STRETCHING CONVEX DOMAINS AND THE HYPERBOLIC METRIC

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Abstract. It is shown that the log-convexity of the density of the hyperbolic metric in a convex planar domain leads to a pointwise comparison between the density of the hyperbolic metric in a convex domain D and that in a domain obtained by stretching D. Applications of this result are given, including estimates for the density of the hyperbolic metric in the domain interior to an ellipse and a lower bound for the density of the hyperbolic metric in a convex domain in terms of the density in a comparison strip. Connections are made with the convexity of related functions on convex regions in space.

1. Introduction

A real-valued function f defined on a region D in $\mathbb{R}^n$ is said to be convex if, whenever the line segment $[w_0, w_1]$ is contained in D,

$$f\big((1-t)w_0 + tw_1\big) \leq (1-t)f(w_0) + tf(w_1), \qquad 0 \leq t \leq 1.$$

Convexity of a region in Euclidean space is known to be equivalent to the convexity of each of several important functions associated with the region. For example, D is convex if and only if the first Dirichlet eigenfunction of the Laplacian on D is log-concave on D. This log-concavity property in a convex domain follows from the fact that the eigenfunction can be obtained as the limit of the distribution of the lifetime of Brownian motion which is log-concave whenever D is convex. Such results are derived directly from the log-concavity of the Gaussian density using multiple convolutions. For precise statements and proofs we refer the reader to [5] and [3], for example. By different techniques it is shown in [10] and [4] that the square root of the expected lifetime of Brownian motion (the square root of the torsion function) is concave whenever the domain D is convex. From this it follows that if a domain D is convex, then the expected lifetime is log-concave as a function of the starting point. We will comment again on these results at the end of this note, whose main focus is on the intrinsic hyperbolic geometry of convex planar domains.

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Kim and Minda [12] have unified results of Keogh [11], Harmelin [8], Yamashita [16], and Gustafsson [7], adding their own proofs and results, in the form of an elegant characterization of convexity. In the case of a hyperbolic planar domain D, that is a domain whose complement contains at least two points, we write σ_D for the density of the hyperbolic metric in D, normalized so that $\sigma_{\mathbb{D}}(z) = 1/(1-|z|^2)$ in the case of the unit disk $\mathbb{D} = \{z : |z| < 1\}$.

Theorem A. *The following are equivalent for a hyperbolic planar domain D:*

(a) *D is convex;*
(b) *$1/\sigma_D$ is concave on D;*
(c) *$\log \sigma_D$ is convex on D;*
(d) *$1/\sigma_D$ is superharmonic in D.*

Kim and Minda include the differential forms of (b) and (c) obtained by considering the sign of the Hessian of $1/\sigma_D$ and that of $\log \sigma_D$, respectively. The background to this theorem and related results are discussed carefully in [12]. Coefficients inequalities for normalized convex univalent functions, in particular the result that $|a_2| \leq 1$ if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is convex univalent in the unit disk, play a key role.

The log-convexity of the density of the hyperbolic metric in a convex domain leads to a comparison result for densities of the hyperbolic metric, as we now explain. We take a line L through the origin in the complex plane $\mathbb{C}$, and the orthogonal projection π onto L. By a *unidirectional dilation* we mean a map $T_r : \mathbb{C} \to \mathbb{C}$ defined by

$$T_r(z) = z + (r-1)\pi(z).$$

If L is the real axis then $T_r(z) = T_r(x+iy) = rx + iy$. If D is a disk then $T_r(D)$ is the interior of an ellipse. In general, r may be any positive real number, but when r is greater than 1 we refer to T_r as a *stretching*.

It is elementary that (for a fixed line L) $T_1(z) = z$, that $\pi\big(T_r(z)\big) = r\pi(z)$, that

$$T_{r_1} \circ T_{r_2} = T_{r_1 r_2} \quad \text{and that} \quad T_r(z_1 + z_2) = T_r(z_1) + T_r(z_2).$$

In particular, $T_r^{-1} = T_{1/r}$. Also, T_r maps line segments to line segments. It follows that a domain D is convex if and only if $T_r(D)$ is convex.

Our main result is a comparison between the density of the hyperbolic metric in a convex domain and that in the corresponding stretched domain.

Theorem 1. *Suppose that D is a convex domain in the plane and that T_r is a stretching. Then, for each z in D,*

$$\sigma_D(z) \geq \sigma_{T_r(D)}\big(T_r(z)\big) \tag{1.1}$$

$$\sigma_D(z) \leq r\,\sigma_{T_r(D)}\big(T_r(z)\big). \tag{1.2}$$

If equality holds in either (1.1) *or* (1.2) *at some point z in D, then equality holds in that inequality at every z in D and, moreover, D is a strip or a half-plane. If equality holds in* (1.1) *then stretching takes place parallel to the boundary of the strip or half-plane, while if equality holds in* (1.2) *then stretching takes place perpendicular to the boundary.*

In Section 2, we apply Theorem 1 to obtain estimates for the density of the hyperbolic metric in an ellipse, and for the hyperbolic distance between points lying along a line of symmetry of a convex domain. We also apply Theorem 1 to prove a pointwise lower estimate for the density of the hyperbolic metric in a convex domain in terms of the density in a comparison strip. This estimate is related to a result of Minda [14, Theorem 5]. We prove Theorem 1 in Section 3, the main ingredients being the log-convexity of the density of the hyperbolic metric in a convex domain and Ahlfors' version of the Schwarz Lemma. In Section 4, we comment on some results in higher dimensions.

2. Two applications of Theorem 1

As a simple application of Theorem 1, we derive bounds for the density of the hyperbolic metric in an ellipse. We also derive a pointwise estimate for the density of the hyperbolic metric in a convex domain in terms of the density in a comparison strip. This result, Theorem 2, may be viewed as a companion to Minda's estimate [14, Theorem 5].

2.1. The hyperbolic metric in an ellipse.

Proposition 1. *The density σ_E of the hyperbolic metric in the elliptical region*

$$E = \left\{ z = x + iy \colon \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \right\}, \quad a \geq b,$$

satisfies

$$\frac{1}{a}\,\frac{1}{1-(x^2/a^2+y^2/b^2)} \;\leq\; \sigma_E(z) \;\leq\; \frac{1}{b}\,\frac{1}{1-(x^2/a^2+y^2/b^2)}. \tag{2.1}$$

Proof. We consider the stretching

$$T(x,y) = \left(x, \frac{a}{b}\, y \right), \quad (x,y) \in \mathbb{R}\times\mathbb{R}.$$

To be precise, the line L in this case is the imaginary axis and $r = a/b \geq 1$. Then $T(E) = D = \{z \colon |z| < a\}$ and $\sigma_D(u,v) = a/(a^2-u^2-v^2)$, so that

$$\begin{aligned} \sigma_{T(E)}\big(T(z)\big) = \sigma_D\left(x, \frac{a}{b}\, y \right) &= \frac{a}{a^2 - x^2 - (a^2/b^2)y^2} \\ &= \frac{1}{a}\,\frac{1}{1-(x^2/a^2+y^2/b^2)}. \end{aligned}$$

The proposition now follows from (1.1) and (1.2). □

Integration of the second inequality in (2.1), with $y = 0$, leads to the bound

$$d(0,x;E) \le \frac{a}{2b}\log\left(\frac{1+x/a}{1-x/a}\right), \tag{2.2}$$

for the hyperbolic distance between the centre of the ellipse and a point x (with $x > 0$) on the major axis of the ellipse. For a point iy (with $y > 0$) on the minor axis of the ellipse, the first inequality in (2.1) leads to

$$d(0,iy;E) \ge \frac{b}{2a}\log\left(\frac{1+y/b}{1-y/b}\right). \tag{2.3}$$

The estimate (2.1) also provides a lower bound for $d(0,x;E)$ and an upper bound for $d(0,iy;E)$, but these are the same as the trivial bounds obtained by comparing distance in E with that in the circumscribed disk and in the inscribed disk, respectively.

Let us consider, more generally, a convex domain D that is symmetric about the real axis. The open line segment in which D meets the real axis is then a hyperbolic geodesic of D. For x_1 and x_2 on this geodesic and a stretching T_r of D,

$$\frac{1}{r}\,d\big(rx_1,rx_2;T_r(D)\big) \;\le\; d(x_1,x_2;D) \;\le\; d\big(rx_1,rx_2;T_r(D)\big). \tag{2.4}$$

In fact, making use of (1.1),

$$\begin{aligned} d(x_1,x_2;D) &= \int_{x_1}^{x_2} \sigma_D(x)\,dx \\ &\ge \int_{x_1}^{x_2} \sigma_{T_r(D)}(T_r(x))\,dx \\ &= \int_{x_1}^{x_2} \sigma_{T_r(D)}(rx)\,dx \\ &= \frac{1}{r}\int_{rx_1}^{rx_2} \sigma_{T_r(D)}(u)\,du \\ &= \frac{1}{r}\,d(rx_1,rx_2;T_r(D)). \end{aligned}$$

The second inequality follows in (2.4) from (1.2) in a similar fashion.

2.2. **A limit form of Theorem 1.** Theorem 2 is a limiting form of Theorem 1 and permits a comparison between the density of the hyperbolic metric at a point in a convex domain and the density in a comparison strip or half-plane: this comparison domain depends on the point. By a *chord* of a convex domain we mean an open Euclidean straight line segment that lies inside the domain but whose endpoints lie on the boundary of the domain: we allow either boundary point to be the point at infinity, so that the chord may be a half-line or, indeed, a line. Starting with such a chord C, we choose w so that it makes a right angle with the line segment C and set

$$\Omega(C) = \big\{z + tw\colon z \in C,\ -\infty < t < \infty\big\}. \tag{2.5}$$

The domain $\Omega(C)$ does not depend on the choice of w and is either a strip, a half-plane or the complex plane, depending on whether C is a bounded line segment, a half-line or a line.

Proposition 2. *Suppose that D is a convex domain and that C is a chord of D. Set $\Omega = \Omega(C)$ to be the domain constructed in* (2.5)*. Then, if $\Omega \neq \mathbb{C}$,*

$$\sigma_D(z) \geq \sigma_\Omega(z), \qquad \textit{for} \quad z \in C. \tag{2.6}$$

Proof. We may assume that the chord of the convex domain D lies along the imaginary axis. In this case, we write I for the open interval on the real line consisting of those y for which iy lies in D and note that the domain Ω constructed in (2.5) is

$$\Omega = \mathbb{R} \times I = \{z\colon \operatorname{Im} z \in I\}.$$

We suppose that $\{r_n\}_{n=1}^\infty$ is any real sequence with $r_n > 1$ for each n and $r_n \to \infty$ as $n \to \infty$. We set $\Omega_n = T_{r_n}(D)$, where stretching takes place in the horizontal direction, and show that the sequence of domains $\{\Omega_n\}_{n=1}^\infty$ converges, in the sense of Carathéodory and with respect to any point in the chord iI, to the domain Ω (see [6, Section 15.4] for a discussion of Carathéodory convergence of domains). In order to verify this convergence, we take any compact subset $[a, b]$ of I. There is a positive δ such that the rectangle

$$(-\delta, \delta) \times (a, b) = \{z\colon |\operatorname{Re} z| < \delta \text{ and } \operatorname{Im} z \in (a, b)\}$$

is contained in D. Then $(-r_n\delta, r_n\delta) \times (a, b)$ is contained in Ω_n, so that, for each positive R, the rectangle $(-R, R) \times (a, b)$ is contained in Ω_n for all sufficiently large n. This shows that the kernel of the sequence of domains $\{\Omega_n\}_{n=1}^\infty$ (with respect to any point of D on the imaginary axis) contains Ω – any compact subset of Ω will be contained in Ω_n for all sufficiently large n. On the other hand, suppose that a point w lies in the kernel of the sequence of domains $\{\Omega_n\}_{n=1}^\infty$. Then there is a positive δ such that the points $w_t = w + it$, $-\delta \leq t \leq \delta$, lie in Ω_n for all large n. It follows that

$$T_{r_n}^{-1}(w_t) = T_{1/r_n}(w_t) = \frac{1}{r_n}\operatorname{Re} w + i\operatorname{Im} w + it$$

lies in D for $|t| \leq \delta$ and for all large n. Letting $n \to \infty$, so that $r_n \to \infty$, we deduce that, for $-\delta \leq t \leq \delta$, $i\operatorname{Im} w + it$ is in the closure $\overline{D}$ of D. Since D is convex, its closure meets the imaginary axis in the closure of the chord iI. Hence $\operatorname{Im} w \in I$ and $w \in \Omega$. Thus Ω is the kernel of the sequence of domains $\{\Omega_n\}_{n=1}^\infty$. Since this is the case irrespective of the choice of the sequence $\{r_n\}_{n=1}^\infty$, each subsequence of $\{\Omega_n\}_{n=1}^\infty$ also has Ω as its kernel. By definition, $\Omega_n \to \Omega$ in the sense of Carathéodory.

To prove (2.6), we fix $y \in I$ and suppose that $I \neq \mathbb{R}$ so that $\Omega \neq \mathbb{C}$. We denote by g the conformal map of $\mathbb{D}$ onto D with $g(0) = iy$ and $g'(0) > 0$, we denote by f_n the conformal map of $\mathbb{D}$ onto Ω_n with $f_n(0) = iy$ and $f_n'(0) > 0$, and by f the conformal map of $\mathbb{D}$ onto Ω with $f(0) = iy$ and $f'(0) > 0$. Since $\{\Omega_n\}_{n=1}^\infty$ converges to Ω in the sense of Carathéodory and

$\Omega \neq \mathbb{C}$, $\{f_n\}_{n=1}^{\infty}$ converges to f uniformly on compact subsets of $\mathbb{D}$. In particular, $f_n'(0) \to f'(0)$ and so

$$\sigma_{\Omega_n}(iy) = \frac{1}{f_n'(0)} \to \frac{1}{f'(0)} = \sigma_\Omega(iy), \text{ as } n \to \infty.$$

Since, by (1.1),

$$\sigma_D(iy) \geq \sigma_{\Omega_n}(iy),$$

we conclude that $\sigma_D(iy) \geq \sigma_\Omega(iy)$. □

We write $\delta_D(z)$ for the distance from z to the boundary of the domain D. In the case when the comparison domain Ω in Proposition 2 is a strip, we may reformulate the result as follows.

Theorem 2. *Suppose that z is a point in a convex domain D and that P is a point on the boundary of D that is closest to z. Assume that the chord C of D through z and P is a bounded line segment and write $2M$ for its length. Then*

$$\sigma_D(z) \geq \left[\frac{4M}{\pi} \sin\left(\frac{\pi\delta_D(z)}{2M}\right)\right]^{-1} \tag{2.7}$$

The right hand side of (2.7) is the density of the hyperbolic metric in the strip $\Omega(C)$. Minda's lower bound [14, Theorem 5] has the same form as (2.7), the difference being that in Minda's result, M represents the inradius of D. That is, the supremum of the radius of all disks contained in the domain. While neither result implies the other, Theorem 2 performs better than [14, Theorem 5] when the length of the chord C is less than twice the inradius of the domain.

3. Proof of Theorem 1

By applying a suitable rotation, we may assume that stretching takes place in the direction of the real axis. The transformation T_r then takes the form

$$T_r(x,y) = (rx, y) \quad \text{for } (x,y) \in \mathbb{R} \times \mathbb{R}.$$

We consider the metric density ρ_1 defined on D by

$$\rho_1(z) = \sigma_{T_r(D)}(T_r(z)) = \sigma_{T_r(D)}(rx, y).$$

We compute the curvature of this metric density, that is $-(\Delta \log \rho_1)/\rho_1^2$. For $z \in D$,

$$(\Delta \log \rho_1)(z) = r^2 \left(\log \sigma_{T_r(D)}\right)_{11}(T_r(z)) + \left(\log \sigma_{T_r(D)}\right)_{22}(T_r(z)) \tag{3.1}$$

where $\left(\log \sigma_{T_r(D)}\right)_{11}$ and $\left(\log \sigma_{T_r(D)}\right)_{22}$ refer to the second derivatives of $\log \sigma_{T_r(D)}$ in the real and in the imaginary directions, respectively, in $T_r(D)$. Thus,

$$(\Delta \log \rho_1)(z) = \left(\Delta \log \sigma_{T_r(D)}\right)(T_r(z)) + (r^2 - 1)\left(\log \sigma_{T_r(D)}\right)_{11}(T_r(z)).$$

The hyperbolic metric in the domain $T_r(D)$ has curvature -4, and so it follows that the curvature of ρ_1 at a point z in D is given by

$$-\frac{(\Delta\log\rho_1)(z)}{\rho_1^2(z)} = -\frac{\big(\Delta\log\sigma_{T_r(D)}\big)\,(T_r(z))}{\sigma^2_{T_r(D)}(T_r(z))} - (r^2-1)\frac{\big(\log\sigma_{T_r(D)}\big)_{11}\,(T_r(z))}{\sigma^2_{T_r(D)}(T_r(z))}$$

$$= -4 - (r^2-1)\frac{\big(\log\sigma_{T_r(D)}\big)_{11}\,(T_r(z))}{\sigma^2_{T_r(D)}(T_r(z))}. \tag{3.2}$$

By Theorem A(c), $\big(\log\sigma_{T_r(D)}\big)_{11}$ is non-negative in $T_r(D)$. Since T_r is a stretching, $r>1$ and it then follows from (3.2) that

$$-(\Delta\log\rho_1)/\rho_1^2 \le -4, \quad z\in D.$$

Thus the curvature of ρ_1 in D is at most -4 and therefore, by Ahlfors' extension of Schwarz's Lemma [1], $\rho\le\sigma_D$ in D. This is the bound (1.1).

The upper bound (1.2) is deduced from the lower bound (1.1). Let S_r denote stretching by r in the direction of the imaginary axis, so that $S_r(x,y)=(x,ry)$ for $(x,y)\in\mathbb{R}\times\mathbb{R}$. Then

$$(S_rT_r)(z) = rz = (T_rS_r)(z).$$

Inequality (1.1) applied in the case of the domain $T_r(D)$ and the stretching S_r implies that, for $w\in T_r(D)$,

$$\sigma_{T_r(D)}(w) \ge \sigma_{S_rT_r(D)}\big(S_r(w)\big) = \sigma_{rD}\big(S_r(w)\big) \tag{3.3}$$

With $w=T_r(z)$, $z\in D$, the preceding inequality becomes

$$\sigma_{T_r(D)}\big(T_r(z)\big) \ge \sigma_{rD}\big(S_rT_r(z)\big) = \sigma_{rD}(rz).$$

But $r\sigma_{rD}(rz)=\sigma_D(z)$ since the map $f(z)=rz$ from D to rD is conformal, and (1.2) follows.

Finally, we deal with the case of equality. While Ahlfors did not discuss the case of equality in his paper [1] on the Schwarz Lemma, the matter was resolved sometime later by a number of authors, first by Heins [9, Theorem 7.1] and subsequently by Royden [15] and Minda [13]. It follows from these results that if equality holds at one point z in (1.1) then equality holds in (1.1) at every point in D. In that case, $\rho_1=\sigma_D$ and hence ρ_1 has constant curvature -4. On examining the expression (3.2) for this curvature, we conclude that $\big(\log\sigma_{T_r(D)}\big)_{11}$ is identically zero in $T_r(D)$. Hence, if $T_r(D)$ meets the line $\operatorname{Im} z = y$ then $\log\sigma_{T_r(D)}$ is linear on the intersection of the domain $T_r(D)$ with that horizontal line. Since the density of the hyperbolic metric tends to infinity at all finite boundary points, it follows that this intersection must be the entire line $\operatorname{Im} z = y$. We have shown that, whenever $T_r(D)$ meets a horizontal line then it contains that line and so is either a strip or a half-plane with boundary parallel to the real axis. Then D itself has the same property, since $D=T_{1/r}\big(T_r(D)\big)$. We note that stretching, in this case, takes place parallel to the boundary of the domain.

Suppose that equality holds in (1.2) for some z in D. Then equality holds in (3.3) for some $w \in T_r(D)$. By the case of equality in (1.1), $T_r(D)$ is either a strip or a half-plane and the stretching S_r is parallel to the boundary of $T_r(D)$. It follows that D itself is a strip or a half-plane and that the stretching T_r is perpendicular to its boundary.

4. A result of Keogh and its consequences

In [11], Keogh obtained a useful geometric characterization of non-convex planar domains. A crescent-shaped domain is one that can be written as $H \setminus \overline{J}$ where H and J are disks whose boundaries cut orthogonally. Keogh refers to the shorter of the two circular arcs that form the boundary of a crescent-shaped domain as the concave arc. He proves

Lemma A. *Suppose that D is a non-convex, planar domain. Then D contains a crescent-shaped domain G with the additional property that the mid-point of its concave arc is a boundary point of D.*

It is a trivial consequence of Keogh's result that if D is a non-convex planar domain then there is a closed rectangle R that lies inside D with the single exception of the mid-point of one if its sides, and this point is a boundary point of D.

Keogh used this geometric lemma to obtain a result on functions analytic in a disk and taking values in a non-convex domain. His lemma, however, has implications in higher dimensions as well. A region D in $\mathbb{R}^n$ is convex if and only if each of its two-dimensional cross-sections is convex. It is now straightforward to prove that certain standard functions associated with domains in $\mathbb{R}^n$ can be convex (or log-convex) only if the domain itself is convex. For example,

Theorem 3. *If the first Dirichlet eigenfunction of a domain D in $\mathbb{R}^n$ is log-concave, then D is convex.*

Proof. We use a technique from [12, Page 113]. Suppose that D in $\mathbb{R}^n$ is not convex. Then some two-dimensional cross-section is not convex either. After suitable rotations and translations, this becomes the cross-section with the $x_1\,x_2$-plane. By Keogh's lemma and after further translations and rotations, we may assume that

$$R = \{(x_1, x_2, 0, \ldots, 0) : 0 \le x_1 \le \epsilon, -\epsilon \le x_2 \le \epsilon\} \setminus \{0\}$$

is contained in D and that 0 is a boundary point of D. If the first Dirichlet eigenfunction ϕ for D were log-concave on D it would follow that, for $0 < x_1 \le \epsilon$,

$$2\log\phi(x_1, 0, 0, \ldots, 0) \ge \log\phi(x_1, \epsilon, 0, \ldots, 0) + \log\phi(x_1, -\epsilon, 0, \ldots, 0).$$

As $x_1 \to 0^+$, the right hand side tends to

$$\log\phi(0, \epsilon, 0, \ldots, 0) + \log\phi(0, -\epsilon, 0, \ldots, 0),$$

which is a finite number since both $(0, \epsilon, 0, \ldots, 0)$ and $(0, -\epsilon, 0, \ldots, 0)$ are points inside D and ϕ is positive in D. However, since ϕ vanishes on the boundary of D, we see that $\log \phi(x_1, 0, 0, \ldots, 0) \to -\infty$ as $x_1 \to 0^+$. This contradiction shows that ϕ cannot be log-concave in D if D is not convex. $\square$

The above argument works without any change to deduce the convexity of the domain from the log-concavity of several other classical potential theoretic functions such as the torsion (expected exit time) function of the domain. The much more difficult result that the first Dirichlet eigenfunction is log-concave if D is convex was first proved by Brascamp and Lieb in [5]. What follows from their result is that if τ_D denotes the exit time of the Brownian motion from D then, for all $t > 0$, the function

$$P_x\{\tau_D > t\} \tag{4.1}$$

is log-concave in D. This, and the fact that for convex domains

$$\lim_{t\to\infty} e^{\lambda t} P_x\{\tau_D > t\} = \phi(x) \int_D \phi(y)\, dy, \tag{4.2}$$

uniformly for $x \in D$ [2], proves the log-concavity of ϕ.

We denote the expectation of τ_D by $E_x(\tau_D)$. As mentioned in the introduction, if D is convex then the function $\sqrt{E_x(\tau_D)}$ is concave, [4, 10]. This then immediately gives that $E_x(\tau_D)$ is log-concave.

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