Four unknown constants

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1 Four inequalities

Let \( \Omega \subset \mathbb{R}^2 \) be an arbitrary simply connected domain in the plane. We define \( R_\Omega = \sup_{z \in \Omega} d_\Omega(z) \) (the inradius of the domain) where \( d_\Omega(z) \) is the distance from \( z \) to the boundary of \( \Omega \). Let \( \sigma_\Omega(z) \) be the density of the hyperbolic metric in \( \Omega \) and let \( \sigma_\Omega = \inf_{z \in \Omega} \sigma_\Omega(z) \). Finally, denote by \( \lambda_1 \) the lowest eigenvalue for the Dirichlet Laplacian in \( \Omega \) and denote by \( \tau_\Omega \) the first exit time of Brownian motion from \( \Omega \). The following four inequalities hold.

1. There exists a positive constant \( C_1 \), independent of the domain, such that for all functions \( u \in C_0^\infty(\Omega) \)

\[
\int_{\Omega} \frac{|u|^2}{d_\Omega^2} \leq C_1 \int_{\Omega} |\nabla u|^2.
\]

This inequality is known as the “Hardy” inequality in the literature. It holds for domains which are more general than simply connected but does not hold for all domains, see [2]. The survey paper [11] contains a detailed account of this inequality as of around 1998. For some recent work, please see [1], [12], [13], [15], [17], [19] and references therein. In the setting of simply connected domains the inequality can be easily reduced to that of the unit disc or half-space with the aid of the Koebe \( \frac{1}{4} \)-theorem. In fact, the Koebe \( \frac{1}{4} \)-theorem proof gives the inequality with \( C_1 = 16 \), (see [2]).

2. There exists a positive constant \( C_2 \), independent of the domain, such that

\[
\frac{C_2}{R_\Omega^2} \leq \lambda_1 \leq \frac{j_0^2}{R_\Omega^2}.
\]

The right hand side inequality is trivial by domain monotonicity of the eigenvalue—the larger the domain the smaller the eigenvalue. The constant \( j_0 \) is the smallest positive zero of the first Bessel function \( J_0 \). Of course, the right hand side inequality is sharp. The left hand side inequality follows from the variational characterization of

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the eigenvalue and the Hardy inequality (1.1). As above, the left hand side inequality holds
for more general domains than just simply connected domains but not all.
(Adding points to a domain has no affect on the eigenvalue but it can have a drastic
affect on the inradius.) This inequality also has a long and interesting history, see [3]
and [4].

3. There exists a positive constant $C_3$, independent of the domain, such that

$$\frac{1}{2} R^2_{\Omega} \leq \sup_{z \in \Omega} E_z(\tau_\Omega) \leq C_3 R^2_{\Omega}. \tag{1.3}$$

Here we use $E_z$ to denote the expectation with respect to the Brownian motion starting
at the point $z \in \Omega$. Again, the lower bound is trivial by domain monotonicity (the
larger the domain the larger the lifetime). A necessary and sufficient condition (which
includes all simply connected domains in $\mathbb{R}^2$) for a domain in $\mathbb{R}^d$ to have (1.3) is given
in [8]. Again, since Brownian motion does not “see” points in two dimensions, the
right hand side inequality cannot hold for all domains.

4. There exist a positive constant $C_4$, independent of the domain, such that

$$\frac{C_4}{R_{\Omega}} \leq \sigma_{\Omega} \leq \frac{1}{R_{\Omega}}. \tag{1.4}$$

As above, the upper bound is obtained by domain monotonicity and the existence of
the constant $C_4$ follows at once from the Koebe $\frac{1}{4}$-theorem since $\sigma_{\Omega}(z) = \frac{1}{|F'(0)|}$, where
$F$ is the conformal mapping from the unit disc onto the domain $\Omega$ with $F(0) = z$.

Problem 1 Identify the extremal constants $C_1, C_2, C_3, C_4$ in the above inequalities and the
geometry of the “extremal” domains (whenever they exist).

2 Convex domains

In the case of convex domains, all constants are known:

1. $C_1(\text{convex}) = 4$ which is the constant for the half space (or even the one dimensional
half-line). For a proof of this, see Davies [11]. There are also other sharper general-
izations such as the one given in [1]. (Please also consult references given in [1] for
more on these kind of extensions.) These results hold for convex domains in $\mathbb{R}^d$.

2. $C_2(\text{convex}) = \pi^2/4$ and the extremal domain is an infinite strip. The same constant
works also for any convex domain in $\mathbb{R}^d$. There are several proofs of this result
including the original one given by J. Hersh in [14]. (See also [1] for a proof based on
the Hardy inequality and other references.)
3. $C_3(\text{convex}) = 1$ (see R. Sperb in [18]). Again, the extremal is given by an infinite strip (which reduces the problem to an interval). Here again, there is a more general inequality which asserts that for any convex domain in $\mathbb{R}^d$ of inradius $R_\Omega$,

$$P_z\{\tau_\Omega > t\} \leq P_0\{\tau_{(-R_\Omega,R_\Omega)} > t\},$$

where $\tau_{(-R_\Omega,R_\Omega)}$ is the exit time from the interval $(-R_\Omega, R_\Omega)$ on the real line. (For this, see [6] and [7].) The inequality (2.1) together with the well-known classical characterization of the eigenvalue as

$$-\lambda_1 = \lim_{t \to \infty} \frac{1}{t} \log P_z\{\tau_\Omega > t\}$$

gives a different proof that $C_2(\text{convex}) = \pi^2/4$. Again, the same results holds in all dimensions where the extremal domain is the infinite slab.

4. $C_4(\text{convex}) = \pi/4$. This result was proved by Szegő in 1923 (see [3] for exact reference). Again, the extremal domain is the infinite strip.

3 Arbitrary simply connected domains

The following estimates for the optimal constants $C_1, C_2, C_3, C_4$ are known.

(3.1) \hspace{0.5cm} 4 \leq C_1 \leq 16

(3.2) \hspace{0.5cm} 0.6194 < C_2 < 2.095

(3.3) \hspace{0.5cm} 1.584 < C_3 < 3.228

(3.4) \hspace{0.5cm} 0.57088 < C_4 < 0.6563937

For the estimates for $C_2$ and $C_3$, and some history on these constants, we refer the reader to [3] and [9]. The paper [3] also contains some examples of simply connected domains which we conjecture are very close to the extremals for these four problems. The problem of determining the best constant $C_4$ (known as the Schlicht Bloch-Landau constant) has a long history in function theory. For the above estimates on $C_4$ we refer the reader to [16] and [10] and [9]. (The reference [10] contains many references to the literature on the Schlicht Bloch-Landau constant.) The upper estimate for $C_3$ follow from the lower estimate on $C_4$ and inequality (3.5) below. From the upper estimate on $C_3$ we get a lower estimate on $C_2$ using (3.6). The lower estimate for $C_3$ and upper estimate on $C_2$ follow from the example in [3], (see Theorems 2 and 3) and the calculations in [9]. For an approach using a Hardy-type inequality with $\sigma_\Omega$ replacing the distance function, see [5].

**Theorem 3.1** ([3]) For any simply connected domain $\Omega \subset \mathbb{R}^2$, we have

$$\frac{1}{2\sigma_\Omega^2} \leq \sup_{z \in \Omega} E_z (\tau_\Omega) \leq \frac{7\zeta(3)}{8\sigma_\Omega^2}$$

and

$$\frac{2}{\sup_{z \in \Omega} E_z (\tau_\Omega)} \leq \lambda_\Omega \leq \frac{7\zeta(3)j_0^2}{8\sup_{z \in \Omega} E_z (\tau_\Omega)},$$

where $7\zeta(3)/8 = \sum_{n=0}^{\infty} (2n + 1)^{-3}$. 

3
References


