Symmetrization of Lévy processes and applications

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Abstract

It is shown that many of the classical generalized isoperimetric inequalities for the Laplacian when viewed in terms of Brownian motion extend to a wide class of Lévy processes. The results are derived from the multiple integral inequalities of Brascamp, Lieb and Luttinger but the probabilistic structure of the processes plays a crucial role in the proofs.

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1 Introduction

Let $D$ be a domain, an open connected set, in $\mathbb{R}^d$ of finite measure. We will denote by $D^*$ the open ball in $\mathbb{R}^d$ centered at the origin 0 with the same Lebesgue measure as $D$, and $|D|$ will denote the Lebesgue measure of $D$. There is a large class of quantities which are related to Brownian motion killed upon leaving $D$ that are maximized, or minimized, by the corresponding quantities for $D^*$. Such results often go by the name of generalized isoperimetric inequalities. They include the celebrated Rayleigh-Faber-Krahn inequality on the first eigenvalue of the Dirichlet Laplacian, inequalities for transition densities (heat kernels), Green functions, and electrostatic capacities (see [2], [14], [15], [16] and [17]).

Many of these isoperimetric inequalities can be beautifully formulated in terms of exit times of the Brownian motion $B_t$ from the domain $D$. For example, if $\tau_D$ is the first exit time of $B_t$ from $D$, then for all $x \in D$

$$P_x \{ \tau_D > 0 \} \leq P_0 \{ \tau_{D^*} > 0 \},$$

(1.1)

where 0 is the origin of $\mathbb{R}^d$. Inequality (1.1) contains not only the classical Rayleigh-Faber-Krahn inequality but inequalities for heat kernels and Green functions as well. This inequality is now classical and can be found in many places in the literature. For one of its first occurrences, using the Brascamp-Lieb-Luttinger multiple integrals techniques, please see Aizenman and Simon [1]. Similar inequalities can be obtained by these methods for domains of fixed inradius rather than fixed volume. For more on this, we refer the reader to [3] and [11]. Also, versions of some of these results hold for Brownian motion on spheres and hyperbolic spaces, see [7] and references therein.

Once these isoperimetric-type inequalities are formulated in terms of exit times of $B_t$, it is natural to enquire as to their extensions to other stochastic processes, and particularly for more general Lévy processes whose generators, as pseudo differential operators, are natural extensions to the Laplacian. Such extensions have been obtained in recent years for the so called “symmetric stable processes” in $\mathbb{R}^d$ and for more general processes obtained from subordination of Brownian motion. We refer the reader to [3], [4], [11], [20].

The purpose of this paper is to show that many of these results continue to hold for very general Lévy processes. At the heart of these extensions are the rearrangement inequalities of Brascamp, Lieb and Luttinger [6]. However, the probabilistic structure of Lévy processes enters in a very crucial way. Of particular importance for our method is the fact, derived from the
Lévy-Khintchine formula, that our processes are weak limits of sums of a compound Poisson process and a Gaussian process.

We begin with a general description of Lévy processes. A Lévy process $X_t$ in $\mathbb{R}^d$, is a stochastic process with independent and stationary increments which is “stochastically” continuous. That is, for all $0 < s < t < \infty$, $A \subset \mathbb{R}^d$,

$$P^x\{X_t - X_s \in A\} = P^0\{X_{t-s} \in A\},$$

for any given sequence of ordered times $0 < t_1 < t_2 < \cdots < t_m < \infty$, the random variables $X_{t_1} - X_0$, $X_{t_2} - X_{t_1}$, $\ldots$, $X_{t_m} - X_{t_{m-1}}$ are independent, and for all $\varepsilon > 0$,

$$\lim_{t \to s} P^x\{|X_t - X_s| > \varepsilon\} = 0.$$

The celebrated Lévy-Khintchine formula \cite{19} guarantees a triple $(\mathbb{E}, \mathbb{A}, \nu)$ such that the characteristic function of the process is given by

$$E^x[e^{i\xi \cdot X_t}] = e^{-t\Psi(\xi)} + i\xi \cdot x,$$

where

$$\Psi(\xi) = -i\langle b, \xi \rangle + \frac{1}{2} \langle \mathbb{A} \cdot \xi, \xi \rangle + \int_{\mathbb{R}^d} \left[ 1 + i\langle \xi, y \rangle \mathbb{I}_B - e^{i\xi \cdot y} \right] d\nu(y).$$

Here, $b \in \mathbb{R}^d$, $\mathbb{A}$ is a nonnegative $d \times d$ symmetric matrix, $\mathbb{I}_B$ is the indicator function of the ball $B$ centered at the origin of radius 1, and $\nu$ is a measure on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} \frac{|y|^2}{1 + |y|^2} d\nu(y) < \infty \text{ and } \nu(\{0\}) = 0.$$

The triple $(b, \mathbb{A}, \nu)$ is called the characteristics of the process and the measure $\nu$ is called the Lévy measure of the process. Conversely, given a triple with such properties we obtain a Lévy process. We will use the fact any Lévy process has a version with paths that are right continuous with left limits, so called “càdlàg” paths.

Next we recall that given a positive measurable function $f$, its symmetric decreasing rearrangement $f^*$ is the unique function satisfying

$$f^*(x) = f^*(y), \text{ if } |x| = |y|,$$

$$f^*(x) \leq f^*(y), \text{ if } |x| \geq |y|,$$

$$\lim_{|x| \to |y|} \frac{f^*(x)}{|x|} = f^*(y),$$

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and

\[ m \{ f > t \} = m \{ f^* > t \}, \quad (1.4) \]

for all \( t \geq 0 \). Following [13], an explicit expression for this function (under the assumption that \( f \) vanishes at infinity) is:

\[ f^*(x) = \int_0^\infty \chi_{\{ f^* > t \}}(x) \, dt. \]

For symmetrization purposes, in this paper we will only consider Lévy measures \( \nu \) that are absolutely continuous with respect to the Lebesgue measure \( m \). It may be that some of the results in this paper hold for more general Lévy processes but at this stage we are not able to go beyond the absolute continuity case. Let \( \phi \) be the density of \( \nu \) and \( \phi^* \) be its symmetric decreasing rearrangement. Since the function

\[ \psi(y) = 1 - \frac{|y|^2}{1 + |y|^2} \]

is a positive, decreasing and radially symmetric, that is, \( \psi^* = \psi \), it follows that (see Theorem 3.4 in [13])

\[ \int_{\mathbb{R}^d} \frac{|y|^2}{1 + |y|^2} \phi^* (y) \, dy \leq \int_{\mathbb{R}^d} \frac{|y|^2}{1 + |y|^2} \phi(y) \, dy < \infty. \quad (1.5) \]

Hence the measure \( \phi^*(y) \, dy \) satisfies (1.3).

We denote the \( d \times d \) identity matrix by \( I_d \) and the determinant of \( \mathbb{A} \) by \( \det \mathbb{A} \). Set \( \mathbb{A}^* = (\det \mathbb{A})^{1/d} I_d \) and define \( X_t^* \) to be the rotational invariant Lévy process in \( \mathbb{R}^d \) associated to the triple \((0, \mathbb{A}^*, \phi^*(y) \, dy)\). We will often refer to \( X_t^* \) as the symmetrization of \( X_t \).

Notice that

\[ E^x \left[ e^{i \xi \cdot X_t^*} \right] = e^{-t \Psi^*(\xi) + i \xi \cdot x}, \quad (1.6) \]

where

\[ \Psi^*(\xi) = \frac{1}{2} \langle \mathbb{A}^* : \xi, \xi \rangle + \int_{\mathbb{R}^d} \left[ 1 - e^{i \xi \cdot y} \right] \phi^*(y) \, dy. \]

The main result of this paper, from which many of the applications will follow, is

**Theorem 1.1.** Suppose \( X_t \) is a Lévy process with Lévy measure absolutely continuous with respect to the Lebesgue measure and let \( X_t^* \) be the symmetrization of \( X_t \). Let \( f_1, \ldots, f_m, \ m \geq 1 \), be nonnegative functions. Then for all \( z \in \mathbb{R}^d \),
\[
E^z \left[ \prod_{i=1}^{m} f_i(X_{t_i}) \right] \leq E^0 \left[ \prod_{i=1}^{m} f_i^*(X_{t_i}^*) \right],
\]
for all \(0 \leq t_1 \leq \ldots \leq t_m\).

As we shall see below, this result implies a generalization of (1.1) to Lévy processes whose Lévy measure is absolutely continuous with respect to the Lebesgue measure. In fact, we will obtain a more general result which applies to Schrödinger perturbations of Lévy semigroups. Let \(D \subset \mathbb{R}^d\) be a domain of finite measure, and consider

\[
\tau^X_X = \inf \{ t > 0 : X_t \notin D \},
\]
the first exit time of \(X_t\) from \(D\). We also have the corresponding quantity \(\tau^{X^*}_{X^*}\) for \(X^*_t\) in \(D^*\). As explained in §5, the following isoperimetric inequality is a consequence of Theorem 1.1.

**Theorem 1.2.** Let \(D\) be a domain in \(\mathbb{R}^d\) of finite measure, \(f\) be a nonnegative function, and \(V\) be nonnegative continuous functions defined on \(D\). If \(X_t\) is a Lévy process with Lévy measure absolutely continuous with respect to the Lebesgue measure and \(X^*_t\) is the symmetrization of \(X_t\). Then for all \(z \in \mathbb{R}^d\),

\[
E^z \left\{ f(X_t) \exp \left( - \int_0^t V(X_s) ds \right) ; \tau^X_X > t \right\} \leq E^0 \left\{ f^*(X^*_t) \exp \left( - \int_0^t V^*(X^*_s) ds \right) ; \tau^{X^*}_{X^*} > t \right\}.
\]

There is a similar result for capacities of Lévy symmetric processes. Let \(C_X(A)\) be the capacity, associated to \(X_t\), of the set \(A\). If \(X_t\) is a Lévy process with Lévy measure absolutely continuous with respect to the Lebesgue measure and \(X^*_t\) is the symmetrization of \(X_t\). Then for all \(z \in \mathbb{R}^d\),

\[
P( X_t \in A ) = P( X_t \in -A ),
\]
for all \(t > 0\). Then T. Watanabe [22] proved that,

\[
C_X(A) \geq C_{X^*}(A^*). \tag{1.8}
\]

This inequality can be obtained, from the methods used on this paper, only in the case that \(X_t\) is an isotropic unimodal Lévy process, see for example [12].

As explained in [3] and [11], if we additionally assume that the process \(X_t\) is isotropic unimodal, Theorem 1.1 is an immediate consequence of Theorem
1.3. Recall that, if $X_t$ is isotropic unimodal, then the transition densities $p^X(t,x,y)$ are of the form

$$p^X(t,x,y) = q(t, |x - y|),$$

where $q$ is a function such that

$$q(t, r_1) \leq q(t, r_2),$$

for all $r_1 \geq r_2$ and all $t > 0$. Thus for such Lévy processes

$$[p^X(t,x,y)]^* = p^X(t,x,y),$$

and $X_t = X_t^*$. This class of Lévy processes includes the Brownian motion, rotational invariant symmetric $\alpha$-stable processes, relativistic stable processes and any other subordinations of the Brownian motion. Notice that in our more general setting we cannot even ensure that

$$[p^X(t,x,y)]^*$$

is the transition density of a Lévy processes.

Our symmetrization results are based on the following now classical rearrangement inequality of Brascamp, Lieb and Luttinger [6].

**Theorem 1.3.** Let $f_1, \ldots, f_m$ be nonnegative functions in $\mathbb{R}^d$ and let $f_1^*, \ldots, f_m^*$ be their symmetric decreasing rearrangements. Then

$$\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{j=1}^m f_j \left( \sum_{i=1}^k b_{ji} x_i \right) \, dx_1 \cdots dx_k \leq \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{j=1}^m f_j^* \left( \sum_{i=1}^k b_{ji} x_i \right) \, dx_1 \cdots dx_k,$$

for all positive integers $k, m$, and any $m \times k$ matrix $B = [b_{ji}]$.

The rest of the paper is organized as follows. In §2 we will prove Theorem 1.1 for Compound Poisson processes. We will consider the case of Gaussian Lévy processes in §3. Theorem 1.1 is proved in §4, using a weak approximation of $X_t$ and $X_t^*$ by Lévy processes of the form $G_t + C_t$, where $G_t$ is a non-degenerated Gaussian process and $C_t$ is an independent compound processes. We will then show some of the applications in §5.
2 Symmetrization of compound Poisson processes

In this section we prove Theorem 1.1 for compound Poisson processes. If $C_t$ is a compound Poisson process, then its characteristic function is given by

$$E \left( e^{i\xi C_t} \right) = e^{-t \Psi_C(\xi)}, \quad (2.1)$$

where

$$\Psi_C(\xi) = c \int_{\mathbb{R}^d} \left[ 1 - e^{i\xi \cdot y} \right] \phi(y) \, dy,$$

and $\phi$ is a probability density. It is well known, see Theorem 4.3 [19], that there exist a Poisson process $N_t$ with parameter $c > 0$, and a sequence of i.i.d. random variables $\{X_n\}_{n=1}^\infty$ such that

1. $\{N_t\}_{t>0}$ and $\{X_n\}_{n=1}^\infty$ are independent,
2. $\phi(y)$ is the density of the distribution of $X_i$, $i \geq 1$,
3. $C_t = S_{N_t}$, where $S_n = X_1 + \ldots + X_n$.

Hence if $f$ is a nonnegative Borel function, then

$$P^x [ f(C_t) ] = P^x [ f(S_{N_t}) ] = \sum_{n=0}^{\infty} P[N_t = n] \, P[f(x + S_n)]. \quad (2.2)$$

Let $\phi^*$ be the symmetric decreasing rearrangement of $\phi$. Since

$$\int_{\mathbb{R}^d} \phi^*(y) \, dy = \int_{\mathbb{R}^d} \phi(y) \, dy = 1,$$

we can consider a new sequence of i.i.d. random variables $\{X^*_n\}_{n=1}^\infty$ independent of $N_t$ such that $\phi^*(y)$ is the density of $X^*_n$. Define $S^*_n = X^*_1 + \ldots + X^*_n$ to be the corresponding random walk and $C^*_t$ the compound Poisson process given by

$$C^*_t = S^*_{N_t}.$$

The next result is a version of Theorem 1.1 for random walks. We label it as a “Theorem” because it may be of some independent interest.
**Theorem 2.1.** Let $f_1, \ldots, f_m$ nonnegative functions, and $k_1 \leq \ldots \leq k_m$ nonnegative integers. Then for all $x_0$,

$$E \left[ \prod_{i=1}^{m} f_i(x_0 + S_{k_i}) \right] \leq E \left[ \prod_{i=1}^{m} f_i^*(S_{k_i}^*) \right]. \quad (2.3)$$

**Proof.** Given that $X_1, \ldots, X_{k_m}$ are i.i.d we can apply Theorem 1.3 to obtain that

$$E \left[ \prod_{i=1}^{m} f_i(x_0 + X_1 + \ldots + X_{k_i}) \right] = E \left[ \prod_{i=1}^{m} f_i(x_0 + X_1 + \ldots + X_{k_i}) \right]$$

$$= \int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} \left[ \prod_{i=1}^{m} f_i \left( \sum_{j=0}^{k_i} x_j \right) \right] \prod_{i=1}^{k_m} \phi(x_i) \ dx_1 \ldots dx_m$$

$$\leq \int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} \left[ \prod_{i=1}^{m} f_i^* \left( \sum_{j=1}^{k_i} x_j \right) \right] \prod_{i=1}^{k_m} \phi^*(x_i) \ dx_1 \ldots dx_m$$

$$= E \left[ \prod_{i=1}^{m} f_i^*(S_{k_i}^*) \right]. \quad (2.4)$$

We can now obtain the desired result for $C_t$. Let $f_1, \ldots, f_m$ be nonnegative functions. Since $N_t$ is independent of $S_k$ and $S_k^*$, we can combine (2.2) and Theorem 2.1 to obtain

$$E^x \left[ \prod_{i=1}^{m} f_i(S_{N_{t_i}}) \right] = \sum_{k_1 \leq k_2 \leq \ldots \leq k_m} P \left[ N_{t_1} = k_1, \ldots, N_{t_m} = k_m \right] E \left[ \prod_{i=1}^{m} f_i(x + S_{k_i}) \right]$$

$$\leq \sum_{k_1 \leq k_2 \leq \ldots \leq k_m} P \left[ N_{t_1} = k_1, \ldots, N_{t_m} = k_m \right] E \left[ \prod_{i=1}^{m} f_i^*(S_{k_i}^*) \right]$$

$$= E^0 \left[ \prod_{i=1}^{m} f_i^*(S_{N_{t_i}}^*) \right]. \quad (2.5)$$

Thus

$$E^x \left[ \prod_{i=1}^{m} f_i(C_t) \right] \leq E^0 \left[ \prod_{i=1}^{m} f_i^*(C_t^*) \right], \quad (2.6)$$

which is the desired result for $C_t$. 7
Notice that the distribution $\mu_t$ of $C_t$ is not absolutely continuous with respect to Lebesgue measure. However, if $C_0 = x_0$ we have the following representation

$$
\mu_t = P[N_t = 0] \delta_{x_0} + \sum_{k=1}^{\infty} P[N_t = k] \mu_k(x), \quad (2.7)
$$

with $\mu_k$ the distribution of $S_k$. That is,

$$
E^{x_0}[f(S_k)] = \int_{\mathbb{R}^d} f(x_0 + y) d\mu_k(y)
= \int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} f \left( \sum_{j=0}^{k} x_j \right) \prod_{i=1}^{k} \phi(x_i) dx_1 \ldots dx_k.
$$

The symmetrization results of this section follow from the fact that $C_t$ can be written in terms of sums of independent random variables.

## 3 Symmetrization of Gaussian processes

Let $G_t$ be a non-degenerate Gaussian process. Then there exist $b \in \mathbb{R}^d$ and a strictly positive definite symmetric $d \times d$ matrix $A$ such that the density of $G_t$ is given by

$$
f_{A,b}(t, x) = \frac{1}{\sqrt{\det A}} \exp \left[ -\frac{1}{2t} \langle x - tb, A^{-1}(x - tb) \rangle \right],
$$

for all $x \in \mathbb{R}^d$ and all $t > 0$.

Let us first assume that $b = 0$. Let $u > 0$, then

$$
\begin{align*}
\{ x \in \mathbb{R}^d : f_{A,0}(t, x) > u \} & = \{ x \in \mathbb{R}^d : \langle x, A^{-1} \cdot x \rangle < t \ln \left( \frac{1}{(2t\pi)^{d/2} u^2 \det A} \right) \} \\
& = \{ x \in \mathbb{R}^d : \langle A^{-1/2} \cdot x, A^{-1/2} \cdot x \rangle < t \ln \left( \frac{1}{(2t\pi)^{d/2} u^2 \det A} \right) \}.
\end{align*}
$$

A change of variables implies that

$$
m \left\{ x \in \mathbb{R}^d : f_{A,0}(t, x) > u \right\} = \frac{1}{\det A^{1/2}} m \left\{ B(r_{A,d,u,t}) \right\},
$$
where

\[ r_{\mathbb{A},d,u,t} = t \ln \left[ \frac{1}{(2t\pi)^d u^2 \det \mathbb{A}} \right] . \]

Consider the diagonal matrix

\[ \mathbb{A}^* = (\det \mathbb{A})^{\frac{1}{d}} \mathbb{I}_d. \]

Then

\[ m\left\{ x \in \mathbb{R}^d : f_{\mathbb{A},0}(t,x) > u \right\} = m\left\{ x \in \mathbb{R}^d : f_{\mathbb{A}^*,0}(t,x) > u \right\}, \]

for all \( u > 0 \). Given that \( f_{\mathbb{A}^*,0}(t,x) \) is rotational invariant and radially decreasing, we conclude that

\[ [ f_{h,b}(t,x) ]^* = [ f_{h,0}(t,x - tb) ]^* = f_{h^*,0}(t,x). \] (3.1)

If \( G_t \) is a degenerate Gaussian process, then

\[ E \left( e^{i \xi \cdot G_t} \right) = \exp \left( itb \cdot \xi - i \frac{t}{2} \langle \mathbb{A} \cdot \xi, \xi \rangle \right), \] (3.2)

where \( \mathbb{A} \) is a positive definite \( d \times d \) matrix such that \( \det \mathbb{A} = 0 \).

Let \( \{ v_1, \ldots, v_d \} \) the orthonormal eigenvectors of \( \mathbb{A} \) with eigenvalues \( \lambda_1, \ldots, \lambda_d \). We can assume that \( \{ \lambda_1, \ldots, \lambda_k \}, 1 \leq k < d, \) are the non-zero eigenvalues of \( \mathbb{A} \). Let \( W \) be the subspace spanned by \( v_1, \ldots, v_k \). Then \( G_t \) can be identified with a non degenerated Gaussian process in the lower dimension space \( W \) and

\[ P^z \left[ G_t \in D \right] = P^z \left[ G_t \in P_W(D) \right], \]

where \( P_W(D) \) is the projection of \( D \) on the space \( W \).

Define \( \mathbb{A}^* \) to be the projection of the symmetric positive defined matrix with eigenvectors \( v_1, \ldots, v_d \) such that

\[ \mathbb{A}^* v_i = 0, \ k < i \leq d, \]

and

\[ \mathbb{A}^* v_i = \lambda v_i, \ 1 \leq i \leq k, \]

where

\[ \lambda = (\lambda_1 \cdots \lambda_k)^{1/k}. \]

The arguments of this section imply that

\[ P^z \left[ G_t \in D \right] = P^z \left[ G_t \in P_W(D) \right] = P^0 \left[ G_t^* \in D_W^* \right]. \]

where \( D_W^* \) is the ball in \( W \), centered at the origin, with the same \( k \)-dimension measure than \( P_W(D) \). Hence the corresponding symmetrization for this processes should be done in lower dimensions.
4 Symmetrization of Lévy processes

We will now consider general Lévy processes whose Lévy measures are absolutely continuous with respect to the Lebesgue measure. Recall that under our assumptions
\[ E^x \left[ e^{i \xi \cdot X_t} \right] = e^{-t \Psi(\xi) + i \xi \cdot x}, \]
where
\[ \Psi(\xi) = -i \langle b, \xi \rangle + \frac{1}{2} \langle A \xi, \xi \rangle + \int_{\mathbb{R}^d} \left[ 1 + i \langle \xi, y \rangle \mathbb{1}_B - e^{i \xi \cdot y} \right] \phi(y) \, dy, \]
\( B \) is the unit ball centered at the origin and \( \phi \) is such that
\[ \int_{\mathbb{R}^d} \frac{|y|^2}{1 + |y|^2} \phi(y) \, dy < \infty. \quad (4.1) \]
Consider the sequence
\[ \phi_n(y) = \phi(y) \mathbb{1}_{\{ t \in \mathbb{R} : \frac{1}{n} < t \}} (|y|), \]
and let \( \phi_n^*(y) \) be its symmetric decreasing rearrangement. Thanks to (4.1),
\[ c_n = \int_{\mathbb{R}^d} \phi_n(y) \, dy < \infty, \]
and
\[ \int_B |y_i| \phi_n(y) \, dy < \infty, \quad 1 \leq i \leq d, \]
where again \( B \) is the unit ball.
Consider \( C_{n,t} \) a compound Poisson process with characteristic function
\[ E \left( e^{i \xi \cdot C_{n,t}} \right) = e^{-t \Psi_{C,n}(\xi)}, \quad (4.2) \]
where
\[ \Psi_{C,n}(\xi) = c_n \int_{\mathbb{R}^d} \left[ 1 - e^{i \xi \cdot y} \right] \frac{\phi_n(y)}{c_n} \, dy. \]
Given that all the eigenvalues of \( A \) are nonnegative, if \( \{ \epsilon_n \}_{n=1}^\infty \) is a sequence of positive numbers converging to zero, then \( A_n = A + \epsilon_n I_d \) is a sequence of nonnegative non-singular matrices. Let \( G_{n,t} \) be a Gaussian process starting at \( x \), independent of \( C_{n,t} \), and associated with the matrix \( A_n \) and the vector
\[ b_n = b - \int_B y \, \phi_n(y) \, dy. \]
Set
\[ X_{n,t} = C_{n,t} + G_{n,t}. \]

Since \( C_{n,t} \) and \( G_{n,t} \) are independent,
\[ E^x \left[ e^{iξ \cdot X_{n,t}} \right] = e^{-tΨ_n(ξ) + iξ x}, \]
where
\[ Ψ_n(ξ) = -i\langle b_n, ξ \rangle + \frac{1}{2} \langle h_n \cdot ξ, ξ \rangle + \int_{\mathbb{R}^d} \left[ 1 - e^{iξ \cdot y} \right] φ_n(y) dy \quad (4.3) \]
\[ = -i\langle b, ξ \rangle + \frac{1}{2} \langle h_n \cdot ξ, ξ \rangle + \int_{\mathbb{R}^d} \left[ 1 + i⟨ξ, y⟩I_B - e^{iξ \cdot y} \right] φ_n(y) dy. \]

Let \( S_{n,k} = X^n_{1} + \ldots + X^n_{k} \) be the random walk associated to \( C_{n,t} \). If \( f_1, \ldots, f_m \) are positive Borel functions and \( t_1 \leq \ldots \leq t_m \), then
\[ E^x \left[ \prod_{i=1}^{m} f_i(X_{n,t_i}) \right] = E^x \left[ \prod_{i=1}^{m} f_i(C_{n,t_i} + G_{n,t_i}) \right] \]
\[ = \sum_{k_1 \leq k_2 \leq \ldots \leq k_m} P[N_{t_1} = k_1, \ldots, N_{t_m} = k_m] E^x \left[ \prod_{i=1}^{m} f_i(S_{n,k_i} + G_{n,t_i}) \right]. \]

Now Theorem 1.3 and equality (3.1) imply that
\[ E^x \left[ \prod_{i=1}^{m} f_i \left( G_{n,t_i} + \sum_{j=1}^{k_i} X^n_j \right) \right] = \int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} \prod_{i=1}^{m} f_i \left( \sum_{j=0}^{k_i} x_j \right) f_{h_n,b_n}(t, x_0 - x) \prod_{j=1}^{k_m} φ(x_j) dx_0 \ldots dx_{k_m} \]
\[ \leq \int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} \prod_{i=1}^{m} f_i^* \left( \sum_{j=0}^{k_i} x_j \right) f_{h_n,0}(t, x_0) \prod_{j=1}^{k_m} φ^*(x_j) dx_0 \ldots dx_{k_m} \]
\[ = \int_{\mathbb{R}^d} \ldots \int_{\mathbb{R}^d} \prod_{i=1}^{m} f_i^* \left( \sum_{j=0}^{k_i} x_j \right) f_{h_n,0}(t, x_0) \prod_{j=1}^{k_m} φ^*(x_j) dx_0 \ldots dx_{k_m} \]
\[ = E^x \left[ \prod_{i=1}^{m} f_i^* \left( G^*_{n,t_i} + S^*_{n,k} \right) \right]. \]

From this we conclude that
\[
E^x \left[ \prod_{i=1}^{m} f_i(X_{n,t_i}) \right] \leq E^0 \left[ \prod_{i=1}^{m} f^*_i(X^*_{n,t_i}) \right]. \quad (4.6)
\]

In the case that \( f_1, \ldots, f_m \) are continuous functions, Theorem 1.1 will be a consequence of (4.6) and the following result.

**Theorem 4.1.** Let \( f_1, \ldots, f_k \) be nonnegative bounded continuous functions, and \( 0 < t_1 < \ldots < t_m \). Then for all \( x \in \mathbb{R}^d \),

\[
\lim_{n \to \infty} E^x \left[ \prod_{i=1}^{k} f_i(X_{n,t_i}) \right] = E^x \left[ \prod_{i=1}^{k} f_i(X_{t_i}) \right], \quad (4.7)
\]

and

\[
\lim_{n \to \infty} E^x \left[ \prod_{i=1}^{k} f_i(X^*_{n,t_i}) \right] = E^x \left[ \prod_{i=1}^{k} f_i(X^*_{t_i}) \right].
\]

**Proof.** We recall once again that the symmetric decreasing rearrangement \( f^* \) of \( f \) is the function satisfying

- \( f^*(x) = f^*(y) \), if \( |x| = |y| \),
- \( f^*(x) \leq f^*(y) \), if \( |x| \geq |y| \),
- \( \lim_{|x| \to |y|^+} f^*(x) = f^*(y) \),

and

\[
m \{ f > t \} = m \{ f^* > t \}, \quad (4.8)
\]

for all \( t \geq 0 \). Given that \( f^* \) is right-continuous as a function of the radius, we have

\[
m \{ f^* > t \} = m \{ B[0, r(f^*, t)] \},
\]

where

\[
r(f^*, t) = \sup \{ r > 0 : f^*(r) > t \}.
\]

Notice that, for all \( t \geq 0, r(f^*, f^*(t)) = t \).

Since

\[
0 \leq \phi_n \leq \phi_{n+1} \leq \phi, \ \text{and} \ \lim_{n \to \infty} \phi_n = \phi,
\]

we have

\[
\lim_{n \to \infty} m \{ \phi_n > t \} = m \{ \phi > t \}. \quad (4.9)
\]
Thus
\[ \lim_{n \to \infty} m \{ \phi_n^* > t \} = m \{ \phi^* > t \}. \]

and
\[ m \{ B(0, r(\phi_{n+1}^*, t)) \} = m \{ \phi_{n+1}^* > t \} \]
\[ \geq m \{ \phi_n^* > t \} \]
\[ = m \{ B(0, r(\phi_n^*, t)) \}. \]

That is,
\[ r(\phi_{n+1}^*, t) \geq r(\phi_n^*, t), \quad (4.10) \]

and
\[ \lim_{n \to \infty} r(\phi_n^*, t) = r(\phi^*, t). \quad (4.11) \]

Let us assume that there exists \( x \in \mathbb{R}^d \) such that \( \phi_n^*(|x|) > \phi_{n+1}^*(|x|) \). Since \( \phi_n^* \) is decreasing and right-continuous as a function of the radius, we have
\[ r(\phi_n^*, \phi_n^*(|x|)) < r(\phi_{n+1}^*, \phi_{n+1}^*(|x|)). \]

On the other hand
\[ |x| = r(\phi_n^*, \phi_n^*(|x|)) \]
\[ \leq r(\phi_n^*, \phi_{n+1}^*(|x|)) \]
\[ \leq r(\phi_{n+1}^*, \phi_{n+1}^*(|x|)) \]
\[ = |x|, \]

which is a contradiction. Thus
\[ 0 \leq \phi_n^* \leq \phi_{n+1}^* \leq \phi^*, \quad (4.12) \]

see (vi) on page 81 of [13]. In a similar way we can prove that
\[ \lim_{n \to \infty} \phi_n^* = \phi^*. \quad (4.13) \]

Notice that for all \( \xi \in \mathbb{R}^d \),
\[ \lim_{n \to \infty} \langle A_n \cdot \xi, \xi \rangle = \langle A \cdot \xi, \xi \rangle. \]

Given that there exists \( C \in \mathbb{R}^+ \) such that,
\[ \left| 1 + i\langle \xi, y \rangle - e^{i\xi \cdot y} \right| \phi_n(y) \leq C |\xi|^2 |y|^2 \phi(y) < \infty, \quad (4.14) \]
for all \( y \in B \), and
\[
1 - e^{i\xi \cdot y} \phi_n(y) \leq 2 \phi(y) < \infty,
\]
for all \( y \in \mathbb{R}^d \setminus B \), it follows from the dominated convergence theorem that
\[
\lim_{n \to \infty} \Psi_n(\xi) = \lim_{n \to \infty} \left( -i \langle b, \xi \rangle + \frac{1}{2} \langle A_n \cdot \xi, \xi \rangle + \int_{\mathbb{R}^d} \left[ 1 + i \langle \xi, y \rangle I_B - e^{i\xi \cdot y} \right] \phi_n(y) \, dy \right).
\]
We conclude that
\[
\lim_{n \to \infty} E^x \left[ e^{i\xi \cdot X_{n,t}} \right] = E^x \left[ e^{i\xi \cdot X_t} \right]. \tag{4.17}
\]
Using the fact that
\[
\lim_{n \to \infty} \det A_n = \det A,
\]
we similarly prove that
\[
\lim_{n \to \infty} E^x \left[ e^{i\xi \cdot X_{n,t}^*} \right] = E^x \left[ e^{i\xi \cdot X_t^*} \right]. \tag{4.18}
\]
Besides, if \( \xi_1, \ldots, \xi_m \in \mathbb{R}^d \), then
\[
\sum_{j=1}^m \xi_j \cdot X_{n,t_j} = (\xi_1 + \ldots + \xi_m) \cdot X_{n,t_1}
\]
\[
+ \sum_{j=2}^m (\xi_m + \ldots + \xi_j) \cdot (X_{n,t_j} - X_{n,t_{j-1}}). \tag{4.19}
\]
Since \( t_1 < \ldots < t_m \) we have that
\[
E^x \left\{ \exp \left[ i \sum_{j=1}^m \xi_j \cdot X_{n,t_j} \right] \right\} = E^x \left\{ \exp \left[ i (\xi_1 + \ldots + \xi_m) \cdot X_{n,t_1} \right] \right\} \tag{4.20}
\]
\[
\times \prod_{j=2}^m E^0 \left\{ \exp \left[ i(\xi_m + \ldots + \xi_j) \cdot (X_{n,t_j} - t_{j-1}) \right] \right\}.
\]
The desired result immediately follows from (4.17), (4.18) and the fact that our characteristic functions are continuous at 0. This last observation follows from the Lévy-Khintchine formula. \( \square \)
Let $f$ be a nonnegative continuous bounded function. Let us assume that $f^*(x)$ is not continuous at a point $x_0$. Since $f^*$ is radially symmetric decreasing and right-continuous as a function of the radius, $x_0 \neq 0$ and there exist $t_1$ such that

$$m \{ f^* > s \} = m \{ f^* > f^*(x_0) \} \neq 0,$$

for all $s \in [f^*(x_0), t_1)$. However, the continuity of $f$ implies that the set

$$\{ f^*(x_0) < f < s \}$$

is nonempty and open. Therefore

$$m \{ f > s \} > m \{ f > f^*(x_0) \},$$

which is a contradiction. We conclude that $f^*$ is also continuous.

Thus, combining (4.6) and Theorem (4.1), we have that for all nonnegative bounded continuous functions $f_1, \ldots, f_m$,

$$E^x \left[ \prod_{i=1}^m f_i(X_{n,t_i}) \right] \leq E^0 \left[ \prod_{i=1}^m f_i^*(X_{n,t_i}^*) \right].$$

Let $F_1, \ldots, F_m$ be closed sets. As explained on page 8 of [5], given $1 \leq i \leq m$, there exists a sequence $\{ f_i^n \}_{n=1}^\infty$ of continuous functions such that $f_i^n \leq 1$, $n \geq 1$, and

$$\lim_{n \to \infty} f_i^n = \mathbb{1}_{F_i}.$$

Since

$$E^x \left[ \prod_{i=1}^k f^n_i(X_{t_i}) \right] \leq E^x \left[ \prod_{i=1}^k (f_i^n)^*(X_{t_i}^*) \right],$$

the dominated convergence theorem implies Theorem 1.1 in the case that $f_1, \ldots, f_k$ are indicator functions of closed sets. The general case is proven using standard approximation methods, and the arguments used to prove (4.12) and (4.13).

5 Some Applications

In this section we give several applications of Theorem 1.1, we begin with the proof of Theorem 1.2. Recall that

$$\tau^X_D = \inf \{ t > 0 : X_t \notin D \}$$

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is the first exit time of $X_t$ from a domain $D$. Let $D_k$ be a sequence of bounded domains with smooth boundaries such that $D_k \subset D_{k+1}$, and $\bigcup_{k=1}^{\infty} D_k = D$. Since any Lévy process has a version with right continuous paths, we have

$$E^0 \left\{ f(X_t) \exp \left( - \int_0^t V(X_s) ds \right) ; \tau_{\overline{D}}^X > t \right\}$$

(5.1)

$$= E^0 \left\{ f(X_t) \exp \left( - \int_0^t V(X_s) ds \right) ; X_s \in D, \forall s \in [0, t] \right\}
= \lim_{m \to \infty} \lim_{k \to \infty} E^{2^0} \left\{ f(X_t) \exp \left( - \frac{t}{m} \sum_{i=1}^m V(X_{it \over m}) \right) ; X_{it \over m} \in D_k, i = 1, \ldots, m \right\}
= \lim_{m \to \infty} \lim_{k \to \infty} E^{2^0} \left\{ f(X_t) \prod_{i=1}^m \exp \left( - \frac{t}{m} V(X_{it \over m}) \right) I_{D_k} \left( X_{it \over m} \right) \right\}.
$$

Since

$$\left[ \exp \left( -sV(x) \right) \right]^* = \exp \left( -sV^*(x) \right),$$

for all $s > 0$ and all $x \in \mathbb{R}^d$. Theorem 1.1 implies that

$$E^{2^0} \left\{ f(X_t) \prod_{i=1}^m \exp \left( - \frac{t}{m} V^*(X_{it \over m}) \right) I_{D_k^*} \left( X_{it \over m} \right) \right\}
\leq E^0 \left\{ f^*(X_t^*) \prod_{i=1}^m \exp \left( - \frac{t}{m} V^*(X_{it \over m}) \right) I_{D_k^*} \left( X_{it \over m} \right) \right\}.
$$

Hence we have the following

$$E^2 \left\{ f(X_t) \exp \left( - \int_0^t V(X_s) ds \right) ; \tau_{\overline{D}}^X > t \right\}$$

(5.2)

$$\leq E^0 \left\{ f^*(X_t^*) \exp \left( - \int_0^t V^*(X_s^*) ds \right) ; \tau_{\overline{D^*}}^X > t \right\},
$$

which is Theorem (1.2). Taking $V = 0$ and $f = 1$, gives

$$P^2 \left\{ \tau_{\overline{D}}^X > t \right\} \leq P^0 \left\{ \tau_{\overline{D^*}}^X > t \right\},$$

(5.3)

which is a generalization of inequality (1.1). Integrating this inequality we obtain

**Corollary 5.1.** If $\psi$ is a nonnegative increasing function, then

$$E^2 \left[ \psi \left( \tau_{\overline{D}}^X \right) \right] \leq E^0 \left[ \psi \left( \tau_{\overline{D^*}}^X \right) \right],
$$

(5.4)
for all \( z \in D \). In particular
\[
E^z \left[ \left( \tau^X_D \right)^p \right] \leq E^0 \left[ \left( \tau^X_{D^*} \right)^p \right],
\]
for all \( 0 < p < \infty \).

Our results imply many isoperimetric inequalities for the potentials and the eigenvalues of Schrödinger operators of the form
\[
H^X_{D,V} = H^X_D + V,
\]
where \( H^X_D \) is the pseudo differential operator associated to \( X_t \) with Dirichlet Boundary conditions on \( D \). For the convenience of the reader we will give a brief description of the operators and semigroups associated to Lévy processes.

For purposes of our formulae below we define the Fourier transform of an \( L(\mathbb{R}^d) \) function as
\[
\hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) \, dx,
\]
with
\[
f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \hat{f}(\xi) \, d\xi.
\]
We define the semigroup associated to the Lévy process \( X_t \) by
\[
T_t f(x) = E^x[ f(X_t) ]
\]
\[
= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i(x+\xi) \cdot \xi} \hat{f}(\xi) \, d\xi
\]
\[
= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix \cdot \xi} E^0 \left[ e^{iX_t \cdot \xi} \right] \hat{f}(\xi) \, d\xi
\]
\[
= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{-t\Psi(\xi)} \hat{f}(\xi) \, d\xi.
\]
This semigroup takes \( C_0(\mathbb{R}^d) \) into itself. That is, it is a Feller semigroup. From this we see that, at least formally for \( f \in \mathcal{S}(\mathbb{R}^d) \), the infinitesimal generator is
\[
H^X f(x) = \left. \frac{\partial T_t f(x)}{\partial t} \right|_{t=0} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \Psi(\xi) \hat{f}(\xi) \, d\xi.
\]
Then the Lévy-Khintchine formula implies that the operator associated to $X_t$ is given by:

$$H^X f(x) = \sum_{j=1}^{d} b_j \partial_j f(x) - \frac{1}{2} \sum_{j,k=1}^{d} a_{j,k} \partial_j \partial_k f(x)$$

$$+ \int_{\mathbb{R}^d} \left[ f(x + y) - f(x) - y \cdot \nabla f(x) \mathbb{1}_{|y|<1} \right] d\nu(y).$$

(5.6)

For instance:

1. If $X_t$ is a standard Brownian motion:
   $$H^X f = -\frac{1}{2} \Delta f.$$

2. If $X_t$ is a symmetric stable processes of order $0 < \alpha < 2$:
   $$H^X f = -\left( -\frac{1}{2} \Delta \right)^{\alpha/2} f.$$

3. If $X_t$ is a Poisson process of intensity $c$:
   $$H^X f(x) = c \left[ f(x + 1) - f(x) \right].$$

4. If $X_t$ is a compound Poisson process with measure $\nu$ and $c = 1$:
   $$H^X f(x) = \int \left[ f(x + y) - f(x) \right] d\nu(y).$$

In this paper we are interested not on the “free” semigroup for $X_t$ but rather on its “killed” semigroup and its perturbation by the potential $V$.

That is, we want properties of the semigroup

$$T^{D,V}_t f(z) = E^z \left\{ f(X_t) \exp \left( -\int_0^t V(X_s) ds \right); \tau^X_D > t \right\},$$

(5.7)

defined for $t > 0$, $z \in D$, and $f \in L^2(D)$.

For the rest of the paper we shall assume that the distribution of $X_t$ has a density $p^X(t, z, w)$ which is continuous in both $z$ and $w$ for all $t > 0$. That
is, $X_t$ satisfies the ACT condition (see page 288 of [19]) and the density is continuous. The semigroup has a heat kernel $p_{D,V}^X(t,z,w)$ satisfying

$$T_t^{D,V} f(z) = \int_D p_{D,V}^X(t,z,w) f(w) \, dw.$$  \quad (5.8)

Inequality (5.2) is equivalent to

$$\int_D f(w) p_{D,V}^X(t,z,w) \, dw \leq \int_{D^*} f^*(w) p_{D^*,V^*}^X(t,0,w) \, dw, \quad (5.9)$$

for all $z \in D$ and all $t > 0$. Since $f$ is an arbitrary Borel function, our continuity assumption gives

$$p_{D,V}^X(t,z,w) \leq p_{D^*,V^*}^X(t,0,0) < \infty. \quad (5.10)$$

If in addition $X_t$ is transient, we can integrate (5.9) in time to obtain the following isoperimetric inequality for the potentials associated to $X_t$ and $X_t^*$.

**Corollary 5.2.** Suppose $X_t$ is transient and has a continuous density for all $t$. Then for all $z \in D$,

$$\int_D f(w) G_{D,V}^X(z,w) \, dw \leq \int_{D^*} f^*(w) G_{D^*,V^*}^X(0,w) \, dw. \quad (5.11)$$

The heat kernel $p_{D,V}^X(t,z,w)$ can also be represented in terms of the multidimensional distributions. One easily proves, see [11], that

$$p_{D,V}^X(t,z,w) = p^X(t,z,w) E^z \left\{ \exp \left[ - \int_0^t V(X_s) ds \right] ; \tau_D^X > t \bigg| X_t = w \right\}. \quad (5.12)$$

If $0 = t_0 < t_1 < \ldots < t_m < t$, the conditional finite dimensional distribution

$$P^{z_0} \left\{ X_{t_1} \in dz_1, \ldots, X_{t_m} \in dz_m \bigg| X_t = w \right\},$$

is given by

$$\frac{p^X(t-t_m, z_m-w)}{p^X(t,z_0-w)} \prod_{i=1}^m p^X(t_i-t_{i-1}, z_i-z_{i-1}) \, dz_1 \ldots dz_m.$$
Combining (5.12) with the arguments used in (5.1) we have that

\[ p_{D,V}^X(t, z, w) = \lim_{m \to \infty} \lim_{k \to \infty} \int_{D_k} \cdots \int_{D_k} e^{-\frac{t}{m} \sum_{i=1}^{m} V(X_{\frac{i}{m}})} \prod_{i=1}^{m+1} p^X \left( \frac{t}{m}, z_i - z_{i-1} \right) dz_1 \cdots dz_m, \]

where \( z_0 = z \) and \( z_{m+1} = w \).

The proof of Theorem 1.1 can be adapted to obtain

\[ \int_D p_{D,V}^X(t, w, w) dw \leq \int_D p_{D*,V*}^X(t, w, w) dw < \infty. \]  

That is, the trace of the Schrödinger semigroup for \( H_{D,V}^X \) is maximized by the trace of the Schrödinger semigroup \( H_{D*,V*}^X \).

As explain in [21], the amount of heat contained in the domain \( D \) at time \( t \), when \( D \) has temperature 1 at \( t = 0 \) and the boundary of \( D \) is kept at temperature 0 at all times, is given by

\[ Q_t(D) = \int_D \int_D p_D^B(t, z, w) dw dz, \]

where \( B \) is a Brownian motion. Also the torsional rigidity of \( D \) is given by

\[ \int_0^\infty Q_t(D) dt = \int_D \int_D G_D^B(z, w) dw dz. \]

Using the representation (5.13), we obtain the following results for the heat content and torsional rigidity of Lévy processes.

Corollary 5.3. Suppose \( X_t \) is transient and has a continuous density for all \( t \). Then for all \( z \in D \) and \( t > 0 \),

\[ \int_D \int_D p_{D,V}^X(t, z, w) dw dz \leq \int_D \int_D p_{D*,V*}^X(t, z, w) dw dz \]

and

\[ \int_D \int_D G_{D,V}^X(z, w) dw dz \leq \int_D \int_D G_{D*,V*}^X(z, w) dw dz. \]

Notice that the operator \( H_{D,V}^X \) is symmetric if and only if the process \( X_t \) is symmetric. That is, for any Borel set \( A \subset \mathbb{R}^d \),

\[ P^0\{X_t \in A\} = P^0\{X_t \in -A\}. \]
In terms of the exponent in the Lévy-Khintchine formula this means that
\[ \Psi(\xi) = \frac{1}{2} \langle A \cdot \xi, \xi \rangle - \int_{\mathbb{R}^d} \left[ \cos(x \cdot \xi) - 1 \right] d\nu(x), \]
where $A$ is a symmetric matrix and $\nu$ is a symmetric Lévy measure $[\nu(A) = \nu(-A)]$.

In this case the symmetric Markovian semigroup generated by $X_t$ gives rise to the self-adjoint generator $H^X$. Recall that $H^X_{T,D}$ is the operator obtained by imposing Dirichlet boundary conditions on $D$ to the Schrödinger operator $H^X + V$. That is, the generator of the killed semigroup $\{T^D,V_t\}_{t \geq 0}$. By (5.10) we have that
\[ \int_D p^X_{D,V}(t, w, w) dw \leq \int_{D^*} p^{X^*}_{D^*,V^*}(t, 0, 0) dw \]
\[ = p^{X^*}_{D^*,V^*}(t, 0, 0) |D^*| < \infty. \]
That is, the semigroup of the killed process has finite trace.

Whenever $D$ is of finite volume, the operator $T^D,V_t$ maps $L^2(D)$ into $L^\infty(D)$ for every $t > 0$. This follows from (5.10), and the general theory of heat semigroups as described in page 59 of [9]. In fact, under these assumptions it follows from [9] that there exists an orthonormal basis of eigenfunctions $\{\varphi^n_{D,V,X}\}_{n=1}^\infty$ for $L^2(D)$ and corresponding eigenvalues $\{\lambda_n(D,V,X)\}_{n=1}^\infty$ for the semigroup $\{T^D,V_t\}_{t \geq 0}$ satisfying
\[ 0 < \lambda_1(D,V,X) < \lambda_2(D,V,X) \leq \lambda_3(D,V,X) \leq \ldots \]
with $\lambda_n(D,V,X) \to \infty$ as $n \to \infty$. That is, the pair $\{\varphi^n_{D,V,X}, \lambda_n(D,V,X)\}$ satisfies
\[ T^D_t \varphi^n_{D,V,X}(z) = e^{-\lambda_n(D,V,X)t} \varphi^n_{D,V,X}(z), \quad z \in D, \quad t > 0. \]
Notice that $\lambda_n(D,V,X)$ is a Dirichlet eigenvalue of $H^X + V$ on $D$ with eigenfunction $\varphi^n_{D,V,X}(z)$. Under such assumptions we have
\[ p^X_{D,V}(t, z, w) = \sum_{n=1}^\infty e^{-\lambda_n(D,V,X)t} \varphi^n_{D,V,X}(z) \varphi^n_{D,V,X}(w). \]
This eigenfunction expansion for $p^X_{D,V}(t, z, w)$ implies that
\[ -\lambda_1(D,V,X) = \lim_{t \to \infty} \frac{1}{t} \log E^z \left\{ \exp \left( -\int_0^t V(X_s) ds \right) ; \tau^X_D > t \right\}. \]
for all domains $D$ of finite volume. Thus we have
Corollary 5.4 (Faber-Krahn inequality for Lévy Processes). Suppose $X_t$ is symmetric, transient and has a continuous transition density for all $t$. Then

$$\lambda_1(D^*, V^*, X^*) \leq \lambda_1(D, V, X)$$  \hspace{1cm} (5.20)

More generally, we also have the trace inequality

$$\sum_{n=1}^{\infty} e^{-t\lambda_n(D, V, X)} \leq \sum_{n=1}^{\infty} e^{-t\lambda_n(D^*, V^*, X^*)},$$

valid for all $t > 0$. Furthermore

$$\int_D \Phi(p_{D, V}^X(t, z, w)) \, dw \leq \int_{D^*} \Phi(p_{D^*, V}^{X^*}(t, w, 0)) \, dw,$$  \hspace{1cm} (5.21)

and

$$\int_D \Phi(G_{D, V}^X(z, w)) \, dw \leq \int_{D^*} \Phi(G_{D^*, V}^{X^*}(w, 0)) \, dw,$$  \hspace{1cm} (5.22)

valid for all $z \in D$, $t > 0$ and all increasing convex functions $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$. The inequalities in this Corollary extend several results of C. Bandle, see p. 214 of [2].

Finally, recall that if $C_X(A)$ is the capacity associated to $X_t$ of the set $A$. T. Watanabe [22] proved that

$$C_X(A) \geq C_{X^*}(A^*).$$  \hspace{1cm} (5.23)

As explained in [12], this inequality can be obtained from the existing rearrangement inequalities of multiple integrals only in the case that $X_t$ is isotropic unimodal. For general Lévy processes we have the following representation of the capacity due to Port and Stone [18]

$$\lim_{t \to \infty} \frac{1}{t} \int P^{z_0} \left( \tau_{A^c}^X \leq t \right) \, dz_0 = C_X(A).$$  \hspace{1cm} (5.24)

Since

$$\int P^{z_0} \left( \tau_{A^c}^X \leq t \right) \, dz_0$$  \hspace{1cm} (5.25)

$$= \lim_{k \to \infty} \lim_{m \to \infty} \int \ldots \int \left[ 1 - \prod_{j=1}^{m} I_{A_k^c}(z_j) \right] \prod_{j=1}^{m} p^X \left( \frac{t}{m}, z_j, z_{j-1} \right) \, dz_0 \ldots dz_m,$$

where $A_k$ is a decreasing sequence of compact sets such that the interior of $A_k$ contains $A$ for all $k$ and $\cap_{k=1}^{\infty} A_k = A$. We would expect to obtained
(5.23) using a result similar to Theorem 1.3 for more general Lévy processes. However the corresponding rearrangement inequality for this type of multiple integrals is only known for radially symmetric decreasing functions. That is, only when $X_t$ is an isotropic unimodal Lévy process.

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**References**


