L^p-multipliers arising from Lévy processes

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A Borel measure $\nu \geq 0$ on $\mathbb{R}^d$ with $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} \min(|z|^2, 1) d\nu(z) < \infty$$

is called a Lévy measure. Let $\mu \geq 0$ be a finite Borel measure on the unit sphere $\mathbb{S} \subset \mathbb{R}^d$, and

$$\varphi : \mathbb{R}^d \to \mathbb{C}, \quad \psi : \mathbb{S} \to \mathbb{C}, \quad \|\phi\|_\infty \leq 1, \|\psi\|_\infty \leq 1$$

Consider the multiplier

$$\mathcal{M}(\xi) = \frac{\int_{\mathbb{R}^d} (1 - \cos \xi \cdot z)\, \varphi(z)\, d\nu(z) + \frac{1}{2} \int_{\mathbb{S}} |\xi \cdot \theta|^2 \psi(\theta)\, d\mu(\theta)}{\int_{\mathbb{R}^d} (1 - \cos \xi \cdot z)\, d\nu(z) + \frac{1}{2} \int_{\mathbb{S}} |\xi \cdot \theta|^2 d\mu(\theta)},$$

Note that $\|\mathcal{M}\|_\infty \leq 1$.

We call φ and ψ **the "jump" and "Gaussian" modulators,** respectively, as they transform the corresponding parts of the Lévy process arising in the **"Lévy-Khinchin formula"**.

We note that

$$\mathcal{M}(\xi) = \frac{\int (1 - \cos \xi \cdot z)\, \varphi(z)\, d\nu(z) + \frac{1}{2}\mathbb{A}\xi \cdot \xi}{\int (1 - \cos \xi \cdot z)\, d\nu(z) + \frac{1}{2}\mathbb{B}\xi \cdot \xi},$$

$$\mathbb{A} = \left[\int_{\mathbb{S}} \varphi(\theta)\, \theta_i \theta_j \, d\mu(\theta) \right]_{i,j=1\ldots d} \quad \text{and} \quad \mathbb{B} = \left[\int_{\mathbb{S}} \theta_i \theta_j \, d\mu(\theta) \right]_{i,j=1\ldots d}$$

with both $\mathbb{A}$ and $\mathbb{B}$ symmetric and $\mathbb{B}$ non–negative definite.

Theorem (R.B. A. Bielaszewsk, K. Bordan (2007, 2009))

For every $1 < p < \infty$ the multiplier operator (use $\mathcal{M}$ for multiplier and operator) is bounded on $L^p(\mathbb{R}^d)$ and

$$\|\mathcal{M}f\|_p \leq (p^* - 1)\|f\|_p,$$

*where here (**and forever more**)* $p^* = \max\{p, \frac{p}{p-1}\}$

Questions

(1) Where do these multipliers come from?

(2) Why study these multipliers?

Answers

(1) They come from a more “sophisticated" stochastic analysis approach to investigate the L^p–norm of the Beurling–Alhfors operator (the (p*-1)–problem) using general Lévy processes.

(2) Same answer as (1)

Conjecture: Tadeusz Iwaniec, 1982

$$Bf(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{f(w)}{(z-w)^2}\, dA(w) \qquad \Rightarrow \|Bf\|_p \leq (p^* - 1)\|f\|_p.$$

Long known (Lehto (1965)) that $(p^* - 1)$ cannot be improved.

- Ba.–Wang (1995):

  $$\|B\| \leq 4(p^* - 1)$$

  Via "Brownian" stochastic integrals, and Burkholder's martingale Inequalities.
- Nazarov–Volberg (2003):

  $$\|B\| \leq 2(p^* - 1)$$

  Via a "Heat Littlewood-Paley inequality" proved with Bellman functions. Burkholder's result is used for the construction of the function–hence not martingale independent either.

- Ba.-Méndez (2003):

$$\|B\| \leq 2(p^* - 1)$$

Exact same proof as the "4" result with Wang except applied to "space-time" Brownian martingales motivated by Nazarov–Volberg.

- Dragičević–Volberg (2005):

$$\|B\| \leq \sqrt{2}(p-1)\left(\frac{1}{2\pi}\int_0^{2\pi} |\cos(\theta)|^p d\theta\right)^{-\frac{1}{p}}, \quad 2 \leq p < \infty$$

- Ba.-Janakiraman (2006):

$$\|B\| \leq \sqrt{2p(p-1)}, \quad 2 \leq p < \infty$$

"More" space-time stochastic integrals, and an improvement to Burkholder's inequality in certain "orthogonal" cases. From this one obtains

$$\|B\| \leq 1.575(p^* - 1), \quad 1 < p < \infty$$

First proof that does not treat B just as sums of second order Riesz!

Definition

A Lévy Process is a stochastic process $X = (X_t), t \geq 0$ with

- X has independent and stationary increments

- $X_0 = 0$ (with probability 1)

- X is *stochastically continuous*: For all $\varepsilon > 0$,

$$\lim_{t \to s} P\{|X_t - X_s| > \varepsilon\} = 0$$

Note: Not the same as a.s. continuous paths. However, it gives "cadlag" paths: Right continuous with left limits.

- **Stationary increments:** $0 < s < t < \infty$, $A \in \mathbb{R}^d$ Borel

$$P\{X_t - X_s \in A\} = P\{X_{t-s} \in A\}$$

- **Independent increments:** For any given sequence of ordered times

$$0 < t_1 < t_2 < \cdots < t_m < \infty,$$

the random variables

$$X_{t_1} - X_0,\ X_{t_2} - X_{t_1}, \ldots, X_{t_m} - X_{t_{m-1}}$$

are independent.

The characteristic function of X_t is

$$\varphi_t(\xi) = E\left(e^{i\xi \cdot X_t}\right) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} p_t(dx)$$

where $p_t(dx)$ is the distribution of the random variable X_t.

The characteristic function has the form $\varphi_t(\xi) = e^{t\rho(\xi)}$, where

$$\rho(\xi) = ib\cdot\xi - \frac{1}{2}\xi\cdot\mathbb{B}\,\xi + \int_{\mathbb{R}^d}\left(e^{i\xi\cdot x} - 1 - i\xi\cdot x 1_{\{|x|<1\}}(x)\right)\nu(dx)$$

for some $b \in \mathbb{R}^d$, a non–negative definite symmetric $d \times d$ matrix $\mathbb{B}$ and a Borel measure ν on $\mathbb{R}^d$ with $\nu\{0\} = 0$ and

$$\int_{\mathbb{R}^d} \min\left(|x|^2, 1\right)\nu(dx) < \infty$$

$\rho(\xi)$ is called the **"Lévy symbol"** of the process or the **characteristic exponent.** The triple $(b, \mathbb{B}, \nu)$ is called the **characteristics of the process.**

Converse also true. Given such a triple we can construct a Lévy process.

Lévy processes can be written as

$$X_t = X_t^c + \sum_{s:0\leq s\leq t} \Delta X_s$$

$$\Delta X_s = X_s - X_{s^-} = \text{jump at time s.}$$

The *Poisson random measure,* aka "jump measure" is

$$N(t,A) = N(t,A,\omega) = \sum_{s:0<s\leq t} \chi_A(\Delta X_s)$$

and the Lévy measure is:

$$\nu(A) = E\left(\sum_{s:0<s\leq 1} \chi_A(\Delta X_s)\right)$$

Lévy-Itô: The Lévy process $X(t)$ has the representation

$$X_t = bt + B_A(t) + \int_{\{0<|x|<1\}} x\left[N(t,dx) - td\nu(x)\right] + \int_{\{|x|\geq|1\}} xN(t,dx)$$

Example (1. Brownian motion or Brownian motion plus drift)

With $(0, I, 0)$, I, the identity matrix, or $(b, I, 0)$, we get

$$X_t = B_t, \qquad \text{Standard Brownian motion}$$

or

$$X_t = bt + B_t, \qquad \text{Brownian motion plus drift}$$

Example (2. Gaussian Processes, "General Brownian motion")

$(0, \mathbb{B}, 0)$, X_t is "generalized" Brownian motion, mean zero, covariance

$$E(X_s^j X_t^i) = b_{ij} \min(s, t)$$

X_t has the normal distribution (assume here that $det(\mathbb{B}) > 0$)

$$\frac{1}{(2\pi t)^{d/2}\sqrt{\det(\mathbb{B})}} \exp\left(-\frac{1}{2t} x \cdot \mathbb{B}^{-1} x\right)$$

Example (3. Poisson Process)

The Poisson Process $X_t = \pi_t(\lambda)$ of intensity $\lambda > 0$ is a Lévy process with $(0, 0, \lambda\delta_1)$ where δ_1 is the Dirac delta at 1.

$$P\{\pi_t(\lambda) = m\} = \frac{e^{-\lambda t}(\lambda t)^m}{m!}, \; m = 0, 1, \ldots$$

$\pi_t(\lambda)$ has continuous paths except for jumps of size 1 at the random times

$$\tau_m = \inf\{t > 0 : \pi_t(\lambda) = m\}$$

Example (4. Compound Poisson Process)

Let $Y_1, Y_2, \ldots$ be i.i.d. and independent of the π_t with distribution ν. The process $X_t = Y_1 + Y_2 + \cdots + Y_{\pi_t(\lambda)} = S_{\pi_t(\lambda)}$ is a Lévy process. By independence

$$E[e^{i\xi \cdot X_t}] = \sum_{m=0}^{\infty} P\{\pi_t = m\} \, E[e^{i\xi \cdot S_m}] = \sum_{m=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^m}{m!} \left(\widehat{\nu}(\xi)\right)^m = e^{\lambda t(\widehat{\nu}(\xi)-1))}$$

$$\Rightarrow \rho(\xi) = \lambda \int_{\mathbb{R}^d} \left(e^{ix\cdot\xi} - 1\right) d\nu(x)$$

Example (5. The rotationally invariant stable processes, $0 < \alpha \leq 2$.)

These are self–similar processes, denoted by X_t^α, in $\mathbb{R}^d$ with symbol

$$\rho(\xi) = -|\xi|^\alpha,$$

with Lévy measures

$$d\nu^\alpha = \frac{c_{\alpha,d}}{|x|^{d+\alpha}} dx, \quad 0 < \alpha < 2, \quad \nu^\alpha = 0, \quad \alpha = 2.$$

$\alpha = 2$ is **Brownian motion,** $\alpha = 1$ **Cauchy processes.** $\alpha = 3/2$ is called the **Haltsmark distribution** used to model gravitational fields of stars.

Example (6. Relativistic Brownian motion)

According to quantum mechanics, a particle of mass $m > 0$ moving with momentum p has kinetic energy

$$E(p) = \sqrt{m^2 c^4 + c^2 |p|^2} - mc^2$$

where c is speed of light. Then $\rho(\xi) = -E(\xi)$ is the symbol of a Lévy process, called *“relativistic Brownian motion."*

Take μ point mass at $1, i, e^{-i\pi/4}, e^{i\pi/4}$ $\quad (\nu = 0)$

$$\psi(1) = 1,\ \varphi(e^{-i\pi/4}) = i,\ \varphi(i) = -1,\ \varphi(e^{i\pi/4}) = -i\ .$$

Write $\xi = |\xi|\,(\cos u, \sin u)$, then we have

$$\begin{aligned}\int_{\mathbb{S}} |\xi \cdot \theta|^2 \psi(\theta) d\mu(\theta) &= \xi_1^2 - \xi_2^2 + i(\xi_1 \frac{1}{\sqrt{2}} - \xi_2 \frac{1}{\sqrt{2}})^2 - i(\xi_1 \frac{1}{\sqrt{2}} + \xi_2 \frac{1}{\sqrt{2}})^2 = \\ &= \xi_1^2 - \xi_2^2 - 2i\, \xi_1 \xi_2 = \bar{\xi}^2,\end{aligned}$$

and

$$\int_{\mathbb{S}} |\xi \cdot \theta|^2 \mu(d\theta) = \xi_1^2 + \xi_2^2 + (\xi_1 \frac{1}{\sqrt{2}} - \xi_2 \frac{1}{\sqrt{2}})^2 + (\xi_1 \frac{1}{\sqrt{2}} + \xi_2 \frac{1}{\sqrt{2}})^2 = 2\,|\xi|^2\ ,$$

$$\mathcal{M}(\xi) = \frac{\int_{\mathbb{S}} |\xi \cdot \theta|^2 \psi(\theta) d\mu(\theta)}{\int_{\mathbb{S}} |\xi \cdot \theta|^2 d\mu(\theta)} = \frac{\overline{\xi^2}}{2\,|\xi|^2}\,,$$

Beurling-Ahlfors bound $2(p^* - 1)$.

Proposition (A disappointing fact)

If μ is a measure on the circle $\mathbb{S}$ in $\mathbb{R}^2$ and $\psi : \mathbb{S} \to \mathbb{C}$ with $\|\psi\|_\infty \leq 1$ and

$$\frac{\int_{\mathbb{S}} |\xi \cdot \theta|^2 \, \psi(\theta) d\mu(\theta)}{\int_{\mathbb{S}} |\xi \cdot \theta|^2 \, d\mu(\theta)} = \frac{\overline{\xi^2}}{c\,|\xi|^2}, \quad \xi \in \mathbb{R}^2 \setminus \{0\},$$

then $|c| \geq 2$.

For $0 < \alpha < 2$, consider a stable Lévy measure in $\mathbb{R}^2$ (in polar coordinates)

$$d\nu^\alpha(r,\theta) = r^{-1-\alpha} dr\, d\sigma(\theta) \quad \text{and} \quad \varphi(z) = \varphi\left(\frac{z}{|z|}\right),$$

σ Lebesgue on unit disc $\mathbb{S}$.

$$\begin{aligned} \Psi_\varphi(\xi) &= \int_{\mathbb{R}^2} (1 - cos(\xi \cdot z))\varphi(z) d\nu^\alpha(z) \\ &= \int_{\mathbb{S}} \int_0^\infty (1 - \cos(r\theta \cdot \xi))\varphi(r\theta) r^{-1-\alpha}\, dr\, d\sigma(\theta) \\ &= \int_{\mathbb{S}} |\xi \cdot \theta|^\alpha \varphi(\theta) \int_0^\infty \frac{1 - \cos(s)}{s^{1+\alpha}} ds d\sigma(\theta) \\ &= c_\alpha \int_{\mathbb{S}} |\xi \cdot \theta|^\alpha \varphi(\theta) d\sigma(\theta), \end{aligned}$$

$$c_\alpha = \int_0^\infty \frac{1 - \cos(s)}{s^{1+\alpha}} ds$$

The Lévy "Stable Multiplier" is: $\mathcal{M}(\xi) = \frac{\Psi_\varphi(\xi)}{\Psi(\xi)} = \frac{\int_{\mathbb{S}} |\xi\cdot\theta|^\alpha \varphi(\theta)\, d\sigma(\theta)}{\int_{\mathbb{S}} |\xi\cdot\theta|^\alpha \, d\sigma(\theta)}$.

With $\theta = (\cos(t), \sin(t))$, set

$$\varphi(\cos(t), \sin(t)) = e^{-i2t}, \qquad \xi = |\xi|\, e^{iu} = |\xi|\,(\cos(u), \sin(u))$$

$$\begin{aligned}
\Psi_\varphi(\xi) &= c_\alpha \int_0^{2\pi} |\xi|^\alpha \, |(\cos(u), \sin(u)) \cdot (\cos(t), \sin(t))|^\alpha \, e^{-i2t} dt \\
&= c_\alpha \, |\xi|^\alpha \int_0^{2\pi} |\cos(t-u)|^\alpha \, e^{-i2t} dt \\
&= c_\alpha \, |\xi|^\alpha \int_0^{2\pi} |\cos(v)|^\alpha \, e^{-i2(u+v)} dv \\
&= c_\alpha \, |\xi|^\alpha \, e^{-i2u} \int_0^{2\pi} |\cos(v)|^\alpha \, e^{-i2v} dv \\
&= c_\alpha \, |\xi|^\alpha \, \frac{\bar{\xi}^2}{|\xi|^2} \int_0^{2\pi} |\cos(v)|^\alpha \, (\cos(2v) - i\sin(2v)) dv.
\end{aligned}$$

$$\int_0^{2\pi} |\cos(v)|^\alpha \sin(2v)dv = 2\int_0^{2\pi} |\cos(v)|^\alpha \sin(v)\cos(v)dv = 0$$

$$\int_0^{\frac{\pi}{2}} \sin^a(v)\cos^b(v)\,dv = \frac{1}{2}\frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{a+b+2}{2}\right)} = \frac{1}{2}\mathcal{B}\left(\frac{a+1}{2},\frac{b+1}{2}\right), a, b > -1$$

$$\begin{aligned}
\int_0^{2\pi} |\cos(v)|^\alpha \cos(2v)dv &= \int_0^{2\pi} |\cos(v)|^\alpha (2\cos^2(v)-1)dv \\
&= 2\int_0^{2\pi} |\cos(v)|^{\alpha+2}\,dv - \int_0^{2\pi} |\cos(v)|^\alpha\,dv \\
&= 4\left(2\int_0^{\frac{\pi}{2}} |\cos(v)|^{\alpha+2}\,dv - \int_0^{\frac{\pi}{2}} |\cos(v)|^\alpha\,dv\right) \\
&= 2\left(2\mathcal{B}\left(\frac{\alpha+3}{2},\frac{1}{2}\right) - \mathcal{B}\left(\frac{\alpha+1}{2},\frac{1}{2}\right)\right) \\
&= 2\left(2\frac{\alpha+1}{\alpha+2}-1\right)\mathcal{B}\left(\frac{\alpha+1}{2},\frac{1}{2}\right) \\
&= 2\frac{\alpha}{\alpha+2}\mathcal{B}\left(\frac{\alpha+1}{2},\frac{1}{2}\right).
\end{aligned}$$

$$\Psi_\varphi(\xi) = 2c_\alpha \, |\xi|^\alpha \, \frac{\bar{\xi}^2}{|\xi|^2} \frac{\alpha}{\alpha+2} \, \mathcal{B}\left(\frac{\alpha+1}{2}, \frac{1}{2}\right),$$

For $\varphi = 1$ then we have

$$\Psi(\xi) = c_\alpha \, |\xi|^\alpha \int_0^{2\pi} |\cos(v)|^\alpha \, dv = 2c_\alpha \, |\xi|^\alpha \, \mathcal{B}\left(\frac{\alpha+1}{2}, \frac{1}{2}\right).$$

Thus,

$$\mathcal{M}(\xi) = \frac{\Psi_\varphi(\xi)}{\Psi(\xi)} = \frac{\bar{\xi}^2}{|\xi|^2} \frac{\alpha}{\alpha+2}$$ **(also disappointing!)**

Therefore

$$\|Bf\|_p \leq \frac{\alpha+2}{\alpha}(p^* - 1)\|f\|_p,$$

and letting $\alpha \to 2$ gives

$$\|B\| \leq 2(p^* - 1)$$

Example (A Marcinkiewicz–type, Stein "Singular Integrals" p. 110)

Let

$$\lambda = \delta_{(1,0,\ldots,0)} + \delta_{(-1,0,\ldots,0)} + \cdots + \delta_{(0,0,\ldots,1)} + \delta_{(0,0,\ldots,-1)} \,.$$

In polar coordinates we define the Lévy measure

$$\nu^\alpha(dr, d\theta) = r^{-1-\alpha} dr\, d\lambda(\theta), \qquad 0 < \alpha < 2.$$

(Symmetric α-stable Lévy process with independent coordinates.)

$$\int_{\mathbb{S}} |\xi \cdot \theta|^\alpha d\lambda(\theta) = |\xi_1|^\alpha + \cdots + |\xi_d|^\alpha$$

Let $\varphi(z_1, \ldots, z_d) = 1$ if $z_k = 0$ for $k \neq j$ and $z_j \neq 0$, and let $\varphi = 0$ otherwise. (That is, observe only the jumps of the jth coordinate process.) The Multiplier is:

$$M(\xi) = \frac{|\xi_j|^\alpha}{|\xi_1|^\alpha + \ldots + |\xi_d|^\alpha}, \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^d \,.$$

"Proof": Enough to do $\mu = 0$, ν finite and symmetric

Case $\mu = 0$ By "symmetrization", we may assume ν is symmetric, i.e., $\nu(B) = \nu(-B)$

$$\tilde{\nu}(E) = \frac{(\nu(E) + \nu(-E))}{2}, \quad \hat{\nu}(E) = \frac{(\nu(E) - \nu(-E))}{2},$$

$$\tilde{\varphi}(z) = \frac{(\varphi(z) + \varphi(-z))}{2}, \quad \hat{\varphi}(z) = \frac{(\varphi(z) - \varphi(-z))}{2}.$$

Since

$$f(z) = \cos \xi \cdot z \qquad \text{is symmetric,}$$

$$\int_{\mathbb{R}^d} (\cos(\xi \cdot z) - 1)\, d\nu(z) = \int_{\mathbb{R}^d} (\cos(\xi \cdot z) - 1)\, d\tilde{\nu}(z)$$

Also (as measures),

$$\varphi\nu = (\tilde{\varphi} + \hat{\varphi})(\tilde{\nu} + \hat{\nu}) = (\tilde{\varphi}\tilde{\nu} + \hat{\varphi}\hat{\nu}) + (\tilde{\varphi}\hat{\nu} + \hat{\varphi}\tilde{\nu})$$

Thus,

$$\begin{aligned} M(\xi) &= \frac{\int_{\mathbb{R}^d} (1-\cos(z\cdot\xi))\, \varphi(z) d\nu(z)}{\int_{\mathbb{R}^d}(1-\cos(z\cdot\xi))d\nu(z)} \\ &= \frac{\int_{\mathbb{R}^d} (1-\cos(z\cdot\xi))\, d\left(\tilde{\varphi}\tilde{\nu}+\hat{\varphi}\hat{\nu}\right)(z)}{\int_{\mathbb{R}^d}(1-\cos(z\cdot\xi))d\tilde{\nu}(z)} \\ &= \frac{\int_{\mathbb{R}^d} (1-\cos(z\cdot\xi))\left(\tilde{\varphi}+k\hat{\varphi}\right)(z) d\tilde{\nu}(z)}{\int_{\mathbb{R}^d}(1-\cos(z\cdot\xi))d\tilde{\nu}(z)} \end{aligned}$$

Observe:

$$\nu = \tilde{\nu}+\hat{\nu} \geq 0, \text{ and } \hat{\nu} = k\tilde{\nu},$$

with an antisymmetric function k such that $|k| \leq 1$. So, the numerator above is integrated against $(\tilde{\varphi}+k\hat{\varphi})\tilde{\nu}$. With $|\varphi| \leq 1$, we also have $|\tilde{\varphi} \pm \hat{\varphi}| \leq 1$ and so

$$\left|\tilde{\varphi}+k\hat{\varphi}\right| = \left|\frac{1+k}{2}(\tilde{\varphi}+\hat{\varphi}) + \frac{1-k}{2}(\tilde{\varphi}-\hat{\varphi})\right| \leq 1.$$

Conclusion: Enough to prove Theorem for symmetric ν when $\mu = 0$.

Given μ and ψ on $\mathbb{S}$, consider the Lévy measures and function ψ on $\mathbb{R}^d$

$$\nu_\varepsilon(drd\theta) = \varepsilon^{-2}\delta_\varepsilon(dr)\mu(d\theta), \qquad \psi(z) = \psi\left(\frac{z}{|z|}\right)$$

We have

$$\int_{\mathbb{R}^d}(1-\cos(\xi\cdot z))\psi(z)d\nu_\varepsilon(z) = \int_{\mathbb{S}}\int_0^\infty(1-\cos(r\theta\cdot\xi))\psi(\theta)\frac{1}{\varepsilon^2}\delta_\varepsilon(dr)d\mu(\theta)$$

$$= \int_{\mathbb{S}}|\xi\cdot\theta|^2\,\psi(\theta)\frac{(1-\cos(\varepsilon\theta\cdot\xi))}{|\varepsilon\theta\cdot\xi|^2}d\mu(\theta) \quad\to\quad \int_{\mathbb{S}}|\theta\cdot\xi|^2\,\frac{1}{2}\psi(\theta)d\mu(\theta), \varepsilon\to 0$$

Let $\mathcal{M}_\varepsilon$ be the multiplier for the Lévy measure

$$\tilde{\nu}_\varepsilon = \mathbf{1}_{\{|z|>\varepsilon\}}\nu + \nu_\varepsilon$$

and define $\varphi_\varepsilon : \mathbb{R}^d \to \mathbb{C}$ by

$$\varphi_\varepsilon = \mathbf{1}_{\{|z|>\varepsilon\}}\varphi + \mathbf{1}_{\{|z|\leq\varepsilon\}}\psi.$$

Letting $\varepsilon\to 0$ gives result.

The Lévy semigroup and generator

Notation: $\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx, \;\; f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \widehat{f}(\xi)d\xi$

$$
\begin{aligned}
T_t f(x) &= E[f(X(t))|X_0 = x] = E_0[f(X(t) + x)] \\
&= \frac{1}{(2\pi)^{d/2}} E_0 \left(\int_{\mathbb{R}^d} e^{i(X_t + x)\cdot\xi} \widehat{f}(\xi) d\xi \right) \\
&= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} E_0 \left(e^{iX_t\cdot\xi} \right) \widehat{f}(\xi) d\xi \\
&= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{t\rho(\xi)} \widehat{f}(\xi) d\xi
\end{aligned}
$$

That is, semigroup and generator are, respectively,

$$\widehat{T_t f}(\xi) = e^{t\rho(\xi)} \hat{f}(\xi) \quad \text{and} \quad \widehat{Af}(\xi) = \rho(\xi)\hat{f}(\xi)$$

Theorem (Self -Adjointness of semigroup)

The semigroup is self-adjoint in L^2 iff X_t is symmetric ($P\{X_t \in B\} = P\{X_t \in -B\}$). This leads to: T_t is self-adjoint iff

$$\rho(\xi) = -\frac{1}{2}\xi \cdot A\xi + \int_{\mathbb{R}^d} \Big(cos(x \cdot \xi) - 1\Big) d\nu(x)$$

A a symmetric matrix and ν a symmetric ($\nu(B) = \nu(-B)$) Lévy measure.

Take $\nu = 0$ and $A = I$. X_t is good old–fashioned Brownian motion.

$$u_f(x, t) = T_t f(x)$$

Good old–fashioned heat extension.

$$F_t = u_f(X_t, T - t) = \textit{martingale}, \quad 0 < t < T$$

The Itô formula gives

$$F_T - F_0 = f(X_T) - u_f(x, T) = \int_0^T \nabla_x u_f(B_t) \cdot dX_t$$

Assume X_t is a Compound Poisson Processes. That is,

$$\rho(\xi) = \int_{\mathbb{R}^d} \Big(cos(x \cdot \xi) - 1 \Big) d\nu(x), \qquad \nu(\mathbb{R}^d) < \infty.$$

Another approximation removes the assumption $\nu(\mathbb{R}^d) < \infty$.

For "very nice" $f's$ on $\mathbb{R}^d$, set

$$u_f(x,t) = T_t f(x) = \text{"}\textit{heat extension}\text{"}$$

and **"space-time" martingale**

$$F_t = u_f(X_t, T - t), \qquad 0 < t < T$$

By **Itô with "jumps"**

$$\begin{aligned} F_t - F_0 &= \sum_{0<s\leq t} [u_f(X_s, T - s) - u_f(X_{s^-}, T - s_i)] \\ &- \int_0^t \int_{\mathbb{R}^d} [u_f(X_{s^-} + z, T - s) - u_f(X_{s^-}, T - s)] \nu(dz) ds. \end{aligned}$$

This is a martingale for $0 < t < T$.

Let $\varphi : \mathbb{R}^d \to \mathbb{C}$ be such that $\|\varphi\|_\infty \leq 1$. Consider the new martingale.

$$\begin{aligned}\varphi(F)_t &= \sum_{0<s\leq t} [u_f(X_s, T-s) - u_f(X_{s^-}, T-s)]\varphi(\Delta X_s) \\ &+ \int_0^t \int_{\mathbb{R}^d} [u_f(X_{s^-}+z, T-s) - u_f(X_{s^-}, T-s)]\varphi(z)\nu(dz)ds\end{aligned}$$

on $0 < t < T$.

Theorem (Wang 1997)

If Y_t and X_t are two general (right continuous, left limits=r.c.l.l.=càdlàg) martingales with $[X, X]_t - [Y, Y]_t$ non-decreasing and nonnegative. Then

$$\|Y\|_{L^p(\Omega)} \leq (p^* - 1)\|X\|_{L^p(\Omega)}, \qquad 1 < p < \infty$$

and $p^* - 1$ is best possible.

From this we get

Theorem (Note the spaces for p-norms)

$$\|\varphi(F)_T\|_{L^p(\Omega)} \leq (p^* - 1)\|F_T\|_{L^p(\Omega)} = (p^* - 1)\|f\|_{L^p(\mathbb{R}^d)}, \qquad 1 < p < \infty$$

Note:

$$E|f(X_T)|^p = \int_{\mathbb{R}^d} E_x|f(X_T)|^p \, dx = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x+y) p_T(dy) \right) dx$$

$$= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x+y))|^p dx \right) p_T(dy) = \int_{\mathbb{R}^d} |f(x)|^p dx$$

Definition

For large and fixed T, define

$$\mathbf{S}_\varphi^\mathbf{T}\mathbf{f}(\mathbf{x}) = \mathbf{E}[\varphi(\mathbf{F})_\mathbf{T}|\mathbf{X}_\mathbf{T} = \mathbf{x}]$$

Corollary

$$\|S_\varphi^T f\|_{L^p(\mathbb{R}^d)} \leq \|\varphi(F)_T\|_{L^p(\Omega)} \leq (p^* - 1)\|F_T\|_{L^p(\Omega)}$$

$$= (p^* - 1)\|f\|_{L^p(\mathbb{R}^d)}, \quad 1 < p < \infty,$$

Proposition

$$\widehat{S^T_\varphi f}(\xi) = \mathcal{M}_T(\xi)\hat{f}(\xi)$$

$$\begin{aligned}\mathcal{M}_T(\xi) &= \left(e^{2T\rho(\xi)} - 1\right)\frac{1}{\rho(\xi)}\int_{\mathbb{R}^d}(1-\cos(z\cdot\xi))\,\varphi(z)\nu(dz)\\ &= \left(e^{2T\rho(\xi)} - 1\right)\frac{\int_{\mathbb{R}^d}(1-\cos(z\cdot\xi))\,\varphi(z)\nu(dz)}{\int_{\mathbb{R}^d}(\cos(z\cdot\xi)-1)\nu(dz)}\end{aligned}$$

With $M_T(\xi) = 0$ *whenever* $\rho(\xi) = 0$.

Lemma (A “Littlewood-Paley" type identity)

$$\begin{aligned}&\int_{\mathbb{R}^d} S^T_\varphi f(x)\overline{g(x)}dx = E[\phi(F)_T\overline{G_T}]\\ = \;&\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left\{\overline{u_g(x+y+z,T-s) - u_g(x+y,T-s)}\right\}\times\\ &\left\{u_f(x+y+z,T-s) - u_f(x+y,T-s)\right\}\varphi(z)d\nu(z)p_T(dy)\;dx\,ds\end{aligned}$$

Recall

$$\widehat{u_f}(\xi,t) = \widehat{T_t f}(\xi) = e^{t\rho(\xi)}\hat{f}(\xi)$$

Do the $\{dx\, p_T(dy)\}$ integral first. Change variables $t = T - s$

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left\{\overline{u_g(x+z,T-s) - u_g(x,T-s)}\right\} \times \\
& \left\{u_f(x+z,T-s) - u_f(x,T-s)\right\}\, dx\ \varphi(z) d\nu(z)\, ds \\
= \ & \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left\{\overline{u_g(x+z,t) - u_g(x,t)}\right\}\left\{u_f(x+z,t) - u_f(x,t)\right\} dx \varphi(z) d\nu(z) d \\
& \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left\{\overline{e^{iz\cdot\xi}\hat{g}(\xi) - \hat{g}(\xi)}\right\} e^{t\rho(\xi)} \left\{e^{iz\cdot\xi}\hat{f}(\xi) - \hat{f}(\xi)\right\} e^{t\rho(\xi)} d\xi \varphi(z) d\nu(z) dt \\
& \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |e^{iz\cdot\xi} - 1|^2 \hat{f}(\xi)\overline{\hat{g}(\xi)} e^{2t\rho(\xi)} d\xi \varphi(z) d\nu(z) dt \\
= \ & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 2\Big(1 - \cos(z\cdot\xi)\Big)\left(e^{2T\rho(\xi)} - 1\right) \frac{1}{2\rho(\xi)} \hat{f}(\xi)\overline{\hat{g}(\xi)} d\xi \varphi(z) d\nu(z) \\
= \ & \int_{\mathbb{R}^d} \left(e^{2T\rho(\xi)} - 1\right) \frac{1}{\rho(\xi)} \left(\int_{\mathbb{R}^d} \Big(1 - \cos(z\cdot\xi)\Big)\varphi(z) d\nu(z)\right) \hat{f}(\xi)\, \overline{\hat{g}(\xi)}\, d\xi
\end{aligned}$$

Remark 1

This was all for finite measures ν. Another limiting procedure extends results to general symmetric Lévy measures.

Remark 2

If $\rho(\xi) = \int_{\mathbb{R}^d}(\cos \xi \cdot z - 1)\nu(dz) = 0$ for $\xi \neq 0$, then $\operatorname{supp} \nu \subset A_\xi$, where

$$A_\xi = \{z \,:\, \xi \cdot z = 2k\pi \text{ for some integer } k\}\,.$$

In particular, A_ξ is discrete in the direction of ξ. By Fubini's theorem $\{\xi \,:\, \rho(\xi) = 0\}$ has zero Lebesgue measure. Thus our convention that $\mathcal{M}(\xi) = 0$ when $\rho(\xi) = 0$, does not influence the definition of $\mathcal{M}$ on L^2 or L^p.