C^*$ -algebra  $A$  which admits an unital embedding of  $\mathcal{Z}$  then satisfies  $A^{\otimes\infty} \otimes \mathcal{Z} \cong A^{\otimes\infty}$  (cf. Theorem 7.2.2 of Rørdam); if Universal Property 6.1 is satisfied by  $\mathcal{Z}$  inside a class  $\mathcal{C}$  of separable, unital, and nuclear  $C^*$ -algebras, then the same is true of Universal Property 6.2. The attraction of condition 6.2(ii), as we shall see, is that it can be verified (with  $A = \mathcal{Z}$ ) for a large class of projectionless  $C^*$ -algebras; there is, to date, no similar confirmation of condition 6.1(ii).

An interesting point: if one takes  $\mathcal{C}$  to be the class of Kirchberg algebras, then Universal Properties 6.1 and 6.2 both identify  $\mathcal{O}_\infty$ ; if one takes  $\mathcal{C}$  to be the class of simple, nuclear, separable, and unital  $C^*$ -algebras satisfying the Universal Coefficients Theorem and containing an infinite projection—a class which properly contains the Kirchberg algebras—then Universal Property 6.2 still identifies  $\mathcal{O}_\infty$ , while Universal Property 6.1 does not (indeed, it is unclear whether 6.1 identifies anything at all in this case).

To prove that  $\mathcal{Z}$  satisfies Universal Property 6.2 among unital and nuclear  $C^*$ -algebras, one must determine whether infinite tensor products of such algebras are  $\mathcal{Z}$ -stable. Formally, the question is reasonable. If  $\text{gr}(A^{\otimes \text{gr}(A)}) = 1$  whenever  $\text{gr}(A) < \infty$ , then why not  $\text{gr}(A^{\otimes\infty}) = 1$  for any  $A$ ? It follows immediately from Definition 2.1 that one has either  $\text{gr}(A^{\otimes\infty}) = 0$  or  $\text{gr}(A^{\otimes\infty}) = \infty$ .

Recall that for natural numbers  $p, q, n$  such that  $p$  and  $q$  divide  $n$ , the *dimension drop interval*  $I[p, n, q]$  is the algebra of functions

$$\{f \in C([0, 1], M_n) \mid f(0) = a \otimes 1_{n/q}, a \in M_p, f(1) = b \otimes 1_{n/p}, b \in M_q\}.$$

If  $p$  and  $q$  are relatively prime and  $n = pq$ , then we say that  $I[p, pq, q]$  is a *prime dimension drop interval*.  $\mathcal{Z}$  is the unique simple and unital inductive limit of prime dimension drop intervals having

$$(K_0, K_0^+, [1]) \cong (\mathbb{Z}, \mathbb{Z}^+, 1), \quad K_1 = 0, \quad T = \{*\},$$

see [16].

The next two propositions are germane to the results in this section. They follow more or less directly from Proposition 2.2 and Theorem 2.3 of [34], respectively.

**Proposition 6.1.** *Let  $A$  be a separable, nuclear, and unital  $C^*$ -algebra. Then,  $\text{gr}(A^{\otimes\infty}) = 1$  if and only if there exists, for any relatively prime natural numbers  $p$  and  $q$ , an unital  $*$ -homomorphism  $\iota: I[p, pq, q] \rightarrow A^{\otimes\infty}$ .*

**Proof.** The “only if” part of the proposition is straightforward—every prime dimension drop interval embeds into  $\mathcal{Z}$ , which in turn embeds into  $A^{\otimes\infty} \otimes \mathcal{Z} \cong A^{\otimes\infty}$ .

Proposition 2.2 of [34] states: if  $B$  is a separable and unital  $C^*$ -algebra and there exists, for each pair of relatively prime natural numbers  $p$  and  $q$ , an unital  $*$ -homomorphism

$$\phi: I[p, pq, q] \rightarrow \frac{\prod_{i=1}^{\infty} B}{\bigoplus_{i=1}^{\infty} B} \cap B',$$

where  $B'$  is the commutant of embedding of  $B \rightarrow \prod_{i=1}^{\infty} B / \bigoplus_{i=1}^{\infty} B$  coming from constant sequences, then  $B \otimes \mathcal{Z} \cong B$ . Equivalently, if one has finite sets  $F_1 \subseteq F_2 \subseteq \dots \subseteq B$  such that  $\bigcup_i F_i$  is dense in  $B$  and, for any relatively prime natural numbers  $p$  and  $q$  and  $i \in \mathbb{N}$  an unital  $*$ -homomorphisms  $\phi_{p,q}: I[p, pq, q] \rightarrow B$  such that  $\text{Im}(\phi_{p,q})$  commutes with  $F_i$  up to  $1/2^i$ , then  $B \otimes \mathcal{Z} \cong B$ .

Put  $B = A^{\otimes \infty}$ , and choose finite sets  $F_i \subseteq A^{\otimes \infty}$  with dense union. We may write

$$A^{\otimes \infty} \cong A^{\otimes \infty} \otimes A^{\otimes \infty} \otimes \dots,$$

and assume that  $F_i$  is contained in the first  $i$  tensor factors of  $A^{\otimes \infty}$  above. By assumption there exists, for any relatively prime natural numbers  $p$  and  $q$ , an unital  $*$ -homomorphism  $\phi: I[p, pq, q] \rightarrow A^{\otimes \infty}$ . By composing  $\phi$  with the embedding of  $A^{\otimes \infty}$  as the  $(i+1)$ th tensor factor of  $A^{\otimes \infty} \otimes A^{\otimes \infty} \otimes \dots$ , we obtain an unital  $*$ -homomorphism from  $I[p, pq, q]$  to  $A^{\otimes \infty}$  whose image commutes with  $F_i$ , as required. Thus,  $A^{\otimes \infty} \otimes \mathcal{Z} \cong A^{\otimes \infty}$  and  $\text{gr}(A^{\otimes \infty}) = 1$ .  $\square$

**Proposition 6.2.** *Let  $A$  be a separable, nuclear, and unital  $C^*$ -algebra. Suppose that  $A$  admits an unital  $*$ -homomorphism  $\iota: M_2 \oplus M_3 \rightarrow A$ . Then,  $\text{gr}(A^{\otimes \infty}) = 1$ .*

**Proof.** Choose finite sets  $F_1 \subseteq F_2 \subseteq \dots \subseteq A^{\otimes \infty}$  with dense union, and with the property that  $F_i$  is contained in the first  $i$  tensor factors of  $A^{\otimes \infty}$ . One can then use  $\iota$  to obtain an unital  $*$ -homomorphism from  $M_2 \oplus M_3$  to the  $(i+1)$ th tensor factor of  $A^{\otimes \infty}$ . In particular, the image of this homomorphism commutes with  $F_i$ . It follows that  $A^{\otimes \infty}$  is approximately divisible, and hence  $\mathcal{Z}$ -stable by Theorem 2.3 of [33].  $\square$

**Corollary 6.1.** *Let  $A$  be a separable, nuclear and unital  $C^*$ -algebra of real rank zero having no one-dimensional representation. Then,  $\text{gr}(A^{\otimes \infty}) = 1$ .*

**Proof.** In Proposition 5.7 of [23], Perera and Rørdam prove that an algebra  $A$  as in the hypotheses of the corollary admits an unital embedding of a finite-dimensional algebra  $F$  having no direct summand of dimension one. Apply Proposition 6.2.  $\square$

It is not known at present whether every simple and unital AH algebra admits an unital embedding of  $\mathcal{Z}$ . We will prove that infinite tensor products of such algebras are nevertheless  $\mathcal{Z}$ -stable whenever they lack one-dimensional representations.

**Lemma 6.1.** *Given any natural number  $N$ , there exists  $\epsilon > 0$  with the following property: if*

$$A := p(C(X) \otimes \mathcal{K})p,$$

*X a connected finite CW-complex, is such that*

$$\frac{\dim(X)}{\text{rank}(p)} < \epsilon,$$

then there is an unital  $*$ -homomorphism

$$\iota : \mathbf{M}_N \oplus \mathbf{M}_{N+1} \rightarrow A.$$

**Proof.** Since  $X$  is a finite CW-complex, the  $K_0$ -group of  $A$  is finitely generated. Write

$$K_0 A \cong G_1 \oplus G_2 \oplus \cdots \oplus G_k,$$

where each  $G_i$  is cyclic, and  $G_1 = \langle [\theta_1] \rangle$  is the free cyclic group generated by the  $K_0$ -class  $[\theta_1]$  of the trivial complex line bundle  $\theta_1$  over  $X$ .

Let  $\text{rank}(p)$  be large enough—equivalently,  $\dim(X)/\text{rank}(p)$  small enough—to ensure the existence of non-negative integers  $a$  and  $b$  such that

$$\text{rank}(p) = aN + b(N + 1), \quad a, b \geq \dim(X)/2.$$

Write

$$[p] = \bigoplus_{j=1}^k g_j, \quad g_j \in G_j, \quad 1 \leq j \leq m.$$

One has, by definition, that  $g_1 = Na[\theta_1] + (N + 1)b[\theta_1]$ . Since  $N$  and  $N + 1$  are relatively prime one also has, for every  $i \geq 2$ , elements  $h_i, r_i$  of  $G_i$  such that

$$g_i = Nh_i + (N + 1)r_i.$$

Set

$$h := h_1 \oplus \cdots \oplus h_k; \quad r := r_1 \oplus \cdots \oplus r_k.$$

Then,  $g = Nh + (N + 1)r$ , and  $h, r \in K_0(A)^+$ —the virtual dimension of these elements exceeds  $\dim(X)/2$ .

Find pairwise orthogonal projections  $P_1, \dots, P_N$  in  $M_\infty(A)$  such that  $[P_i] = h_i$ ,  $1 \leq i \leq N$ . Similarly, find pairwise orthogonal projections  $Q_1, \dots, Q_{N+1}$  such that  $[Q_j] = r_j$ ,  $1 \leq j \leq N + 1$ . Since  $(\bigoplus_i P_i) \oplus (\bigoplus_j Q_j)$  is Murray–von Neumann equivalent to  $p$ , we may assume that the  $P_i$ s and  $Q_j$ s are in  $A$ . Furthermore,  $P_i$  and  $P_k$  are Murray–von Neumann equivalent for any  $i$  and  $k$ , and similarly for  $R_i$  and  $R_k$ . One may then easily find a system of matrix units for  $M_N$  and  $M_{N+1}$  using the partial isometries implementing the equivalences among the  $P_i$ s and  $R_j$ s. It follows that there is an unital embedding of  $M_N \oplus M_{N+1}$  into  $A$ .  $\square$

**Proposition 6.3.** *Let  $A$  be a separable, unital  $C^*$ -algebra. Let*

$$B = \bigoplus_{i=1}^n p_i (C(X_i) \otimes \mathcal{K}) p_i$$

*satisfy  $\text{rank}(p_i) \geq 2$ . If there is an unital  $*$ -homomorphism  $\phi : B \rightarrow A$ , then  $\text{gr}(A^{\otimes \infty}) = 1$ .*

**Proof.** For any natural number  $k$  one has an unital  $*$ -homomorphism

$$\phi^{\otimes k} : B^{\otimes k} \rightarrow A^{\otimes k}.$$

Let  $\iota_k : A^{\otimes k} \rightarrow A^{\otimes\infty}$  be the map obtained by embedding  $A^{\otimes k}$  as the first  $k$  factors of  $A^{\otimes\infty}$ . Setting  $\gamma_k = \iota_k \circ \phi^{\otimes k}$ , one has an unital  $*$ -homomorphism

$$\gamma_k : B^{\otimes k} \rightarrow A^{\otimes\infty}.$$

Recall that

$$p(C(X) \otimes \mathcal{K})p \otimes q(C(Y) \otimes \mathcal{K})q \cong (p \otimes q)(C(X \times Y) \otimes \mathcal{K})(p \otimes q) \quad (10)$$

for compact Hausdorff spaces  $X$  and  $Y$  and projections  $p \in C(X) \otimes \mathcal{K}$ ,  $q \in C(Y) \otimes \mathcal{K}$ . Let  $Z$  be any connected component of the spectrum of  $B^{\otimes k}$ , and let  $p_Z \in B^{\otimes k}$  be the projection which is equal to the unit of  $B^{\otimes k}$  at every point in  $Z$ , and equal to zero at every other point in the spectrum of  $B^{\otimes k}$ . It follows from Eq. (10) that

$$\frac{\dim(Z)}{\text{rank}(p_Z)} \leqslant \frac{k(\max_{1 \leq i \leq n} \dim(X_i))}{(\min_{1 \leq i \leq n} \text{rank}(p_i))^k} \leqslant \frac{k(\max_{1 \leq i \leq n} \dim(X_i))}{2^k} \xrightarrow{k \rightarrow \infty} 0.$$

Thus, for a fixed  $N \in \mathbb{N}$ , there is some  $k \in \mathbb{N}$  such that every homogeneous direct summand of  $B^{\otimes k}$  with connected spectrum satisfies the hypothesis of Lemma 6.1 for the corresponding value of  $\epsilon$ . It follows that there is an unital  $*$ -homomorphism

$$\psi : M_N \oplus M_{N+1} \rightarrow B^{\otimes k}.$$

The composition  $\gamma_k \circ \psi$  yields an unital  $*$ -homomorphism from  $M_N \oplus M_{N+1}$  to  $A^{\otimes\infty}$  (we may assume that  $N \geq 2$ ). It follows that there is an unital  $*$ -homomorphism from  $M_2 \oplus M_3$  to  $M_N \oplus M_{N+1}$ , and hence an unital  $*$ -homomorphism from  $M_2 \oplus M_3$  to  $A^{\otimes\infty}$ . Apply Proposition 6.2 to conclude that  $(A^{\otimes\infty})^{\otimes\infty} \cong A^{\otimes\infty}$  is  $\mathcal{Z}$ -stable.  $\square$

An algebra  $B$  as in the statement of Proposition 6.3 need not have any non-trivial projections. Take, for instance, the algebra  $p(C(S^4) \otimes \mathcal{K})p$ , where  $p$  is the higher Bott projection;  $p$  has no non-zero Whitney summands by a Chern class argument. On the other hand, the proof of Proposition 6.3 shows that if  $A$  satisfies the hypotheses of the same, then  $A^{\otimes\infty}$  has many projections.

**Theorem 6.1.** *Let  $A$  be an unital AH algebra having no one-dimensional representation. Then,  $\text{gr}(A^{\otimes\infty}) = 1$ .*

**Proof.** Write

$$A = \lim_{i \rightarrow \infty} \left( A_i := \bigoplus_{l=1}^{n_i} p_{i,l}(C(X_{i,l}) \otimes \mathcal{K})p_{i,l}, \phi_i \right),$$

where  $\phi_i : A_i \rightarrow A_{i+1}$  is unital. Define  $J_i := \{l \in \mathbb{N} \mid \text{rank}(p_{i,l}) = 1\}$ , and put

$$B_i := \bigoplus_{l \in J_i} p_{i,l}(C(X_{i,l}) \otimes \mathcal{K})p_{i,l}.$$

Define  $\psi_i : B_i \rightarrow B_{i+1}$  by restricting  $\phi_i$  to  $B_i$ , then cutting down the image by the unit of  $B_{i+1}$ . Put  $\psi_{i,j} := \psi_{j-1} \circ \cdots \circ \psi_i$ . Notice that for reasons of rank, each  $\psi_i$  is unital—the only summands of  $A_i$  whose images in  $B_{i+1}$  may be non-zero are the summands which already lie in  $B_i$ .

If, for some  $i \in \mathbb{N}$ , one has  $\psi_{i,j} \neq 0$  for every  $j > i$ , then one may find, for every  $j > i$ , a rank one projection  $q_j \in \{p_{j,l}\}_{l=1}^{n_j}$  such that the cut-down of the image of  $\psi_j|_{q_j B_j q_j}$  by  $q_{j+1}$  gives an unital  $*$ -homomorphism from  $q_j B_j q_j$  to  $q_{j+1} B_{j+1} q_{j+1}$ . Let  $Y_j$  be the spectrum of  $q_j B_j q_j$ . There is a continuous map  $\theta_j : Y_{j+1} \rightarrow Y_j$  such that  $\psi_j(f)(y) = f(\theta_j(y))$  for every  $y \in Y_{j+1}$  and  $f \in q_j B_j q_j$ . Choose a sequence of points  $y_j \in Y_j$ ,  $j > i$ , with the property that  $\theta_j(y_{j+1}) = y_j$ . Let  $\gamma_j : A_j \rightarrow \mathbb{C}$  be given by  $\gamma_j(f) = f(y_j)$ . Then  $(\gamma_j)_{j>i}$  defines an unital inductive limit  $*$ -homomorphism  $\gamma : A \rightarrow \mathbb{C}$ ;  $A$  has a one-dimensional representation. We conclude that for every  $i \in \mathbb{N}$  there exists  $j > i$  such that  $\psi_{i,j} = 0$ . It follows that  $B_j = \{0\}$ , so that  $A_j$  has no one-dimensional representations. Apply Proposition 6.3 to conclude that  $\text{gr}(A^{\otimes \infty}) = 1$ .  $\square$

Theorem 6.1 is interesting in light of the fact that there are unital AH algebras which are not weakly divisible (every algebra constructed in [36] has this deficiency, for instance), so Proposition 6.2 cannot be applied to them.

In the case of simple, unital AH algebras, infinite tensor products are not only  $\mathcal{Z}$ -stable, but classifiable as well. The next proposition has been observed independently by Bruce Blackadar and the author.

**Proposition 6.4.** *Let  $A$  be a simple, unital AH algebra. Then,  $A^{\otimes \infty}$  has very slow dimension growth in the sense of [14].*

**Proof.** Write  $A = \lim_{i \rightarrow \infty} (A_i, \phi_i)$  where, as usual,

$$A_i = \bigoplus_{l=1}^{m_i} p_{i,l} (C(X_{i,l}) \otimes \mathcal{K}) p_{i,l}.$$

Define

$$n_i := \min_{1 \leq l \leq m_i} \{\text{rank}(p_{i,l})\}, \quad k_i := \max_{1 \leq l \leq m_i} \{\dim(X_{i,l})\}.$$

Let  $\epsilon_1, \epsilon_2, \dots$  be a sequence of positive tolerances converging to zero. Set  $r_1 = 1$ , and choose for each  $i \in \mathbb{N}$  a natural number  $r_i \in \mathbb{N}$  satisfying

$$\frac{n_i^{r_i}}{(k_i r_i)^3} < \epsilon_i.$$

$A^{\otimes \infty}$  can be decomposed as follows:

$$A^{\otimes \infty} = \lim_{i \rightarrow \infty} (A_i^{\otimes r_i}, \phi_i^{\otimes r_i} \otimes 1_{A_{i+1}}^{\otimes r_{i+1}-r_i}).$$

The maximum dimension of a connected component of the spectrum of  $A_i^{\otimes r_i}$  is  $r_i k_i$ , while the minimum rank of the unit of a homogeneous direct summand of  $A_i^{\otimes r_i}$  corresponding to such a connected component is  $n_i^{r_i}$ . It follows that the decomposition for  $A^{\otimes \infty}$  above has very slow

dimension growth, whence the limit is approximately divisible [10],  $\mathcal{Z}$ -stable (Theorem 2.3 of [34]), and classifiable via the Elliott invariant [12].

Certain infinite tensor products of  $C^*$ -algebras of real rank zero are also classifiable. The proposition below is direct consequence of recent work by Brown [5].

**Proposition 6.5.** *Let  $A$  be a simple, unital, inductive limit of type-I  $C^*$ -algebras with a unique tracial state. If there is an unital  $*$ -homomorphism  $\phi : M_2 \oplus M_3 \rightarrow A$ , then  $A^{\otimes\infty}$  is tracially AF.*

**Proof.**  $A^{\otimes\infty}$  is  $\mathcal{Z}$ -stable by an application of Proposition 6.2, whence it has weakly unperforated  $K_0$ -group (Theorem 1 of [15]) and stable rank one (Theorem 6.7 of [30]). Since the proof of Proposition 6.2 actually shows that  $A^{\otimes\infty}$  is approximately divisible, we conclude that it has real rank zero by the main theorem of [3]. We have thus collected the hypotheses of Corollary 7.9 of [5], whence  $A^{\otimes\infty}$  is tracially AF.  $\square$

Notice that algebras satisfying the hypotheses of Proposition 6.5 need not be tracially AF, even if one excludes the trite example of a finite-dimensional algebra with no one-dimensional representation. Examples include  $M_2(A)$  for any of the algebras produced in Theorem 4.1 or any algebra constructed in [36], hence algebras of arbitrary growth rank or stable rank.

The infinite tensor products considered so far have all contained non-trivial projections. We turn now to certain potentially projectionless infinite tensor products.

**Definition 6.1.** Let there be given a homogeneous  $C^*$ -algebra  $M_k(C(X))$  ( $X$  is not necessarily connected) and closed pairwise disjoint sets  $X_1, \dots, X_n \subseteq X$ . Let  $F$  be a finite-dimensional  $C^*$ -algebra, and let  $\iota_i : F \rightarrow M_k$ ,  $1 \leq i \leq n$ , be unital  $*$ -homomorphisms. Define

$$\phi_i : F \rightarrow M_k(C(X_i)) \cong C(X_i) \otimes M_k$$

by  $\phi_i := \mathbf{1} \otimes \iota_i$ , and put

$$A := \{f \in M_k(C(X)) \mid f|_{X_i} \in \text{Im}(\phi_i)\}.$$

We call  $A$  a *generalised dimension drop algebra*.

Separable and unital direct limits of direct sums of generalised dimension drop algebras are, in general, beyond the scope of current methods for classifying approximately subhomogeneous (ASH) algebras via  $K$ -theory. The only such algebras which are known to admit an unital embedding of  $\mathcal{Z}$  are those for which classification theorems exist, and these form a quite limited class. But for infinite tensor products, we can prove the following:

**Theorem 6.2.** *Let  $A$  be a separable, unital, and nuclear  $C^*$ -algebra. Suppose that for every  $n \in \mathbb{N}$  there is a finite direct sum of generalised dimension drop algebras  $B_n$  having no representation of dimension less than  $n$ , and an unital  $*$ -homomorphism  $\gamma_n : B_n \rightarrow A$ . Then,  $A^{\otimes\infty} \otimes \mathcal{Z} \cong A^{\otimes\infty}$ .*

Theorem 6.2 follows directly from Proposition 6.1 and Lemma 6.2.

**Lemma 6.2.** Let  $I[p, pq, q]$  be a fixed prime dimension drop interval, and let  $B$  be a generalised dimension drop algebra. There exists  $N \in \mathbb{N}$  such that if every non-zero finite-dimensional representation of  $B$  has dimension at least  $N$ , then there is a unital  $*$ -homomorphism  $\gamma : I[p, pq, q] \rightarrow B$ .

**Proof.** Let  $N \geq p - q$ . It is well known that for any natural number  $M \geq N$  there are non-negative integers  $a_M$  and  $b_M$  such that  $a_M p + b_M q = M$ . Since  $B$  has no representations of dimension less than  $N$ , we may assume that every simple direct summand of the finite-dimensional algebra  $F$  associated to  $B$  has matrix size at least  $N$ . There is an unital  $*$ -homomorphism  $\psi : I[p, pq, q] \rightarrow F$  defined as follows: given a simple direct summand  $M_{k_j}$  of  $F$ ,  $1 \leq j \leq m$ , define a map

$$\psi_j : I[p, pq, q] \rightarrow M_{k_j}$$

by

$$\psi_j(f) = \bigoplus_{l=1}^{a_{k_j}} f(0) \oplus \bigoplus_{r=1}^{b_{k_j}} f(1);$$

put

$$\psi := \bigoplus_{j=1}^m \psi_j.$$

Adopt the notation of Definition 6.1 for  $B$ . Find pairwise disjoint open sets  $O_i \supseteq X_i$ ,  $1 \leq i \leq n$ , and put  $C = (\bigcup_i O_i)^c$ . Since  $X$  is normal, there is a continuous function  $f : X \rightarrow [0, 1]$  such that  $f = 0$  on  $C$  and  $f = 1$  on  $\bigcup_i X_i$ . Define a map

$$\gamma_1 : I[p, pq, q] \rightarrow M_k(C(C \cup O_1))$$

by

$$\gamma_1 := (\mathbf{1}_{M_k(C(C \cup O_1))} \otimes \iota_1) \circ \psi.$$

For each  $2 \leq i \leq n$  define similar maps

$$\gamma_i : I[p, pq, q] \rightarrow M_k(C(X_i))$$

by

$$\gamma_i := (\mathbf{1}_{M_k(C(X_i))} \otimes \iota_i) \circ \psi.$$

Lemma 2.3 of [16] shows that the space of unital  $*$ -homomorphisms from  $I[p, pq, q]$  to  $M_k$  is path connected. Choose, then, for each  $2 \leq i \leq n$ , a homotopy

$$\omega_k : [0, 1] \times I[p, pq, q] \rightarrow M_k$$

such that  $\omega_i(0, g) = \gamma_1(g)$  and  $\omega_i(1, g) = \gamma_i(g)$ ,  $\forall g \in I[p, pq, q]$ . Define an unital  $*$ -homomorphism  $\gamma : I[p, pq, q] \rightarrow B$  by

$$\gamma(g)(x) = \begin{cases} \gamma_1(g), & x \in C \cup O_1 \\ \gamma_i(g), & x \in X_i, 2 \leq i \leq n, \\ \omega_i(t, g), & x \in O_i \setminus X_i \text{ and } f(x) = t. \end{cases} \quad \square$$

The hypotheses of Theorem 6.2 are less general than one would like—replacing generalised dimension drop algebras with recursive subhomogeneous algebras would be a marked improvement. On the other hand, they do not even require that the algebra  $A$  be approximated locally on finite sets by generalised dimension drop algebras, and are satisfied by a wide range of  $C^*$ -algebras:

- (i) simple and unital limits of inductive sequences  $(A_i, \phi_i)$ ,  $i \in \mathbb{N}$ , where each  $A_i$  is a finite direct sum of generalised dimension drop algebras—these encompass all approximately subhomogeneous (ASH) algebras for which the Elliott conjecture is confirmed;
- (ii) for every weakly unperforated instance of the Elliott invariant  $I$ , an unital, separable, and nuclear  $C^*$ -algebra  $A_I$  having this invariant (see the proof of the main theorem of Section 7 in [11]);
- (iii) simple, unital, separable, and nuclear  $C^*$ -algebras having the same Elliott invariant as  $\mathcal{Z}$  for which there are no classification results (the main theorem in Section 7 of [11] provides a construction of a simple, unital, separable, and nuclear  $C^*$ -algebra with the same Elliott invariant as  $\mathcal{Z}$ ; there are no ASH classification results which cover this algebra, yet it satisfies the hypotheses of Theorem 6.2).

## 7. The growth rank and other ranks

In this last section we explore the connections between the growth rank and other ranks for nuclear  $C^*$ -algebras: the stable rank ( $\text{sr}(\bullet)$ ), the real rank ( $\text{rr}(\bullet)$ ), the tracial topological rank ( $\text{tr}(\bullet)$ ), and the decomposition rank ( $\text{dr}(\bullet)$ ).

Growth rank one is the condition that  $A$  absorbs  $\mathcal{Z}$  tensorially. If, in addition,  $A$  simple and unital, then it is either stably finite or purely infinite by Theorem 3 of [15]. If  $A$  is purely infinite, then it has infinite stable rank and real rank zero (see [28], for instance). If  $A$  is finite, then it has stable rank one by Theorem 6.7 of [30]. The bound  $\text{rr}(A) \leq 2\text{sr}(A) - 1$  holds in general, so one also has  $\text{rr}(A) \leq 1$  [4].

As mentioned at the end of Section 4, the AH algebras of [35] with perforated ordered  $K_0$ -groups of bounded perforation and those of [36] having finite stable rank all have growth rank two. For each natural number  $n$  there is an algebra from either [35] or [36] with stable rank  $n$ . Thus, there is no restriction on the stable rank of a  $C^*$ -algebra with growth rank two other than the fact that it is perhaps not infinite. The algebra in [36] of stable rank  $n \geq 2$  has real rank equal to  $n$  or  $n - 1$ , so algebras of growth rank two may have more or less arbitrary finite non-zero real rank.

We can compute the stable rank, real rank, decomposition rank, and tracial topological rank of the algebras constructed in Theorem 4.1.

**Proposition 7.1.** *Let  $A_n$  be the algebra of Theorem 4.1 such that  $\text{gr}(A) = n \in \mathbb{N} \cup \{\infty\}$ . Then,*

- (1)  $\text{sr}(A_1) = 1$ ,  $\text{sr}(A_2) < \infty$ , and  $\text{sr}(A_n) = \infty$  for all  $n \geq 3$ ;

- (2)  $\text{rr}(A_1) = 1$ ,  $\text{rr}(A_2) < \infty$ , and  $\text{rr}(A_n) = \infty$  for all  $n \geq 3$ ;
- (3)  $\text{tr}(A_1) = 0$  and  $\text{tr}(A_n) = \infty$  for all  $n \geq 2$ ;
- (4)  $\text{dr}(A_1) = 0$  and  $\text{dr}(A_n) = \infty$  for all  $n \geq 2$ .

**Proof.** Since  $A_1$  was taken to be a UHF algebra, it has stable rank one. Inspection of the construction of  $A_2$  shows that the ratio

$$\frac{\dim(X_i)}{\dim(p_i)}$$

is bounded above by a constant  $K \in \mathbb{R}^+$ . In [22], Nistor proves that

$$\text{sr}(p(C(X) \otimes \mathcal{K})p) = \left\lceil \frac{\lfloor \dim(X)/2 \rfloor}{\text{rank}(p)} \right\rceil + 1,$$

where  $X$  is a compact Hausdorff space. It follows that each building block in the inductive sequence for  $A_2$  has stable rank less than  $K$ , whence  $\text{sr}(A_2) < K < \infty$  by Theorem 5.1 of [26]. For  $n \geq 2$  the algebra  $A_n$  is similar to the algebra of infinite stable rank constructed in Theorem 12 of [36]. In fact, the proof of the latter can be applied directly to show that  $\text{sr}(A_n) = \infty$ —one only needs to know that  $e(\xi_{I_1^j})^j \neq 0$ , which follows from Proposition 3.2 of [29] and the fact that  $|I_1^j| \geq j$ .

The case of the real rank is similar to that of the stable rank.  $A_1$  is UHF, and so  $\text{rr}(A_0) = 0$ . Since the bound  $\text{rr}(\bullet) \leq 2\text{sr}(\bullet) - 1$  holds in general, we have  $\text{rr}(A_2) < \infty$ . Finally, in a manner analogous to the stable rank case, the proof of Theorem 13 of [36] can be applied directly to  $A_n$  whenever  $n \geq 2$  to show that  $\text{rr}(A_n) = \infty$ .

All UHF algebras have  $\text{tr} = 0$ , whence  $\text{tr}(A_1) = 0$  [19]. Theorem 6.9 of [19] asserts that an unital simple  $C^*$ -algebra  $A$  with  $\text{tr}(A) < \infty$  must have stable rank one. Since  $\text{sr}(A_n) > 1$  for all  $n \geq 2$ , we conclude that  $\text{tr}(A_n) = \infty$  for all such  $n$ . The proof of Theorem 8 of [36] can be applied directly to  $A_2$  to show that  $\text{sr}(A_2) \geq 2$ , whence  $\text{tr}(A_2) = \infty$ .

All UHF algebras have  $\text{dr} = 0$  by Corollary 6.3 of [37], whence  $\text{dr}(A_1) = 0$ . The same corollary implies that  $\text{dr}(A_n) = \infty$  for all  $n \geq 1$  since, by construction, these  $A_n$ s have unique trace and contain projections of arbitrarily small trace.  $\square$

Thus, the growth rank is able to distinguish between simple  $C^*$ -algebras which are undifferentiated by other ranks. We offer a brief discussion of these other ranks as they relate to Elliott's classification program for separable nuclear  $C^*$ -algebras, and argue that the growth rank meshes most naturally with this program. It must be stressed, however, that these other ranks have been indispensable to the confirmation of Elliott's conjecture over the years.

Simple, nuclear, unital and separable  $C^*$ -algebras of real rank zero have so far confirmed Elliott's conjecture, but as the conjecture has also been confirmed for large classes of simple, nuclear  $C^*$ -algebras of real rank one, one must conclude that real rank zero is at best too strong to characterise the simple, separable and nuclear  $C^*$ -algebras satisfying the Elliott conjecture. There is a counterexample to Elliott's conjecture having  $\text{rr} = \text{sr} = 1$  in [32], so the conditions  $\text{rr} \leq 1$  and  $\text{sr} = 1$  also fail to characterise classifiability.

The condition  $\text{tr} = 0$  has been shown to be sufficient for the classification of large swaths of simple, separable, nuclear  $C^*$ -algebras of real rank zero, but the fact that algebras with  $\text{tr} < \infty$

must have small projections means that this condition will not characterise classifiability—the Elliott conjecture has been confirmed for large classes of projectionless  $C^*$ -algebras.

The condition  $\text{dr} < \infty$  is so far promising as far as characterising classifiability is concerned at least in the stably finite case. It may be true that

$$\text{dr} < \infty \Leftrightarrow \text{gr} = 1$$

for simple, separable, nuclear, and stably finite  $C^*$ -algebras. On the other hand it remains unclear whether the decomposition rank takes more than finitely many values for such algebras, and it is unlikely to distinguish non- $\mathcal{Z}$ -stable algebras.

The growth rank has strong evidence to recommend it as the correct notion of rank vis a vis classification: all simple, separable, nuclear and non-elementary  $C^*$ -algebras for which the Elliott conjecture is confirmed have  $\text{gr} = 1$ ; all known counterexamples to the conjecture have  $\text{gr} > 1$ ; the growth rank achieves every possible value in its range on simple, nuclear and separable  $C^*$ -algebras and is well behaved with respect to common operations. Furthermore, the classification results available for  $\mathcal{Z}$ -stable  $C^*$ -algebras are very powerful. Let  $\mathcal{E}$  denote the class of simple, separable, nuclear, and unital  $C^*$ -algebras in the bootstrap class  $\mathcal{N}$  which are  $\mathcal{Z}$ -stable. (In light of known examples,  $\mathcal{E}$  is the largest class of simple unital algebras for which one can expect the Elliott conjecture to hold.) Then:

- (i) the subclass of  $\mathcal{E}_{\text{inf}}$  of  $\mathcal{E}$  consisting of algebras containing an infinite projection satisfies the Elliott conjecture [17,24];
- (ii) the subclass of  $\mathcal{E} \setminus \mathcal{E}_{\text{inf}}$  consisting of algebras with real rank zero and locally finite decomposition rank satisfies the Elliott conjecture [38].

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