Three Applications of the Cuntz Semigroup

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Building on work of Elliott and coworkers, we present three applications of the Cuntz semigroup:

(i) for many simple C∗-algebras, the Thomsen semigroup is recovered functorially from the Elliott invariant, and this yields a new proof of Elliott’s classification theorem for simple, unital AI algebras;

(ii) for the algebras in (i), classification of their Hilbert modules is similar to the von Neumann algebra context;

(iii) for the algebras in (i), approximate unitary equivalence of self-adjoint operators is characterised in terms of the Elliott invariant.

1 Introduction

The Cuntz semigroup (see [6], [9], [12], [13] for definitions and basic properties) has recently become quite popular. In this note we extend the main theorem of [3] to stable C∗-algebras. By combining this result with those of Coward-Elliott-Ivanescu [5] and Elliott-Ciuperca [4], we obtain the applications of the abstract directly (see

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Theorems 4.1 and 4.2 for the Thomsen semigroup result, Theorem 3.4 for Hilbert modules and Theorem 5.1 for unitary orbits of self-adjoints. For the reader interested primarily in Elliott’s classification program, we emphasize that most of our results are formulated in terms of the Elliott invariant—the Cuntz semigroup is a powerful technical tool used only in proofs. This paper is a natural sequel to [3] and [12], and the latter contain the requisite definitions, notation, and basic facts employed herein. Finally, we thank the referee for a number of helpful comments and suggestions.

2 Computation of $\mathcal{W}(A \otimes \mathcal{K})$

Throughout this paper A will denote a unital simple separable C*-algebra with tracial states. Let $\mathcal{W}(A)$ denote the Cuntz semigroup of $A$ and let $\mathcal{T}(A)$ denote the simplex of tracial states. Since $\mathcal{T}(A) \neq \emptyset$, $A$ is stably finite. It follows that $\mathcal{W}(A)$ can be decomposed into the disjoint union of $\mathcal{V}(A)$ (the Murray-von Neumann semigroup of equivalence classes of projections) and the set $\mathcal{W}(A)^+$ of Cuntz classes of positive elements which are not equal to the class of a projection. If $\mathcal{Laff}_b(\mathcal{T}(A))^+$ denotes the bounded, lower semicontinuous, affine, strictly positive functions on $\mathcal{T}(A)$, then there is a canonical map

$$\iota: \mathcal{W}(A)^+ \rightarrow \mathcal{Laff}_b(\mathcal{T}(A))^+$$

given by

$$\iota(x)(\tau) = d_\tau(x),$$

where $d_\tau(x) := \lim_{n \to \infty} \tau \otimes \text{Tr}_k(x^{1/n})$ for an element $x \in A \otimes M_k(\mathbb{C})$. (Here Tr$_k$ is the non-normalized trace on $M_k(\mathbb{C})$.) The main theorem of [3] was that $\iota$ is an order isomorphism, whence

$$\mathcal{W}(A) \cong \mathcal{V}(A) \sqcup \mathcal{Laff}_b(\mathcal{T}(A))^+$$  \hspace{1cm} (2.1)

as partially ordered semigroups for two important classes of C*-algebras: simple unital exact finite C*-algebras which absorb the Jiang-Su algebra $\mathbb{Z}$ tensorially, and simple unital AH algebras with slow dimension growth. (We refer the reader to [12] for the definition of the order structure on $\mathcal{V}(A) \sqcup \mathcal{Laff}_b(\mathcal{T}(A))^+$. As usual, $\mathbb{Z}$ denotes the Jiang-Su algebra—see [8].) In this section we prove a structure theorem similar to (2.1) for $\mathcal{W}(A \otimes \mathcal{K})$, with $A$ as above.
Recall that $A$ has strict comparison if $x \preceq y$ whenever $d_{\tau}(x) < d_{\tau}(y)$ for all $\tau \in T(A)$. ($\preceq$ denotes Cuntz’s relation and $x \sim y$ means $x \preceq y$ and $y \preceq x$.) When $A$ is unital simple exact and has strict comparison, the map $\iota$ is an isomorphism whenever it is surjective (cf. [12, Proposition 3.3]).

**Lemma 2.1.** Let $A$ be a simple unital exact $C^*$-algebra, and let $\langle a \rangle \in W(A)_+$ be given. It follows that for any $\epsilon > 0$, there exists $\delta > 0$ and a continuous affine function $f : T(A) \to \mathbb{R}^+$ such that

$$d_{\tau}((a - \epsilon)_+) < f(\tau) < d_{\tau}((a - \delta)_+), \forall \tau \in T(A).$$

Proof. First note that zero must be an accumulation point of the spectrum $\sigma(a)$ (otherwise, functional calculus would provide a projection with the same Cuntz class as $\langle a \rangle \in W(A)_+$, which is impossible). Choose points $\delta < \eta \in (0, \epsilon) \cap \sigma(a)$ so that each of $(\delta, \eta)$ and $(\eta, \epsilon)$ are nonempty. Since $A$ is simple, each trace and hence each lower semicontinuous dimension function is faithful. It follows from a functional calculus argument that

$$d_{\tau}((a - \epsilon)_+) < d_{\tau}((a - \eta)_+) < d_{\tau}((a - \delta)_+), \forall \tau \in T(A).$$

Let $\mu_{\tau}$ be the (regular Borel) measure induced on $\sigma(a)$ by $\tau \in T(A)$. The affine map $h : T(A) \to \mathbb{R}^+$ given by

$$h(\tau) := \mu_{\tau}([\epsilon, \infty) \cap \sigma(a))$$

is upper semicontinuous by the Portmanteau Theorem ([1]). From the inclusions

$$(\epsilon, \infty) \cap \sigma(a) \subseteq [\epsilon, \infty) \cap \sigma(a) \subseteq (\eta, \infty) \cap \sigma(a)$$

we have the following inequalities:

$$d_{\tau}((a - \epsilon)_+) \leq h(\tau) \leq d_{\tau}((a - \eta)_+) < d_{\tau}((a - \delta)_+), \forall \tau \in T(A).$$

The affine map $\tau \mapsto d_{\tau}((a - \delta)_+)$ is strictly positive and lower semicontinuous. Since $T(A)$ is a metrizable compact convex set, this map is the pointwise supremum of a strictly increasing sequence of continuous affine maps, say $(f_n)_{n=1}^\infty$. A straightforward argument using compactness then shows that there is some $n_0 \in \mathbb{N}$ such that

$$f_n(\tau) > h(\tau), \forall \tau \in T(A), \forall n \geq n_0.$$ 

Setting $f(\tau) = f_{n_0}(\tau)$ completes the proof. \qed
Let $A$ be a unital $C^*$-algebra and $a \in A \otimes \mathcal{K}$ be positive. Let $\{e_n\} \subset \mathcal{K}$ be an increasing sequence of projections with rank$(e_n) = n$, and put $P_n = 1 \otimes e_n \in A \otimes \mathcal{K}$. Then,

$$P_1 a P_1 \preceq P_2 a P_2 \preceq P_3 a P_3 \preceq \cdots$$

in $W(A \otimes \mathcal{K})$ and $P_n a P_n \to a$ in norm. Let $b = \sup_n \langle P_n a P_n \rangle \in W(A \otimes \mathcal{K})$ (suprema of increasing sequences in $W(A \otimes \mathcal{K})$ always exist by [5, Theorem 1]). Then, given $\epsilon > 0$, there is some $n \in \mathbb{N}$ such that

$$(a - \epsilon)_+ \preceq P_n a P_n \preceq b.$$ 

It follows that $a \preceq b$. Since $P_n a P_n \preceq a$ for each $n$, we also have that $b \preceq a$, which shows $a - b$, i.e. $\langle a \rangle = \sup_n \langle P_n a P_n \rangle$.

**Lemma 2.2.** Let $A$ be a simple unital exact $C^*$-algebra, and let $a \in A \otimes \mathcal{K}$ be a positive element such that $\langle a \rangle \in W(A \otimes \mathcal{K})$. It follows that there is a sequence $(a_n)_{n=1}^{\infty}$ of positive elements in $A \otimes \mathcal{K}$ satisfying the following conditions:

(i) $\langle a \rangle = \sup_n \langle a_n \rangle$;
(ii) $a_n \in A \otimes M_{k(n)}$ for some $k(n) \in \mathbb{N}$;
(iii) for each $n$ there is a continuous affine function $f_n : T(A) \to \mathbb{R}$ such that

$$d_\tau(a_n) < f(\tau) < d_\tau(a_{n+1}), \quad \forall \tau \in T(A).$$

□

**Proof.** Let $P_n$ be the unit of $A \otimes M_n$ (as above) and define $b_n := P_n a P_n$. The sequence $b_n$ satisfies parts (i) and (ii) of the conclusion of the lemma by construction. Note that $b_n \preceq b_{n+1}$.

**Case I.** Let us first address the case where infinitely many of the $b_n$s are Cuntz equivalent to a projection. By passing to a subsequence, we may assume that every $b_n$ is Cuntz equivalent to a projection (this does not affect the validity of (i) and (ii)). If infinitely many of the $b_n$s are Cuntz equivalent to a fixed projection $p \in A \otimes \mathcal{K}$, then we have

$$\langle a \rangle = \sup_n \langle b_n \rangle = \langle p \rangle;$$

this contradicts our assumption that $a$ is not Cuntz equivalent to a projection. Thus, each Cuntz class $\langle b_m \rangle$, $m \in \mathbb{N}$ occurs at most finitely many times in the sequence $(\langle b_n \rangle)_{n=1}^{\infty}$. Passing to a subsequence again, we may assume that $\langle b_m \rangle \neq \langle b_n \rangle$ whenever $m \neq n$. 


Put \( a_n = b_n \). As noted, \( (a_n)_{n=1}^\infty \) satisfies parts (i) and (ii) of the conclusion of the lemma already. The map \( \tau \mapsto d_\tau(a_n) \) is continuous since \( a_n \) is Cuntz equivalent to a projection. The fact that \( a_n \) is Cuntz equivalent to a projection also means that it is complemented inside \( a_{n+1} \), i.e., there is a projection \( p_n \) in \( A \otimes M_{k(n+1)} \) such that \( \langle a_n \rangle + \langle p_n \rangle = \langle a_{n+1} \rangle \) ([12, Proposition 2.2]). Since \( A \) is simple, the map \( \tau \mapsto d_\tau(p_n) \) is continuous and strictly positive on \( T(A) \). Setting \( f_n(\tau) = d_\tau(a_n) + (1/2)d_\tau(p_n) \) then gives condition (iii).

**Case II.** Now we may assume that none of the \( b_n \)'s is equivalent to a projection. Given any \( \epsilon_1 > 0 \), we may use Lemma 2.1 to find \( \delta_1 > 0 \) and a continuous affine map \( f_1 : T(A) \to \mathbb{R}^+ \) such that

\[
d_\tau((b_1 - \epsilon_1)_+) < f_1(\tau) < d_\tau((b_1 - \delta_1)_+), \quad \forall \tau \in T(A).
\]

Assume that we have found sequences \( \epsilon_1, \ldots, \epsilon_n \) and \( \delta_1, \ldots, \delta_n \) of strictly positive tolerances satisfying the following conditions:

(a) \( (b_k - \epsilon_k)_+ \precsim (b_k - \delta_k)_+, \quad k \in \{1, \ldots, n\} \);

(b) there is a continuous affine map \( f_k : T(A) \to \mathbb{R} \) such that

\[
d_\tau((b_k - \epsilon_k)_+) < f_k(\tau) < d_\tau((b_k - \delta_k)_+), \quad \forall \tau \in T(A);
\]

(c) \( (b_k - \epsilon_k/l)_+ \precsim (b_l - \epsilon_l)_+, \quad 1 \leq k < l \) and \( l \in \{1, \ldots, n\} \);

(d) \( (b_k - \delta_k)_+ \precsim (b_{k+1} - \epsilon_{k+1})_+, \quad k \in \{1, \ldots, n\} \).

Using the basic properties of Cuntz’s comparison relation, we can find \( \epsilon_{n+1} \) satisfying (c) and (d) above (with \( n + 1 \) in place of \( l \) and \( n \) in place of \( k \), respectively). Applying Lemma 2.1, we can find \( \delta_{n+1} \) satisfying (a) and (b) with \( n + 1 \) in place of \( k \). Thus, our sequences \( \epsilon_1, \ldots, \epsilon_n \) and \( \delta_1, \ldots, \delta_n \) can be extended, inductively, to sequences \( \epsilon_i \) and \( \delta_l \) satisfying (a)–(d), as appropriate.

Set \( a_n = (b_n - \epsilon_n)_+ \). Let us verify condition (i) for this choice of \( a_n \). Since \( a_n \precsim a \) for each \( n \), we have

\[
\sup_n a_n \precsim a.
\]

On the other hand, (c) gives \( (b_k - \epsilon_k/n)_+ \precsim a_n \) for every \( 1 \leq k < n \) and \( n \in \mathbb{N} \). It follows that

\[
\sup_n \langle a_n \rangle \geq \langle (b_k - \epsilon_k/n)_+ \rangle, \quad \forall n, k \in \mathbb{N}.
\]
In particular,
\[ \sup_n \langle a_n \rangle \geq \langle b_k \rangle, \quad \forall k \in \mathbb{N}, \]
and so
\[ \sup_n \langle a_n \rangle \geq \sup_k \langle b_k \rangle = \langle a \rangle. \]
Condition (ii) is satisfied by construction, while condition (iii) is satisfied by the functions \( f_n \) from (b) above. This completes the proof. \( \blacksquare \)

Definition 2.3. For every positive element \( a \in A \otimes K \), define an affine function \( \iota(a) : T(A) \to \mathbb{R}^+ \cup \{ \infty \} \) by
\[ \iota(a)(\tau) = \sup_n d_\tau(P_n a P_n), \]
for each trace \( \tau \in T(A) \), where \( P_n \) are the projections defined before Lemma 2.2.

In analogy with previous notation, we now observe that \( a \mapsto \iota(a) \) drops to a well-defined map (denoted by \( \iota \)) on \( W(A \otimes K) \). (Indeed, if \( a \in M_n(A \otimes K)_+ \), then we can identify it with a positive element in \( (A \otimes K)_+ \) and hence define \( \iota(a) \); since \( \iota(\cdot) \) is independent of the projections used in its definition, it is not hard to check that our recipe for extending \( \iota \) to \( M_n(A \otimes K)_+ \) is independent of the identification \( M_n(K) \cong K \).

Lemma 2.4. If \( a, b \in (A \otimes K)_+ \) and \( \langle a \rangle = \langle b \rangle \in W(A \otimes K) \) then \( \iota(a) = \iota(b) \). Moreover, \( \iota(a) \) is independent of the choice of projections \( P_n \).

Proof. Assume \( a \sim b \). For each \( \epsilon > 0 \) and \( n \in \mathbb{N} \) there exists a \( \delta > 0 \) and \( m \in \mathbb{N} \) such that
\[ (P_n a P_n - 2\epsilon)_+ \preceq (a - \epsilon)_+ \preceq (b - \delta)_+ \preceq P_m b P_m. \]
It follows that for any \( \tau \in T(A) \),
\[ \iota(b)(\tau) \geq d_\tau(P_m b P_m) \geq d_\tau(P_n a P_n - 2\epsilon)_+. \]
Since \( n \) and \( \epsilon \) were arbitrary, we conclude that \( \iota(b)(\tau) \geq \iota(a)(\tau) \). Similarly, \( \iota(a)(\tau) \geq \iota(b)(\tau) \).

For the second assertion, let \( \{ e_n \}, \{ f_n \} \subset K \) be increasing sequences of projections with \( \text{rank}(e_n) = \text{rank}(f_n) = n \), and put \( P_n = 1 \otimes e_n, Q_n = 1 \otimes f_n \in A \otimes K \). Fix \( n \in \mathbb{N} \) and...
and $\epsilon > 0$. Since $\lim_{k \to \infty} \|P_nQ_kP_n - P_n\| = 0$, we can find $k$ such that $\|P_nQ_kaQ_kP_n - P_naP_n\| < \epsilon$. It follows that $(P_n(aP_n - \epsilon)) \lesssim Q_k(aQ_k)$ for all sufficiently large $k$. In particular, $d_\tau((P_n(aP_n - \epsilon)) \leq \sup_k d_\tau(Q_k(aQ_k))$ for every $\epsilon > 0$. Since $d_\tau(P_n(aP_n)) = \sup_\epsilon d_\tau((P_n(aP_n - \epsilon))$, the lemma follows.

**Proposition 2.5.** Let $A$ be a unital simple exact C*-algebra with strict comparison of positive elements. If $\langle a \rangle, \langle b \rangle \in W(A \otimes \mathcal{K})_+$, then $a \sim b$ if and only if $\iota(a) = \iota(b)$.

**Proof.** The forward implication is contained in Lemma 2.4, so suppose that $\iota(a) = \iota(b)$. Find, using Lemma 2.2, sequences $(a_n)^{\infty}_{n=1}$ and $(b_n)^{\infty}_{n=1}$ corresponding to $a$ and $b$, respectively; let $f_n$ and $g_n$ denote, respectively, the functions provided by part (iii) of the conclusion of Lemma 2.2. By a compactness argument, for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for every $\tau \in T(A)$ we have the following inequalities:

$$d_\tau(a_n) < f_n(\tau) < d_\tau(b_m); \quad d_\tau(b_n) < g_n(\tau) < d_\tau(a_m).$$

Since $A$ has strict comparison, $a_n \lesssim b_m$ and $b_n \lesssim a_m$. It follows that

$$\langle a \rangle = \sup_n \langle a_n \rangle = \sup_m \langle b_m \rangle = \langle b \rangle,$$

as desired.

Let $\text{SAff}(T(A))$ denote the set of functions on $T(A)$ which are pointwise suprema of increasing sequences of continuous, affine, and strictly positive functions on $T(A)$. Define an addition operation on the disjoint union $V(A) \cup \text{SAff}(T(A))$ as follows:

(i) if $x, y \in V(A)$, then their sum is the usual sum in $V(A)$;
(ii) if $x, y \in \text{SAff}(T(A))$, then $x + y$ is the pointwise sum in $\text{SAff}(T(A))$;
(iii) if $x \in V(A)$ and $y \in \text{SAff}(T(A))$, then their sum is the usual (pointwise) sum of $x$ and $y$ in $\text{SAff}(T(A))$, where $x(\tau) = \tau(p)$ for some projection $p$ with $\langle p \rangle = x$, $\forall \tau \in T(A)$.

Equip $V(A) \cup \text{SAff}(T(A))$ with the partial order $\leq$ which restricts to the usual partial orders on $V(A)$ (i.e. Murray-von Neumann) and $\text{SAff}(T(A))$ (i.e. $f \leq g \iff f(\tau) \leq g(\tau)$ for all $\tau \in T(A)$), and which satisfies the following conditions for $x \in V(A)$ and $y \in \text{SAff}(T(A))$:

(i) $x \leq y$ if and only if $x(\tau) \leq y(\tau)$, $\forall \tau \in T(A)$;
(ii) $y \leq x$ if and only if $y(\tau) \leq x(\tau)$, $\forall \tau \in T(A)$. 

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Theorem 2.6. Let $A$ be a unital simple exact and tracial $C^*$-algebra with strict comparison. Assume that $\iota: W(A)_+ \to \text{LAff}_b(T(A))^{++}$ is surjective. It follows that

$$W(A \otimes \mathcal{K}) \cong V(A) \sqcup \text{SAff}(T(A)),$$

as ordered semigroups.

\begin{proof}
Define

$$\phi: W(A \otimes \mathcal{K}) \to V(A) \sqcup \text{SAff}(T(A))$$

by $\text{id}_{V(A \otimes \mathcal{K})}$ on $V(A \otimes \mathcal{K})$ and by $\iota$ on $W(A \otimes \mathcal{K})_+$ (that is, $x \mapsto \iota(a)$, where $x = \langle a \rangle$). Let us first prove that $\phi$ is a bijection. Since $A$ is stably finite, $W(A \otimes \mathcal{K}) = V(A) \sqcup W(A \otimes \mathcal{K})_+$, hence it suffices to show $\iota: W(A \otimes \mathcal{K})_+ \to \text{SAff}(T(A))$ is a bijection.

Injectivity of $\iota$ follows from Proposition 2.5. Surjectivity follows from two facts: (i) the range of $\iota$ contains $\text{LAff}_b(T(A))^{++}$ and (ii) $W(A \otimes \mathcal{K})$ has suprema (cf. [5]). Indeed, given $f \in \text{SAff}(T(A))$ we find continuous affine functions $f_n \leq f_{n+1} \leq \cdots$ converging up to $f$ pointwise. Letting $a_n \in A \otimes \mathcal{K}$ be positive elements such that $\bar{a}_n = f_n$, we let $x = \sup_n \langle a_n \rangle \in W(A \otimes \mathcal{K})$ (we have used strict comparison here to ensure $\langle a_n \rangle$ is an increasing sequence in $W(A \otimes \mathcal{K})$). Then it is clear that $\iota(x) = f$.

To complete the proof, we must show that $\phi$ is order preserving. Suppose that $x \leq y$, $x, y \in W(A \otimes \mathcal{K})$. There are four cases to consider.

(a) If $x, y \in V(A \otimes \mathcal{K})$, then $\phi(x) \leq \phi(y)$ since $\phi|_{V(A \otimes \mathcal{K})} = \text{id}_{V(A \otimes \mathcal{K})}$.

(b) If $x, y \in W(A \otimes \mathcal{K})$, then $\phi(x) \leq \phi(y)$ since $\phi|_{W(A \otimes \mathcal{K})} = \iota$ and $\iota$ is order-preserving. (The proof of this last fact follows from the proof of the first implication in Proposition 2.5.)

(c) If $x \in V(A \otimes \mathcal{K})$ and $y \in W(A \otimes \mathcal{K})_+$, then we apply [12, Proposition 2.2] to find $z \in W(A \otimes \mathcal{K})$ such that $x + z = y$. It follows that $\iota(x)(\tau) < \iota(y)(\tau)$, $\forall \tau \in T(A)$ (note that $\iota(x)(\tau) < \infty$ in this case), whence $\phi(x) \leq \phi(y)$.

(d) If $x \in W(A \otimes \mathcal{K})_+$ and $y \in V(A \otimes \mathcal{K})$, then $\phi(x) \leq \phi(y)$ since $\iota$ is order-preserving.

The theorem above holds for all simple unital AH algebras with slow dimension growth, and for the class of simple unital exact stably finite $C^*$-algebras which absorb $\mathbb{Z}$ ([3, Theorems 5.3 and 5.5], [14, Corollary 4.6], [16, Corollary 4.6]).
3 Classifying Hilbert modules

Let $E, F$ be countably generated Hilbert modules over a separable, unital $C^*$-algebra $A$. By Kasparov’s stabilization theorem, there are projections $P_E, P_F \in L(H_A)$ such that $E$ is isomorphic to $P_E H_A$ and $F$ is isomorphic to $P_F H_A$. (Here $H_A = \ell^2 \otimes A$ is the standard Hilbert module over $A$ and $L(H_A)$ is the set of bounded adjointable operators on $H_A$; see [10] for more.) Since $L(H_A) = M(A \otimes \mathcal{K})$ (the multiplier algebra of $A \otimes \mathcal{K}$), we can find strictly positive elements $a \in P_E (A \otimes \mathcal{K}) P_E$ and $b \in P_F (A \otimes \mathcal{K}) P_F$. According to [5, Theorem 3], if we further assume $A$ has stable rank one,

$$E \cong F \text{ if and only if } \langle a \rangle = \langle b \rangle \in W(A \otimes \mathcal{K}).$$

In this section we’ll reformulate this result in terms of the projections $P_E$ and $P_F$.

First, an alternate formula for $\langle a \rangle \in \text{SAff}(T(A))$ will be handy. Let $\mathcal{F} \subset \mathcal{K}$ denote the finite-rank operators and $A \otimes \mathcal{F}$ be the algebraic tensor product of $A$ and $\mathcal{F}$ (which we identify with the “finite-rank” operators on $H_A$).

**Lemma 3.1.** For every $0 \leq a \in A \otimes \mathcal{K}$ and $\tau \in T(A)$ we have

$$\langle a \rangle(\tau) = \sup \{ d_\tau(b) : 0 \leq b \in A \otimes \mathcal{F}, b \not\preceq a \}. \quad \square$$

Proof. If $P = 1 \otimes e$ for some finite rank projection $e \in \mathcal{K}$, then $PaP \in A \otimes \mathcal{F}$ and $PaP \not\preceq a$; hence, the inequality $\leq$ is immediate.

For the other direction, fix $b \in A \otimes \mathcal{F}$ such that $b \not\preceq a$, and fix $\epsilon > 0$. Choose $\delta > 0$ such that $d_\tau(b) - \epsilon \leq d_\tau(\langle b - \delta \rangle_+)$ and find $x \in A \otimes \mathcal{K}$ such that $\|x^*ax - b\| < \delta$. By density, we may assume $x \in A \otimes M_n(\mathbb{C})$ for some large $n \in \mathbb{N}$. It follows that $\langle b - \delta \rangle_+ \not\preceq x^*ax$. Now, let $P_n = 1 \otimes e_n$, for some increasing finite-rank projections $e_n$, such $P_n x = x = xP_n$ for all $n$. We have that $\langle b - \delta \rangle_+ \not\preceq x^*ax = x^*(P_n a P_n)x$. Hence,

$$d_\tau(b) - \epsilon \leq d_\tau(\langle b - \delta \rangle_+) \leq d_\tau(P_n a P_n),$$

and, by Lemma 2.4, this completes the proof of the lemma. \[\square\]

**Definition 3.2.** For any projection $Q \in M(A \otimes \mathcal{K})$ and tracial state $\tau \in T(A)$ we define

$$\hat{Q}(\tau) = \sup \{ \tau \otimes \text{Tr}(b) : 0 \leq b \in A \otimes \mathcal{F}, b \leq Q \},$$

where $\text{Tr}$ is the (unbounded) trace on $\mathcal{F}$. 

**Lemma 3.3.** Assume $A$ is unital with stable rank one. For any projection $Q \in M(A \otimes \mathcal{K})$, strictly positive element $a \in Q(A \otimes \mathcal{K})Q$ and $\tau \in T(A)$, we have

$$\hat{Q}(\tau) = \iota(a)(\tau).$$

**Proof.** Since \{b : b \in A \otimes \mathcal{F}, b \leq P\} \subset \{b : b \in A \otimes \mathcal{F}, b \precsim a\} (cf. [9, Proposition 2.7(ii)]), and $\tau \otimes \text{Tr}(b) \leq \lim_n \tau \otimes \text{Tr}(b^{1/n}) = d_\tau(b)$, the previous lemma implies that $\hat{Q}(\tau) \leq \iota(a)(\tau)$.

For the opposite inequality, fix $b \in A \otimes \mathcal{F}$ such that $b \precsim a$, and $\epsilon > 0$. Choose $\delta > 0$ such that $d_\tau(b) - \epsilon \leq d_\tau((b - \delta)_+).$ Since $A$ has stable rank one, so does $(A \otimes \mathcal{K})$ (the unitization of $A \otimes \mathcal{K}$). Hence, by [13, Proposition 2.4], we can find a unitary $u \in (A \otimes \mathcal{K})$ such that $u^*(b - \delta)_+, u \leq Q$. Since $u^*(b - \delta)_+, u \in A \otimes \mathcal{F}$, the following inequalities complete the proof:

$$d_\tau(b) - \epsilon \leq d_\tau((b - \delta)_+, u) = \lim_n \tau \otimes \text{Tr}([u^*(b - \delta)_+, u]^{1/n}) \leq \hat{Q}(\tau).$$

Recall that if $M \subset B(L^2(M))$ is a $\text{II}_1$-factor in standard form, then isomorphism classes of modules over $M$ (i.e. normal representations $M \subset B(H)$) are completely determined by the traces of the corresponding projections in $M \otimes B(H)$. Our next theorem is analogous to this classical result.

**Theorem 3.4.** Let $A$ be a unital simple exact $C^*$-algebra with strict comparison and stable rank one. Given two countably generated Hilbert modules $E, F$ over $A$, the following are equivalent:

(i) $E$ is isomorphic to $F$;

(ii) $PE$ is Murray-von Neumann equivalent to $PF$;

(iii) Either $(PE) = (PF) \in V(A)$ (in the case $PE, PF \in A \otimes \mathcal{K}$), or $\hat{PE} = \hat{PF}$.

In particular, if neither $E$ nor $F$ is a finitely generated projective module, then $E \cong F$ if and only if $\hat{PE} = \hat{PF}$.

**Proof.** In the case that both $PE, PF \in A \otimes \mathcal{K}$, the equivalence of the three conditions is a well-known exercise; when neither $PE$ nor $PF$ belong to $A \otimes \mathcal{K}$, the first two conditions are easily seen to be equivalent. Hence, we assume neither $PE$ nor $PF$ belong to $A \otimes \mathcal{K}$ and will show the first and third statements to be equivalent.

Let $a$ (resp. $b$) be a strictly positive element in $PE(A \otimes \mathcal{K})PE$ (resp. $PF(A \otimes \mathcal{K})PF$). If $E \cong F$ then $\langle a \rangle = \langle b \rangle \in W(A \otimes \mathcal{K})$ (by [5, Theorem 3]), and hence the previous lemma implies that $\hat{PE} = \hat{PF}$. Conversely, if we know $\hat{PE} = \hat{PF}$ then (by the previous lemma)
ι(⟨a⟩) = ι(⟨b⟩), so Proposition 2.5 ensures that ⟨a⟩ = ⟨b⟩ ∈ W(A ⊗ K). Then [5, Theorem 3] implies E ∼= F.

Remark 3.5. The theorem above is, in a certain sense, best possible: we really need strict comparison. More precisely, the hypotheses are satisfied by simple AH algebras with slow dimension growth (and Z-stable algebras—cf. [2, Theorem 1], [14, Corollary 4.6], [16, Corollary 4.6]), but the result cannot be extended to all AH algebras. Indeed, the reader will find in [17] a pair of positive elements in a simple unital AH algebra of stable rank one such that the corresponding Hilbert modules, say E and F, are not isomorphic but do satisfy Pf = Pf.

It is also worth remarking that the result above gives a complete parametrization of isomorphism classes of countably generated Hilbert modules over A in terms of K0 and traces.

4 From Elliott to Thomsen and the classification of simple AI algebras

Theorem 4.1. Let A be a unital simple C∗-algebra of stable rank one for which W(A ⊗ K) ∼= V(A) ⊔ SAff(T(A)). Then, the Thomsen semigroup of A (cf. [15]) can be functorially recovered from the Elliott invariant of A.

This theorem follows immediately from [4, Theorems 4 and 10]. The result applies to any algebra satisfying the hypotheses of Theorem 2.6—in particular, by [3, Theorems 5.3 and 5.5], A could be a simple unital AH algebra with slow dimension growth, or a simple unital exact and Z-stable C∗-algebra. The assumption of simplicity in the theorem is actually redundant. The assumption on the structure of W(A ⊗ K) guarantees that every trace on A is faithful, whence A is simple.

Theorem 4.2 (Elliott, [7]). Let A and B be simple unital inductive limits of algebras of the form F ⊗ C[0, 1], where F is finite dimensional. Then A ∼= B if and only if Ell(A) ∼= Ell(B).

Proof. If Ell(A) ∼= Ell(B) then W(A ⊗ K) ∼= W(B ⊗ K), by Theorem 2.6 and [3, Theorem 5.3] (since AI algebras have no dimension growth). From [4, Theorem 4] it follows that the Thomsen semigroups of A and B are isomorphic too. Hence, by [15, Theorem 1.5], A ∼= B.

This theorem is the best possible in the sense that the Elliott invariant is not complete for non-simple AI algebras (cf. [15, pg. 48]). The Cuntz semigroup, however, is a complete invariant in the non-simple case, as shown in [4, Theorem 11].
Unitary orbits of self-adjoints in simple, unital, exact C*-algebras

Let $a \in A$ be self-adjoint with spectrum $\sigma(a)$. Let $\phi_a : C(\sigma(a)) \to A$ be the canonical homomorphism induced by sending the generator $z$ of $C(\sigma(a))$ to $a \in A$, and denote by $\text{Ell}(a)$ the following pair of induced maps:

$$K_*(\phi_a) : K_1(C(\sigma(a))) \to K_1(A); \phi_a^\sigma : T(A) \to T(C(\sigma(a))).$$

As in Theorem 4.1, the hypotheses of the next result guarantee the simplicity of $A$.

**Theorem 5.1.** Let $A$ be a simple unital exact C*-algebra with strict comparison and stable rank one. Let $a, b \in A$ be self-adjoint. It follows that $a$ and $b$ are approximately unitarily equivalent if and only if $\sigma(a) = \sigma(b)$ and $\text{Ell}(a) = \text{Ell}(b)$.

**Proof.** The “only if” statement is routine, so assume $\sigma(a) = \sigma(b)$ and $\text{Ell}(a) = \text{Ell}(b)$.

First, we handle the case that $\sigma(a) = \sigma(b) \subset (0, \infty)$, i.e., that both $a$ and $b$ are positive and invertible. Let $X = (\sigma(a) = \sigma(b)$ and $W_a : W(C(X)) \to W(A \otimes \mathcal{K})$ (resp. $W_b : W(C(X)) \to W(A \otimes \mathcal{K})$) denote the Cuntz-semigroup map induced by the canonical homomorphism $C(X) \to A \otimes \mathcal{K}$ sending $z \mapsto a \otimes e_{1,1}$ (resp. $z \mapsto b \otimes e_{1,1}$). We claim that $W_a = W_b$.

So, let $h \in M_n(C(X))$ be positive and $h_a \in M_n(A)$ (resp. $h_b \in M_n(A)$) denote the image of $h$ under the canonical inclusion $M_n(C(X)) \subset M_n(A)$ sending $C(X) \to C^*(a)$ (resp. $C(X) \to C^*(b)$). If $h \sim p$ for some projection in matrices over $C(X)$, then $h_a \sim p_a$ and $h_b \sim p_b$ (where $p_a$ and $p_b$ are the respective images of $p$ under the maps induced by $a$ and $b$). Since $\text{Ell}(a) = \text{Ell}(b)$, $[p_a] = [p_b] \in V(A)$ and thus $\langle p_a \rangle = \langle p_b \rangle \in W(A \otimes \mathcal{K})$—i.e. $W_a(h) = W_b(h)$.

If $h$ is not equivalent to a projection in matrices over $C(X)$, then neither $h_a$ nor $h_b$ are equivalent to projections (in matrices over $A$); indeed, since $A$ has stable rank one, [11, Proposition 3.12] implies that if $h_a$ was equivalent to a projection then zero would not be an accumulation point of $\sigma(h_a) = \sigma(h)$, hence $h$ would have to be equivalent to a projection as well, contrary to our assumption. In other words, $\langle h_a \rangle, \langle h_b \rangle \in W(A \otimes \mathcal{K})$, and hence Proposition 2.5 implies that it suffices to show $d_\tau(h_a) = d_\tau(h_b)$ for every $\tau \in T(A)$. However, if $\mu$ is a measure on $\sigma(h)$ then $d_\mu(h) = \mu(\sigma(h) \setminus \{0\})$. Since $\text{Ell}(a) = \text{Ell}(b)$, the maps on tracial spaces agree—i.e. for each $\tau \in T(A)$ the measures induced by restriction agree on $\sigma(h_a) = \sigma(h_b)$—and hence $d_\tau(h_a) = d_\tau(h_b)$ for every $\tau \in T(A)$, as desired.

Knowing that $W_a = W_b$, it now follows from [4] that $a \otimes e_{1,1}$ is approximately unitarily equivalent to $b \otimes e_{1,1}$ in the unitization of $A \otimes \mathcal{K}$. So, let $v_n \in (A \otimes \mathcal{K})^+$ be unitaries such that $v_n(a \otimes e_{1,1})v_n^* \to b \otimes e_{1,1}$. Since $a$ is invertible, for every $\varepsilon > 0$ there exists a
polynomial \( p \) such that \( \| p(a) - 1 \| < \varepsilon \); since \( \sigma(a) = \sigma(b) \), \( \| p(b) - 1 \| < \varepsilon \) as well. Hence, for large \( n \), \( \| v_n(1 \otimes e_{1,1})v_n^* - 1 \otimes e_{1,1} \| < C\varepsilon \) for some constant \( C \) depending only on \( \sigma(a) \). If \( \varepsilon \) is sufficiently small, this implies that \( (1 \otimes e_{1,1})v_n(1 \otimes e_{1,1}) \) is almost a unitary in \( A \)—hence can be perturbed to an honest unitary \( u_n \). A routine exercise now confirms that \( a \) is approximately unitarily equivalent to \( b \) (in \( A \)).

For the case of general self-adjoints \( a, b \in A \), we deduce the theorem from a simple trick. Namely, fix some constant \( c \) such that \( a + c1 \) is positive and invertible. Then \( b + c1 \) is also positive and invertible. By the case handled above, \( a + c1 \) and \( b + c1 \) are approximately unitarily equivalent, hence the same is true of \( a \) and \( b \).

The theorem above holds for all simple unital AH algebras with slow dimension growth, and for the class of simple unital exact stably finite \( \mathbb{Z} \)-stable \( C^* \)-algebras (see [2, Theorem 1], [14, Corollary 4.6], [16, Corollary 4.6]).

Another version of Theorem 5.1 holds for simple unital exact and stably finite \( C^* \)-algebras (without the strict comparison or stable rank assumptions):

**Theorem 5.2.** Let \( a \) and \( b \) be self-adjoint elements of a simple unital exact and stably finite \( C^* \)-algebra \( A \). Then \( a \) and \( b \) are approximately unitarily equivalent in \( A \otimes \mathbb{Z} \)—i.e. there exist unitaries \( u_n \in A \otimes \mathbb{Z} \) such that \( \| u_n(a \otimes 1)u_n^* - b \otimes 1 \| \to 0 \)—if and only if \( \sigma(a) = \sigma(b) \) and \( \text{Ell}(a) = \text{Ell}(b) \).

The proof of this result is a tiny perturbation of the proof of Theorem 5.1. The result is also, in some sense the best possible: in [17] a pair of positive elements in a simple unital AH algebra were constructed which have identical Elliott data but which are not Cuntz equivalent (hence not unitarily equivalent). For the interested reader, the elements in question are \( f(\tau^*(\xi) \times \tau^*(\xi)) \) and \( f\theta_1 \oplus f\theta_1 \), constructed in Section 3 of [17].

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**References**


1. Nathanial P. Brown and Andrew S. Toms


