Brownian Motion in Cones

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1. Introduction and Statement of Results. For $x \in \mathbb{R}^n \setminus \{0\}$, we let $\varphi(x)$ be the angle between x and the point $(1, 0, \ldots, 0)$. A right circular cone of angle $0 < \theta < \pi$ is the open connected set Γ given by $\{x \in \mathbb{R}^n : \varphi(x) < \theta\}$. We let $\{B_t : t \ge 0\}$ be n-dimensional Brownian motion and denote by E_x and P_x the expectation and probability associated with this motion starting at x. Finally $\tau_{\Gamma} = \inf\{t > 0 : B_t \notin \Gamma\}$ is the first exit time from Γ . The following result was proved by D. Burkholder [4].

Theorem A. There is a number $p(\theta, n)$, defined in terms of the smallest zero of a certain hypergeometric function, such that

(1.1)
$$E_x(\tau_{\Gamma}^p) < \infty, \quad x \in \Gamma,$$

if and only if $p < p(\theta, n)$.

In [10], D. DeBlassie used Burkholder's result and techniques from partial differential equations to find an exact formula for $P_x\{\tau_{\Gamma} > t\}$ as an infinite series involving confluent hypergeometric functions. From this DeBlassie was able to find the exact asymptotics in t for $P_x\{\tau_{\Gamma} > t\}$. Furthermore, his result is also for more general cones in \mathbb{R}^n . Recently, B. Davis and B. Zhang [8] proved an analogue of Burkholder's result for conditioned Brownian motion in Γ . More precisely, let E_x^{ξ} denote the expectation of Brownian motion started at $x \in \Gamma$ and conditioned to exit the cone at $\xi \in \partial \Gamma$. That is, E_x^{ξ} is the expectation associated with the Doob h-process for $h(z) = K(z,\xi)$, where $K(z,\xi)$ is the Poisson kernel with pole at ξ . The Davis–Zhang [8] result, is

(1.2)
$$E_x^{\xi}(\tau_{\Gamma}^p) < \infty, \quad x \in \Gamma, \quad \xi \in \partial \Gamma,$$

if and only if $p < 2p(\theta, n) + \frac{n-2}{2}$, where $p(\theta, n)$ is the same number as in Theorem A.

The purpose of this paper is to provide a uniform proof for all of the above results based on an explicit formula for the Dirichlet heat kernel for general cones in \mathbb{R}^n . Furthermore, our results also give much more information on the distribution of τ_{Γ} and hold for a wider class of cones. We will denote by S^{n-1} the unit sphere in \mathbb{R}^n and its points by θ or η . If Dis a proper open connected subset of S^{n-1} , the generalized cone C generated by D is the set of all rays emanating from the origin 0 and passing through D. We shall assume throughout that D is regular for the Dirichlet problem with respect to $L_{S^{n-1}}$, the Laplace–Beltrami operator on S^{n-1} . With this assumption (see Chavel [6]) we have a complete set of orthonormal eigenfunctions m_j with corresponding eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 < \ldots$ satisfying

(1.3)
$$\begin{cases} L_{S^{n-1}}m_j = -\lambda_j m_j & \text{on } D\\ m_j = 0 & \text{on } \partial D. \end{cases}$$

For the rest of the paper,

$$\alpha_j = \sqrt{\lambda_j + \left(\frac{n}{2} - 1\right)^2}.$$

The confluent hypergeometric function is, with b > 0,

$$_{1}F_{1}(a,b,z) = 1 + \frac{a}{b}\frac{z}{1!} + \frac{a(a+1)}{b(b+1)}\frac{z^{2}}{2!} + \dots$$

Theorem 1. Let C be a generalized cone in \mathbb{R}^n . Then

(1.4)
$$P_x\{\tau_C > t\} = \sum_{j=1}^{\infty} B_j \left(\frac{|x|^2}{2t}\right)^{a_j/2} {}_1F_1\left(\frac{a_j}{2}, a_j + \frac{n}{2}, \frac{-|x|^2}{2t}\right) m_j\left(\frac{x}{|x|}\right),$$

uniformly for $(x,t) \in K \times (T,\infty)$, where $K \subset C$ is compact and T > 0. Here $a_j = \alpha_j - (\frac{n}{2} - 1)$ and

$$B_j = \frac{\Gamma(\frac{a_j+n}{2})}{\Gamma(a_j+\frac{n}{2})} \int_D m_j(\theta) d\sigma(\theta).$$

Using the notation $a(t) \sim b(t)$ to mean that $a(t)/b(t) \to 1$ as $t \to \infty$, we have

Corollary 1. Let C be a generalized cone in \mathbb{R}^n . Then for each $x \in C$,

(1.5)
$$P_x\{\tau_C > t\} \sim B_1 m_1\left(\frac{x}{|x|}\right) \left(\frac{|x|^2}{2}\right)^{a_1/2} t^{-a_1/2}$$

and

(1.6)
$$E_x(\tau_C^p) < \infty$$
 if and only if $p < a_1/2$.

In addition, for each ray ℓ emanating from the origin and passing through D and fixed t > 0,

(1.7)
$$P_x\{\tau_C > t\} \sim B_1(2t)^{-a_1/2} m_1\left(\frac{x}{|x|}\right) |x|^{a_1}$$

as $|x| \to 0, x \in \ell$.

Theorem 1 and Corollary 1 were first proved by DeBlassie [10] under somewhat stronger assumptions on the cones, (see his hypothesis 1.1). We only require that the generating set D be regular for $L_{S^{n-1}}$. Later in [11], DeBlassie obtained the asymptotics in (1.5) under the same general assumption on D that we make above. (1.6) also follows from Lemma 3.1 in R. Bass and K. Burdzy [2]. The argument in [11] (or the results in [2]), however, do not give (1.4). In the case of circular cones Γ , $a_1/2 = p(\theta, n)$ (see [10]) and so (1.6) is just Burkholder's result in that case. We should mention here also that in \mathbb{R}^2 , formulas for $P_x\{\tau_{\Gamma} > t\}$ have existed for many years. Indeed, F. Spitzer [17] in his study of the winding of two dimensional Brownian motion derives an expression for $P_x\{\tau_{\Gamma} > t\}$ from which the two dimensional case of (1.1) and (1.5) follow, (see his Theorem 2, p192).

Next, we will discuss a version of Theorem 1 for Brownian motion conditioned to exit the cone at its vertex 0. For $x = \rho \theta \in C$, $\rho > 0$, $\theta \in S^{n-1}$, we set

$$K(x,0) = \frac{1}{|x|^{\beta}} m_1\left(\frac{x}{|x|}\right) = \frac{1}{\rho^{\beta}} m_1(\theta),$$

where

$$(1.8)\qquad \qquad \beta = a_1 + n - 2$$

We will prove below that K(x, 0) is (up to normalizing constants) the Poisson kernel for the cone with pole at 0. The corresponding Doob *h*-process for h(x) = K(x, 0)is Brownian motion in *C* conditioned to exit at 0. We denote the corresponding probability measure by P_x^0 .

Theorem 2. Let C be a generalized cone in \mathbb{R}^n . Then for any $x \in C$,

(1.9)
$$P_x^0\{\tau_C > t\} = \frac{1}{\Gamma(\alpha_1 + 1)} \left(\frac{|x|^2}{2t}\right)^{\alpha_1} {}_1F_1\left(\alpha_1, \ \alpha_1 + 1, \ \frac{-|x|^2}{2t}\right) = \frac{1}{\Gamma(\alpha_1)} \int_0^{\frac{|x|^2}{2t}} u^{\alpha_1 - 1} e^{-u} du,$$

(1.10)
$$\lim_{t \to \infty} t^{\alpha_1} P_x^0 \{ \tau_C > t \} = \frac{1}{\Gamma(\alpha_1 + 1)} \left(\frac{|x|^2}{2} \right)^{\alpha_1},$$

and

(1.11)
$$E_x^0(\tau_C^p) = \begin{cases} \frac{|x|^{2p}\Gamma(\alpha_1 - p)}{2^p\Gamma(\alpha_1)}, & \text{if } p < \alpha_1\\ \infty, & \text{if } p \ge \alpha_1. \end{cases}$$

If C is a circular cone then $a_1 = 2p(\theta, n)$, as mentioned above. Therefore $\alpha_1 = 2p(\theta, n) + (\frac{n}{2} - 1)$, which is the exponent given by Davis and Zhang [8] in that case. Also notice that the distribution of τ_C under P_x^0 only depends on x through its radial part ρ and it is fully determined by α_1 . For the special case of the upper half space in \mathbb{R}^2 (the cone of angle π) the independence of the angle is also shown in K. Burdzy [3]. We thank K. Burdzy for pointing this out to us and for bringing [2] to our attention.

The paper is organized as follows. In §1, we present the formula for the heat kernel of a cone. In §3, we prove Theorems 1, 2, Corollary 1 and the more general statement that for Lipschitz cones, $E_x^{\xi}(\tau_C^p) < \infty$ for any $x \in C$, $\xi \in \partial C$ if and only if $p < \alpha_1$. We end §2 by proving that for a generalized cone, $E_x^y(\tau_C^p) < \infty$ for any $x, y \in C$, if and only if $p < \alpha_1$. In §4, we give another application of our formula for the heat kernel by computing the distribution of the last time before 1 that Brownian motion was in a cone having started at its vertex. This formula can be viewed as a generalization of Lévy's First Arcsine Law. In §5, we examine the distribution of τ_C for finite cones C under P_x^0 and prove, in \mathbb{R}^2 , that it only depends on x through its radial part. Throughout the paper c is a constant which may change from line to line.

2. The Heat Kernel for Cones in \mathbb{R}^n .

We will denote by $P_t^C(x, y)$ the heat kernel for $\frac{1}{2}\Delta$ in C with Dirichlet boundary conditions. That is, $P_t^C(x, y)$ are the transition densities for Brownian motion in C killed on the boundary. We will use $I_{\nu}(z)$ to denote the modified Bessel function of order ν satisfying the differential equation

(2.1)
$$I_{\nu}''(z) + \frac{1}{z}I_{\nu}'(z) = (1 + \frac{\nu^2}{z^2})I_{\nu}(z).$$

Recall that m_j and λ_j are the Dirichlet eigenfunctions and eigenvalues for the spherical Laplacian on the generating set $D \subset S^{n-1}$ and $\alpha_j = \sqrt{\lambda_j + (\frac{n}{2} - 1)^2}$.

Lemma 1. Let C be a generalized cone in \mathbb{R}^n . Write $x = \rho \theta$, $y = r\eta$, ρ , r > 0, θ , $\eta \in S^{n-1}$. Then the heat kernel for C, $P_t^C(x, y)$, is given by the sum

(2.2)
$$\frac{e^{-\frac{(\rho^2+r^2)}{2t}}}{t(\rho r)^{\frac{n}{2}-1}}\sum_{j=1}^{\infty}I_{\alpha_j}\left(\frac{\rho r}{t}\right)m_j(\theta)m_j(\eta).$$

The convergence in (1.2) is uniform for $(t, x, y) \in (T, \infty) \times \{x \in C : |x| < R\} \times C$, for any positive constants T and R.

In the case of a cone in two dimensions this formula can be found in H. Carslaw and J. Jaeger [7, p. 379]. The two dimensional formula immediately suggests the one for several dimensions and indeed this is the way we discovered (2.2). Later we learned from M. van den Berg that the formula is a special case of a result of J. Cheeger [5] for more general cone-type manifolds. Once the formula has been written down, it is not too difficult to verify that it does indeed give the Dirichlet heat kernel for C. One only needs to show that it satisfies the heat equation and that it has the correct boundary and initial conditions. Cheeger [5] refers to a paper of Cheeger, Gromov and Lawson for the verification of these properties. It seems that this formula is often quoted in the literature; see for example J.S. Dowker [9, p. 770] where it is stated that by separation of variables "the eigenfunction form of the heat Kernel is then easily manipulated into" the form given in (1.2). We were not able to locate the Cheeger-Gromov-Lawson paper in the literature nor to obtain a copy from the authors. For the sake of completeness, we will outline a proof of Lemma 1, leaving some of the details to the interested reader. First, we will prove the uniform convergence of the sum and then show in Lemmas 2 and 3 that it satisfies the heat equation and that it has the correct initial conditions. It clearly has the correct boundary conditions. A probabilisitic representation of the sum in (2.2) is given in (1.20) below.

Since the m_j 's are normalized by $||m_j||_2 = 1$, we have by Theorem 8, p. 102 in Chavel [6], that

(2.3)
$$||m_j||_{\infty}^2 \le c(n)\lambda_j^{\frac{n-1}{2}}.$$

Moreover, the integral representation of $I_{\nu}(z)$ ([15, p. 119]) together with Stirling's Formula gives

(2.4)
$$I_{\nu}(z) = \frac{(z/2)^{\nu}}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_{-1}^{1} (1 - t^2)^{\nu - 1/2} \cosh(zt) dt$$
$$\leq \frac{c(z/2)^{\nu}}{(\nu + \frac{1}{2})^{\nu}} e^{\nu - 1/2} e^z = \frac{c(\frac{z}{2})^{\nu} e^{\nu} e^z}{(\nu + \frac{1}{2})^{\nu}}.$$

With $\nu = \alpha_j$ and $z = \frac{\rho r}{t}$ we get from (2.2) and (2.3) and the definition of α_j

(2.5)

$$P_{t}^{C}(\rho\theta, r\eta) \leq \frac{e^{-\frac{(\rho^{2}+r^{2})}{2t}}}{t(\rho r)^{n/2-1}} \sum_{j=1}^{\infty} |I_{\alpha_{j}}\left(\frac{\rho r}{t}\right) m_{j}(\theta)m_{j}(\eta)$$

$$\leq \frac{e^{-\frac{(\rho^{2}+r^{2})}{2t}}}{t(\rho r)^{n/2-1}} \sum_{j=1}^{\infty} \frac{\left(\frac{\rho r}{t}\right)^{\alpha_{j}} e^{\alpha_{j}} \alpha_{j}^{n-1}}{(\alpha_{j}+1/2)^{\alpha_{j}}} e^{\frac{\rho r}{2t}}$$

Next, we will show that the quantity on the right hand side of (2.5) is dominated by

(2.6)
$$\frac{cP\left(\frac{\rho r}{t}\right)}{t(\rho r)^{\frac{n-1}{2}}} e^{\left\{-\frac{(\rho^2+r^2)}{2t}+\frac{4\rho r}{2t}\right\}},$$

where P is a polynomial of degree depending on n. The bound (2.6) not only proves the announced uniform convergence but it also allows us to integrate the sum term by term. To prove (1.6), set $M = \frac{\rho r e}{t}$. From the Weyl's asymptotic formula (Chavel [6, p. 172]), it follows that there are constants c_1 and c_2 , depending only on n, such that

$$c_1 j^{\frac{1}{n-1}} \le \alpha_j \le c_2 j^{\frac{1}{n-1}}.$$

From this it follows that the sum on the right hand side of (2.5) is dominated by

$$(2.7) \qquad \sum_{j=1}^{\infty} \frac{M^{cj^{\frac{1}{n-1}}} \left(j^{\frac{1}{n-1}}\right)^{n-1}}{\left(j^{\frac{1}{n-1}} + \frac{1}{2}\right)^{j^{\frac{1}{n-1}}}} = \sum_{j=1}^{\infty} \frac{M^{cj^{\frac{1}{n-1}}}j}{\left(j^{\frac{1}{n-1}} + \frac{1}{2}\right)^{j^{\frac{1}{n-1}}}} \\ \le 2\int_{1}^{\infty} \frac{M^{cx^{\frac{1}{n-1}}} \cdot x}{(x^{1/n-1} + 1/2)^{x^{1/n-1}}} \, dx + \sum_{j \le M^{(n-1)c}} \frac{M^{cj^{\frac{1}{n-1}}}j}{\left(j^{\frac{1}{n-1}} + \frac{1}{2}\right)^{j^{\frac{1}{n-1}}}}.$$

Substituting $u = cx^{1/n-1}$ shows that the integral above becomes

$$c \int_{c}^{\infty} \frac{M^{u}}{(u+\frac{1}{2})^{u}} u^{2n-3} du \leq c \int_{0}^{\infty} \frac{M^{u} u^{2n-3}}{(u+\frac{1}{2})^{u}} du$$

$$\leq c\{1+M+M^{2n-3} \int_{1}^{\infty} \left(\frac{M}{u}\right)^{u-2n+3} du\}$$

$$(2.8) = c\left\{1+M+M^{2n-3} \left(\int_{1}^{M} \left(\frac{M}{u}\right)^{u-2n+3} du + \int_{M}^{\infty} \left(\frac{M}{u}\right)^{u-2n+3} du\right)\right\}$$

$$\leq c\left\{1+M+M^{2n-2} e^{cM/e} + M^{2n-3} \int_{M}^{\infty} \left(\frac{M}{u}\right)^{u-2n+3} du\right\},$$

where we use the fact that the maximum of $\left(\frac{M}{u}\right)^{u-2n+3}$ occurs at u = cM/e. Now, the integral on the last line of (2.8) has a decreasing integrand and thus the quantity in the last bracket is dominated by

$$c\left\{1 + M + M^{2n-2}e^{cM/e} + M^{2n-3} \sum_{j=[\ln_2 M]}^{\infty} \int_{2^j}^{2^{j+1}} \left(\frac{M}{u}\right)^{u-2n+3} du\right\}$$

$$(2.9) \leq c\left\{1 + M + M^{2n-2}e^{cM/e} + M^{2n-3} \sum_{j=0}^{\infty} 2^j \left(\frac{M}{2^j}\right)^{2^j-2n+3}\right\}$$

$$= c\left\{1 + M + M^{2n-2}e^{cM/e} + M^{2n-3} \sum_{j=0}^{\infty} \frac{M^{2^j-2n+3}}{(2^j)^{2^j-2n+2}}\right\}.$$

However, by comparing terms it follows that the sum in the last expansion is dominated by $P(M)e^{M/e}$, where P(M) is a polynomial in M. Putting together (2.7), (2.8) and (2.9) gives (2.6) and completes the uniform convergence of the sum.

Before we prove that it satisfies the heat equation, we observe that if x and t are fixed and $y \to 0$, that is, if $r \to 0$, then since $\alpha_j = \sqrt{\lambda_j + (\frac{n}{2} - 1)^2} > \frac{n}{2} - 1$, we can factor the term $(\rho r/2t)^{\alpha_1}$ in (1.5) and obtained that $P_t^C(x, y) \to 0$, as it should.

Lemma 2. Fix $y \in C$ and define u(t, x) to be the sum in (2.2). Then u satisfies the heat equation $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u$.

Proof. Arguing as in Lemma 1 we can show that the differential operator $\frac{\partial}{\partial t} - \frac{1}{2}\Delta$ can be taken inside the sum. We thus only need to prove that each term in the

sum satisfies the heat equation. If we set, for $y = r\eta$ fixed,

(2.10)
$$v_j(t,x) = \frac{1}{t} \frac{1}{(\rho r)^{\frac{n}{2}-1}} e^{-\frac{(\rho^2 + r^2)}{2t}} I_{\alpha_j}\left(\frac{\rho r}{t}\right) m_j(\theta) m_j(\eta),$$

we obtain

(2.11)
$$\frac{\partial v_j}{\partial t}(t,x) = -\frac{1}{t^2(\rho r)^{\frac{n}{2}-1}} e^{-\frac{(\rho^2+r^2)}{2t}} I_{\alpha_j}\left(\frac{\rho r}{t}\right) m_j(\theta) m_j(\eta) + \frac{(\rho^2+r^2)}{2t^3(\rho r)^{\frac{n}{2}-1}} e^{-\frac{(\rho^2+r^2)}{2t}} I_{\alpha_j}\left(\frac{\rho r}{t}\right) m_j(\theta) m_j(\eta) - \frac{\rho r}{t^3(\rho r)^{\frac{n}{2}-1}} e^{-\frac{(\rho^2+r^2)}{2t}} I'_{\alpha_j}\left(\frac{\rho r}{t}\right) m_j(\theta) m_j(\eta).$$

Next we recall that in polar coordinates

(2.12)
$$\frac{1}{2}\Delta = \frac{1}{2}\left(\frac{1}{\rho^2}L_{S^{n-1}} + \frac{1}{\rho^{n-1}}\frac{\partial}{\partial\rho}\left(\rho^{n-1}\frac{\partial}{\partial\rho}\right)\right).$$

Applying this operator to v_j we obtain

(2.13)

$$\frac{1}{2}\Delta v_{j} = \frac{1}{2} \frac{1}{\rho^{2}} \frac{1}{t(\rho r)^{\frac{n}{2}-1}} e^{-\frac{(\rho^{2}+r^{2})}{2t}} I_{\alpha_{j}}\left(\frac{\rho r}{t}\right) m_{j}(\eta) L_{S^{n-1}}m_{j}(\theta)
+ \frac{1}{2\rho^{n-1}} \left\{ \frac{m_{j}(\theta)m_{j}(\eta)}{t} \frac{e^{-\frac{r^{2}}{2t}}}{(\rho r)^{\frac{n}{2}-1}} \right\} \left\{ \frac{-\rho}{t} e^{-\frac{\rho^{2}}{2t}} \rho^{\frac{n}{2}-1}g(\rho) + \left(\frac{n}{2}-1\right) e^{-\rho^{2}/2t} \rho^{\frac{n}{2}-2}g(\rho) + e^{\frac{-\rho}{2t}} \rho^{\frac{n}{2}-1}g'(\rho) \right\},$$

where

(2.14)
$$g(\rho) = \left(1 - \frac{n}{2}\right) I_{\alpha_j}\left(\frac{\rho r}{t}\right) - \frac{\rho^2}{t} I_{\alpha_j}\left(\frac{\rho r}{t}\right) + \frac{\rho r}{t} I'_{\alpha_j}\left(\frac{\rho r}{t}\right).$$

If we now recall that $L_{S^{n-1}}m_j = -\lambda_j m_j$, and the relation (2.1) satisfied by I_{α_j} , we get from (2.13) and (2.14) that

$$\begin{split} \frac{1}{2}\Delta v_{j} &= -\frac{\lambda_{j}}{2} \frac{1}{\rho^{2}} \frac{1}{t(\rho r)^{\frac{n}{2}-1}} e^{-\frac{(\rho^{2}+r^{2})}{2t}} I_{\alpha_{j}}\left(\frac{\rho r}{t}\right) m_{j}(\eta) m_{j}(\theta) \\ &+ \frac{1}{2} \left\{ \frac{m_{j}(\eta)m_{j}(\theta)}{t} \frac{e^{-\frac{r^{2}}{2t}}}{r^{\frac{n}{2}-1}} \right\} \left\{ \frac{e^{-\frac{\rho^{2}}{2t}}}{\rho^{\frac{n}{2}-1}} \right\} \left\{ \frac{e^{-\frac{(\rho^{2}+r^{2})}{2t}}}{t^{2}} I_{\alpha_{j}}\left(\frac{r\rho}{t}\right) \\ &- \frac{2\rho r}{t^{2}} I_{\alpha_{j}}'\left(\frac{r\rho}{t}\right) - \frac{2}{t} I_{\alpha_{j}}\left(\frac{r\rho}{t}\right) + \frac{\lambda_{j}}{\rho^{2}} I_{\alpha_{j}}\right\} \\ &= \frac{1}{2} \left\{ \frac{m_{j}(\eta)m_{j}(\theta)}{t} \frac{e^{-\frac{(\rho^{2}+r^{2})}{2t}}}{(\rho r)^{\frac{n}{2}-1}} \right\} \left\{ \frac{\rho^{2}+r^{2}}{t^{2}} I_{\alpha_{j}}\left(\frac{\rho r}{t}\right) - \frac{2\rho r}{t} I_{\alpha_{j}}'\left(\frac{\rho r}{t}\right) - \frac{2}{t} I_{\alpha_{j}}\left(\frac{\rho r}{t}\right) \right\}, \end{split}$$

which is the same as the expression for $\frac{\partial v_j}{\partial t}$ in (2.11), completing the proof of Lemma 2.

Next we will show that the function given by the sum in (2.2) satisfies the correct initial conditions thus finishing the proof that it is the heat kernel for the cone. Before doing this, we recall that the transition density for the δ -dimensional Bessel process ([16] pg. 415) is given by

(2.15)
$$P_t^{\delta}(\rho, r) = t^{-1} \left(\frac{r}{\rho}\right)^v r e^{-\frac{(\rho^2 + r^2)}{2t}} I_v\left(\frac{\rho r}{t}\right)$$

where $v = \frac{\delta}{2} - 1$. Thus for every continuous function f(r) of compact support we have

(2.16)
$$\lim_{t \to 0} \int_0^\infty P_t^{\delta}(\rho, r) f(r) dr = f(\rho).$$

Using (1.15) we can write the sum in (2.2) as

(2.17)
$$\frac{\rho}{(\rho r)^{\frac{n}{2}}} \sum_{j=1}^{\infty} P_t^{2\alpha_j+2}(\rho, r) \left(\frac{\rho}{r}\right)^{\alpha_j} m_j(\theta) m_j(\eta).$$

From this representation one can already see several properties of heat kernel, such as the semigroup property. This formula also gives that the sum in (2.2) satisfies the initial condition.

Lemma 3. Let f be a continuous function of compact support in C and let $P_t^C(x, y)$ be the sum in (2.17) (or (2.2)). Then,

$$\lim_{t \to 0} \int_C P_t^C(x, y) f(y) dy = f(x).$$

Proof. We first note that by density we can assume f is of the form $f^*(r)g(\eta)$, where both f^* and g have compact support. By the density of the eigenfunctions, we may assume f has the form, $f = f^*(r) \sum_{s=1}^k c_s m_s(\eta)$ where f^* is a compactly supported radial function. Then using the orthogonality of the eigenfunctions we obtain

$$\lim_{t \to 0} \int_{C} P_{t}^{C}(x,y)f(y)dy = \lim_{t \to 0} \int_{0}^{\infty} \int_{D} P_{t}^{C}(\rho\theta,r\eta)f^{*}(r) \sum_{s=1}^{k} c_{s}m_{s}(\eta)r^{n-1}d\sigma(\eta)dr$$
$$= \sum_{s=1}^{k} c_{s}m_{s}(\theta) \{\lim_{t \to 0} \int_{0}^{\infty} P_{t}^{2\alpha_{j}+2}(\rho,r)\frac{\rho}{(\rho r)^{\frac{n}{2}}}(\frac{\rho}{r})^{\alpha_{j}}r^{n-1}f^{*}(r)dr\}.$$

Since f^* is of compact support, we get from (2.16) that the quantity in $\{ \}$ equals $f^*(\rho)$, from which the lemma follows.

We have thus proved that the quantity $P_t^C(x, y)$ given by the sum in (2.2) is indeed the Dirichlet heat kernel for the cone C.

We shall now give a probabilistic formula for the expression given in (1.2). Let R_s be the Bessel process of dimension n (parameter $\nu = \frac{n}{2} - 1$) with generator $\frac{1}{2}\frac{d^2}{dr^2} + \frac{n-1}{2r}\frac{d}{dr}$ and set $T(t) = \int_0^t \frac{ds}{R_s^2}$. With the notation of (2.15), we denote by $P_t^n(\rho, r)$ the transition probabilities for this process and by E_ρ the corresponding expectation. By Theorem 4.7 in M. Yor [18, p. 80],

(2.18)
$$E_{\rho}\left(\exp(-\frac{\gamma^2}{2}T(t))|R_t = r\right) = \frac{I_{((\frac{n}{2}-1)^2 + \gamma^2)^{1/2}}(\frac{\rho r}{t})}{I_{\frac{n}{2}-1}(\frac{\rho r}{t})},$$

where γ is any real number.

Next, we denote by $P_t^D(\theta, \eta)$ be the heat kernel for the operator $\frac{1}{2}L_{S^{n-1}}$ with Dirichlet boundary conditions on D. (Recall, $D \in S^{n-1}$ is regular for the Dirichlet problem.) We have the eigenfunction expansion

(2.19)
$$P_t^D(\theta,\eta) = \sum_{j=1}^{\infty} e^{-\lambda_j \frac{t}{2}} m_j(\theta) m_j(\eta),$$

where λ_j are as in (1.3). Multiplying and dividing the expression in (2.2) by $P_t^n(\rho, r)$ and recalling that $\alpha_j = \sqrt{(\frac{n}{2} - 1)^2 + \lambda_j}$ and that $a_j = \alpha_j - (\frac{n}{2} - 1)$, it follows from (2.15), (2.18) that this quantity can be written as

$$\frac{1}{r^{n-1}} P_t^n(\rho, r) \sum_{j=1}^{\infty} \frac{P_t^{2a_j+n}(\rho, r)}{P_t^n(\rho, r)} \left(\frac{\rho}{r}\right)^{a_j} m_j(\theta) m_j(\eta)$$
$$= \frac{1}{r^{n-1}} P_t^n(\rho, r) \sum_{j=1}^{\infty} E_\rho \left(e^{-\lambda_j \frac{T(t)}{2}} | R_t = r\right) m_j(\theta) m_j(\eta)$$

From this and (2.19) we arrive at the following probabilisitic representation for the heat kernel for the cone:

(2.20)
$$P_t^C(x,y) = \frac{1}{r^{n-1}} P_t^n(\rho,r) E_\rho\left(P_{T(t)}^D(\theta,\eta) | R_t = r\right),$$

with $x = \rho \theta, y = r\eta$.

The formula (2.20) reflects the skew product representation of the Brownian motion as the pair $(R_t, \Theta_{T(t)})$ where Θ_t is the Brownian motion on S^{n-1} generated by $\frac{1}{2}L_{S^{n-1}}$. Notice that the positivity of the sum in (2.2) is now trivial from this formula. The fact that the sum satisfies the initial condition (Lemma 3) can also be easily obtained from (2.20).

3. Proofs of Theorems 1 and 2.

We first recall some properties of special functions which will be used in the proofs of Theorems 1 and 2 below. With $_1F_1$ as defined in the introduction, we define the Whittaker function ([12, p. 386]) by

(3.1)
$$M_{k,\mu}(z) = z^{1/2+\mu} e^{-\frac{1}{2}z} {}_1F_1(\mu - k + \frac{1}{2}, 2\mu + 1, z).$$

Also if

$$f(s) = s^{\mu - \frac{1}{2}} I_{2\nu}(2\sqrt{as})$$

we have

(3.2)
$$\int_0^\infty e^{-ps} f(s) ds = \frac{\Gamma(\mu + \nu + \frac{1}{2})e^{\frac{a}{2p}}}{\sqrt{a} \ \Gamma(2\nu + 1)p^{\mu}} \ M_{-\mu,\nu}\left(\frac{a}{p}\right),$$

by [11, p. 197]. Finally, we will also use the relation, ([14, p. 267]),

(3.3)
$${}_{1}F_{1}(a,b,z) = e^{z}{}_{1}F_{1}(b-a, b, -z).$$

Proof of Theorem 1. Because of the estimates (2.5) and (2.6), we may integrate $P_t^C(x, y)$ by bringing the integral inside the sum in (2.2). With $x = \rho \theta$ and $y = r\eta$, we obtain by integrating in polar coordinates,

$$(3.4)$$

$$P_{x}\{\tau_{C} > t\} = \sum_{j=1}^{\infty} \int_{0}^{\infty} \int_{D} \frac{r^{n-1}}{t(\rho r)^{\frac{n}{2}-1}} e^{-\frac{(\rho^{2}+r^{2})}{2t}} I_{\alpha_{j}}\left(\frac{\rho r}{t}\right) m_{j}(\theta) m_{j}(\eta) d\sigma(\eta) dr$$

$$= \sum_{j=1}^{\infty} \frac{e^{-\frac{\rho^{2}}{2t}}}{t\rho^{\frac{n}{2}-1}} \left\{ \int_{0}^{\infty} r^{\frac{n}{2}} e^{-\frac{r^{2}}{2t}} I_{\alpha_{j}}\left(\frac{\rho r}{t}\right) dr \right\} m_{j}(\theta) \int_{D} m_{j}(\eta) d\sigma(\eta)$$

$$= \sum_{j=1}^{\infty} \left(\frac{2t}{\rho^{2}}\right)^{\frac{n}{4}} e^{-\frac{\rho^{2}}{4t}} \left\{ \frac{\sqrt{\rho^{2}/2t^{2}}}{t^{n/4}} e^{-\frac{\rho^{2}}{4t}} \int_{0}^{\infty} e^{-u/t} u^{\frac{n}{4}-\frac{1}{2}} I_{\alpha_{j}}\left(2\sqrt{\frac{\rho^{2}}{2t^{2}}} u^{1/2}\right) du \right\} m_{j}(\theta) \int_{D} m_{j}(\eta) d\sigma(\eta),$$

where the last equality follows by changing variables with $u = r^2/2$. Now, since $\alpha_j = a_j - 1 + \frac{n}{2}$, we can apply (3.1) and (3.2) with $\mu = \frac{n}{4}$, $\nu = \frac{a_j - 1}{2} + \frac{n}{4}$, $p = \frac{1}{t}$ and $a = \frac{\rho^2}{2t^2}$ to obtain that the right hand side of (3.4) is

$$\begin{split} &= \sum_{j=1}^{\infty} \frac{\Gamma(\frac{a_j}{2} + \frac{n}{2})}{\Gamma(a_j + \frac{n}{2})} (\frac{2t}{\rho^2})^{\frac{n}{4}} e^{-\frac{\rho^2}{4t}} M_{-\frac{n}{4}, \frac{a_j-1}{2} + \frac{n}{4}} \left(\frac{\rho^2}{2t}\right) m_j(\theta) \int_D m_j(\eta) d\sigma \\ &= \sum_{j=1}^{\infty} \frac{\Gamma(\frac{a_j}{2} + \frac{n}{2})}{\Gamma(a_j + \frac{n}{2})} {}_1F_1\left(\frac{a_j+n}{2}, a_j + \frac{n}{2}, \frac{\rho^2}{2t}\right) e^{-\frac{\rho^2}{2t}} \left(\frac{\rho^2}{2t}\right)^{\frac{a_j}{2}} m_j(\theta) \int_D m_j(\eta) d\sigma \\ &= \sum_{j=1}^{\infty} B_j \left(\frac{\rho^2}{2t}\right)^{\frac{a_j}{2}} {}_1F_1\left(\frac{a_j}{2}, a_j + \frac{n}{2}, \frac{-\rho^2}{2t}\right) m_j(\theta), \end{split}$$

where we have used (3.3). This completes the proof of Theorem 1.

Corollary 1 follows immediately from Theorem 1 and the obvious properties of ${}_1F_1(a, b, z)$. We shall now proceed to the proof of Theorem 2. We first need

Lemma 4. Let $\beta = a_1 + n - 2 = \alpha_1 + \frac{n}{2} - 1$ and set

$$K(x,0) = \frac{1}{|x|^{\beta}} m_1\left(\frac{x}{|x|}\right) = \frac{1}{\rho^{\beta}} m_1(\theta).$$

Then K is the Poisson kernel for C with pole at the vertex 0.

Proof. We must show that K vanishes on the boundary of C, at infinity, that it blows up as x approaches 0 and finally, that it is harmonic in C. The boundary behavior of K is clear from m_1 and the fact that $\beta > 0$. Using (2.12) we obtain that

$$\Delta K(x,0) = -\rho^{-\beta-2}\lambda_1 m_1(\theta) + m_1(\theta) \frac{1}{\rho^{n-1}} \frac{\partial}{\partial \rho} \left(\rho^{n-1} \frac{\partial}{\partial \rho} \rho^{-\beta} \right)$$
$$= -\rho^{-\beta-2}\lambda_1 m_1(\theta) + m_1(\theta)(-\beta)(-\beta+n-2)\rho^{-\beta-2}$$
$$= \rho^{-\beta-2} m_1(\theta) \{ -\lambda_1 + \beta(\beta+2-n) \} = 0,$$

by our definition of α_1 and β .

Proof of Theorem 2. As before, we can interchange the integral with the sum

in our expression for the heat kernel because of (2.6). We thus obtain,

$$P_x^0\{\tau_C > t\} = \frac{1}{K(x,0)} \int_C P_t^C(x,y) K(y,0) dy$$

= $\frac{\rho^\beta}{m_1(\theta)} \sum_{j=1}^\infty \int_0^\infty \left(\int_D m_j(\theta) m_j(\eta) m_1(\eta) d\sigma(\eta) \right) I_{\alpha_j}\left(\frac{\rho r}{t}\right) \frac{e^{-\frac{(\rho^2 + r^2)}{2t}}}{t(\rho r)^{\frac{n}{2} - 1}} r^{n-1-\beta} dr$
= $\rho^\beta \int_0^\infty I_{\alpha_1}\left(\frac{\rho r}{t}\right) \frac{e^{-\frac{(\rho^2 + r^2)}{2t}}}{t(\rho r)^{\frac{n}{2} - 1}} r^{n-1-\beta} dr,$

where we have used the fact that $\{m_j\}$ are orthonormal. Making the substitution we obtain as before that the last quantity is

$$= \frac{\rho^{\beta - \frac{n}{2} + 1} e^{-\frac{\rho^2}{2t}}}{t} \int_0^\infty I_{\alpha_1} \left(\sqrt{2} \ \frac{\rho}{t} \sqrt{u} \right) e^{-\frac{u}{t}} (2u)^{\frac{n}{4} - \frac{\beta}{2} - \frac{1}{2}} du$$

$$= \frac{2^{\frac{n}{4} - \frac{\beta}{2} - \frac{1}{2}}}{t} \rho^{\beta - \frac{n}{2} + 1} e^{-\frac{\rho^2}{4t}} \left\{ \frac{\Gamma(\frac{n}{4} - \frac{\beta}{2} + \frac{\alpha_1}{2} + \frac{1}{2})}{2^{-1/2} \frac{\rho}{t} \Gamma(\alpha_1 + 1) t^{-\frac{n}{4} + \frac{\beta}{2}}} M_{-\frac{n}{4} + \frac{\beta}{2}, \frac{\alpha_1}{2}} \left(\frac{\rho^2}{2t} \right) \right\},$$

where again we have used (3.2) with $\mu = \frac{n}{4} - \frac{\beta}{2}$, $\nu = \frac{\alpha_1}{2}$, $p = \frac{1}{t}$, $a = \frac{\rho^2}{2t^2}$. Again recalling the definition of β in terms of α_1 and the relation (3.1), the above expression

$$= \left(\frac{\rho^2}{2t}\right)^{\frac{\beta}{2} - \frac{n}{4}} \frac{e^{-\frac{\rho^2}{4t}}}{\Gamma(\alpha_1 + 1)} \left(\frac{\rho^2}{2t}\right)^{\frac{\alpha_1}{2} + \frac{1}{2}} e^{\frac{-\rho^2}{4t}} {}_1F_1\left(\frac{n}{4} - \frac{\beta}{2} + \frac{\alpha_1}{2} + \frac{1}{2}, \alpha_1 + 1, \frac{\rho^2}{2t}\right)$$

$$= \frac{\left(\frac{\rho^2}{2t}\right)^{\alpha_1} e^{-\frac{\rho^2}{2t}}}{\Gamma(\alpha_1 + 1)} {}_1F_1\left(1, \alpha_1 + 1, \frac{\rho^2}{2t}\right)$$

$$= \frac{1}{\Gamma(\alpha_1 + 1)} \left(\frac{\rho^2}{2t}\right)^{\alpha_1} {}_1F_1\left(\alpha_1, \alpha_1 + 1, \frac{-\rho^2}{2t}\right),$$

where we have used (3.3) for the last equality.

Finally, by the definition of $_1F_1$, the last expression above is

$$\begin{split} &= \frac{1}{\alpha_1 \Gamma(\alpha_1)} \left(\frac{\rho^2}{2t}\right)^{\alpha_1} \left\{ 1 + \frac{\alpha_1}{\alpha_1 + 1} \left(\frac{-\rho^2}{2t}\right) + \frac{\alpha_1(\alpha_1 + 1)}{(\alpha_1 + 1)(\alpha_1 + 2)} \frac{1}{2!} \left(\frac{-\rho^2}{2t}\right)^2 + \dots \right\} \\ &= \frac{1}{\alpha_1 \Gamma(\alpha_1)} \left(\frac{\rho^2}{2t}\right)^{\alpha_1} \left\{ \alpha_1 \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_1 + 1} \left(\frac{-\rho^2}{2t}\right) + \frac{1}{2!(\alpha_1 + 2)} \left(\frac{-\rho^2}{2t}\right)^2 + \dots \right) \right\} \\ &= \frac{1}{\Gamma(\alpha_1)} \left(\frac{\rho^2}{2t}\right)^{\alpha_1} \sum_{j=0}^{\infty} \frac{1}{(\alpha_1 + j)j!} \left(\frac{-\rho^2}{2t}\right)^j \\ &= \frac{1}{\Gamma(\alpha_1)} \sum_{j=0}^{\infty} \frac{(-1)^j}{(\alpha_1 + j)j!} \left(\frac{\rho^2}{2t}\right)^{j+\alpha_1}. \end{split}$$

Setting

$$H(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(\alpha_1 + j)j!} z^{j+\alpha_1}$$

we obtain that

$$H'(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} z^{j+\alpha_1-1} = z^{\alpha_1-1} e^{-z}$$

and

$$H(z) = H(z) - H(0) = \int_0^z H'(u) du = \int_0^z u^{\alpha_1 - 1} e^{-u} du.$$

Hence,

$$P_x^0\{\tau_C > t\} = \frac{1}{\Gamma(\alpha_1)} \int_0^{\frac{\rho^2}{2t}} u^{\alpha_1 - 1} e^{-u} du,$$

which proves (1.9).

Since

$$P_x^0\{\tau_C > t\} = \frac{1}{\Gamma(\alpha_1 + 1)} \left(\frac{\rho^2}{2t}\right)^{\alpha_1} {}_1F_1\left(\alpha_1, \ \alpha_1 + 1, \ \frac{-\rho^2}{2t}\right),$$

(1.10) follows from $_{1}F_{1}(\alpha_{1}, 1 + \alpha_{1}, 0) = 1$.

By Fubini's Theorem,

$$\begin{split} E_x^0(\tau_C^p) &= p \int_0^\infty t^{p-1} P_x^0 \{\tau_C > t\} dt \\ &= p \int_0^\infty \int_0^{\frac{\rho^2}{2t}} \frac{1}{\Gamma(\alpha_1)} t^{p-1} u^{\alpha_1 - 1} e^{-u} du dt \\ &= \frac{p}{\Gamma(\alpha_1)} \int_0^\infty \left(\int_0^{\frac{\rho^2}{2u}} t^{p-1} dt \right) u^{\alpha_1 - 1} e^{-u} du \\ &= \frac{1}{\Gamma(\alpha_1)} \left(\frac{\rho^2}{2} \right)^p \int_0^\infty u^{\alpha_1 - p - 1} e^{-u} du \\ &= \begin{cases} \frac{\Gamma(\alpha_1 - p)}{\Gamma(\alpha_1)} \left(\frac{\rho^2}{2} \right)^p, & \text{if } p < \alpha_1 \\ \infty, & \text{if } p \ge \alpha_1, \end{cases} \end{split}$$

which proves (1.11) and completes the proof of Theorem 2.

Next, we wish to extend the above results to Brownian motion conditioned to exit the cone at any $\xi \in \partial C$. For this, we make the additional assumption that the cone is simply connected with a Lipschitz boundary. Such cones will be called Lipschitz cones. We start with a lemma which allows us to estimate all the eigenfunctions of D in terms of the first one. **Lemma 5.** If C is a Lipschitz cone, then for $D = S^{n-1} \cap C$, with Dirichlet eigenfunctions $m_j(\eta)$, there exists a constant c, independent of j and η , such that

$$m_j^2(\eta) \le \frac{cm_1^2(\eta)}{I_{\alpha_j}(1)}$$

Proof. Consider a large ball B = B(0, R) and its intersection, $S_2 = B \cap C$, with the cone C. If we fix |x| = 1 and t = 1, we can apply the parabolic boundary Harnack inequality (see Fabes, Garofalo and Salsa [13]) to compare uniformly $P_t^C(x, y)$ and $P_t^{S_2}(x, y)$, at |x| = 1, t = 1. That is there exists a constant c_1 so that

(3.5)
$$P_1^C(x,x) = e^{-1} \sum_{j=1}^{\infty} I_{\alpha_j}(1) m_j^2(\eta) \le c_1 P_1^{S_2}(x,x).$$

Since S_2 is intrinsically ultracontractive, (Bañuelos [1]), it follows that $P_1^{S_2}(x, x) \leq C\varphi_1^2(x)$, where $\varphi_1^2(x)$ is the first Dirichlet eigenfunction on S_2 . Using the fact that $\varphi_1(x) = f(r)m_1(\eta)$, with $x = r\eta$, we obtain from (3.5),

$$\sum_{j=1}^{\infty} I_{\alpha_j}(1) m_j^2(\eta) \le c_1 P_1^{S_2}(x, x) \le C \varphi_1^2(x) = c m_1^2(\eta)$$

and the lemma follows.

Next, recall that by $a(t) \sim b(t)$ we mean $a(t)/b(t) \to 1$ as $t \to \infty$.

Theorem 3. Suppose C is a Lipschitz cone. Fix $x \in C$ and $\xi \in \partial C$. Then

$$P_x^{\xi}\{\tau_C > t\} \sim f(x,\xi)t^{-\alpha_1}.$$

where $f(x,\xi)$ is a function of x and ξ alone.

Proof. By scaling we may assume |x|, $|\xi| < 1/2$. (See [8] for more details on this type of scaling argument.) With $x = r\theta$ and $z = s\eta$ we have

$$P_{x}^{\xi}\{\tau_{C} > t\} = \frac{1}{K(x,\xi)} \left\{ \int_{|z|<1} P_{t}^{C}(x,z)K(z,\xi)dz + \int_{|z|>1} P_{t}^{C}(x,z)K(z,\xi)dz \right\}$$

$$(3.6) \qquad = \frac{1}{K(x,\xi)} \int_{0}^{1} \int_{D} \sum_{j=1}^{\infty} \frac{1}{t(rs)^{\frac{n}{2}-1}} I_{\alpha_{j}}\left(\frac{rs}{t}\right) m_{j}(\theta)m_{j}(\eta)K(s\eta,\xi)s^{n-1}d\eta ds$$

$$+ \frac{1}{K(x,\xi)} \int_{|z|>1} P_{t}^{C}(x,z)K(z,\xi)dz$$

The boundary Harnack principle for Lipschitz domains (see Jerison–Kenig [14]) gives that for |z| > 1, $z \in C$, $K(z,\xi) \approx K(z,0)$, where by $a(z) \approx b(z)$ we mean that their ratio is bounded above and below by absolute constants. Then using Theorem 2 we see that the second integral in (3.6) decays like $t^{-\alpha_1}$. What remains then is to examine the decay of

(3.7)
$$\frac{1}{K(x,\xi)} \int_0^1 \int_D \sum_{j=1}^\infty \frac{1}{t(rs)^{\frac{n}{2}-1}} I_{\alpha_j}\left(\frac{rs}{t}\right) m_j(\theta) m_j(\eta) K(s\eta,\xi) s^{n-1} d\eta ds.$$

Since x, ξ are fixed |x| = r < 1 and s < 1 this integral is majorized by $(I_{\alpha_j}$ are increasing)

(3.8)
$$\frac{1}{K(x,\xi)} \int_0^1 \int_D \sum_{j=1}^\infty I_{\alpha_j}\left(\frac{1}{t}\right) \cdot \frac{1}{t} |m_j(\theta)| |m_j(\eta)| K(s\eta,\xi) s^{n/2} d\eta ds.$$

At this point we can apply Lemma 5 and dominate (3.8) by

(3.9)
$$\frac{1}{K(x,\xi)} \frac{c}{t} \int_0^1 \int_D \sum_{j=1}^\infty I_{\alpha_j}(1/t) |m_j(\theta)| \frac{|m_1(\eta)|}{\sqrt{I_{\alpha_j}(1)}} |K(s\eta,\xi)| s^{n/2} d\eta ds.$$

By (2.3), $|m_j(\theta)| \leq c\lambda_j^d$, with $d = \frac{n-2}{2}$. The last quantity is thus bounded by

(3.10)
$$\sum_{j=1}^{\infty} \frac{c}{tK(x,\xi)} \frac{\lambda_j^d I_{\alpha_j}(1/t)}{\sqrt{I_{\alpha_j}(1)}} \int_0^1 \int_D |m_1(\eta)| K(s\eta,\xi) s^{n/2} d\eta ds$$

Using lemma 3 this equal

(3.11)
$$\sum_{j=1}^{\infty} \frac{c}{tK(x,\xi)} \frac{\lambda_j^d I_{\alpha_j}(1/t)}{\sqrt{I_{\alpha_j}(1)}} \int_0^1 \int_D K(s\eta,0) K(s\eta,\xi) s^{\frac{n}{2}+\beta} d\eta ds.$$

As $1/t \to 0$, $I_{\alpha_j}(1/t) \sim \frac{t^{-\alpha_j}}{2^{\alpha_j}\Gamma(1+\alpha_j)}$ so that the last sum is like

$$\frac{1}{K(x,\xi)}\sum_{j=1}^{\infty}t^{-\alpha_j-1}\frac{\lambda_j^d}{2^{\alpha_j}\Gamma(1+\alpha_j)\sqrt{I_{\alpha_j}(1)}}\int_0^1\int_D K(s\eta,0)K(s\eta,\xi)s^{n/2+\beta}d\eta ds.$$

The last expression is $\approx t^{-\alpha_1-1}$, provided we can show that

(3.12)
$$\int_0^1 \int_D K(s\eta, 0) K(s\eta, \xi) s^{\frac{n}{2} + \beta} d\eta ds < \infty.$$

Since |z| < 1 we invoke the boundary Harnack principle one last time to have $K(z,0) \approx K^{S_2}(z,0)$ and $K(z,\xi) \approx K^{S_2}(z,\xi)$ uniformly in $B(0,1) \cap C$, where $S_2 = B(0,R) \cap C$ for a large R and $K^{S_2}(z,\xi)$ is the Poisson kernel for S_2 with pole at ξ . Thus the integral in (3.12) is bounded by

$$\int_0^1 \int_D K^{S_2}(s\eta, 0) K^{S_2}(s\eta, \xi) s^{\frac{n}{2} + \beta} d\eta ds \le CR^{\alpha_1} \int_{S_2} K^{S_2}(z, 0) K^{S_2}(z, \xi) dz,$$

since $\beta = \frac{n}{2} - 1 + \alpha_1$. The last integral is finite since it is, up to a constant, the expected lifetime of conditioned Brownian motion from 0 to ξ in $S_2; S_2$ being a bounded Lipschitz domain, (see Bañuelos [1]).

Corollary 2. Suppose C is a Lipschitz cone. Then

$$E_x^{\xi}(\tau_C^p) < \infty, \quad x \in C, \quad \xi \in \partial C,$$

if and only if $p < \alpha_1$.

In the case of right circular cones Γ , Corollary 2 is due to Davis and Zhang [8].

Next, let us denote the Green's function for C by $G_C(x, y)$, $x, y \in C$. Fixing x and y in C, we can use the Green's function to construct the Brownian motion starting at x and conditioned to hit y before τ_C . If we denote by P_x^y the probability measure associated with this motion it follows that

(3.13)
$$P_x^y\{\tau_C > t\} = \frac{1}{G_C(x,y)} \int_C P_t^C(x,z) G_C(z,y) dz$$

Differentiating (3.13) with respect to t and using the basic properties of $P_t^C(x, z)$ and $G_C(z, y)$, it follows that the density of τ_C under P_x^y is given by

(3.14)
$$D_x^y(t) = \frac{P_t^C(x,y)}{G_C(x,y)}.$$

The proof of Lemma 1 shows that for $x, y \in C$ fixed,

(3.15)
$$P_t^C(x,y) \sim h(x,y)t^{-\alpha_1-1},$$

where h(x, y) is a function of x and y alone. Therefore (3.14) and (3.15) give

Corollary 3. Let C be a generalized cone in \mathbb{R}^n . Then

$$E_x^y(\tau_C^p) < \infty, \qquad x, y \in C,$$

if and only if $p < \alpha_1$.

4. A Generalized Arcsine Law.

The classical first arcsine law states that if w_t is one dimensional Brownian motion and if

$$g_1 = \sup\{t \le 1 : w_t = 0\}$$

then ([16], p. 107),

(4.1)
$$P_0\{g_1 \le s\} = \frac{2}{\pi} \arcsin(\sqrt{s}).$$

If C is a generalized cone in \mathbb{R}^n we define

$$(4.2) L = \sup\{t \le 1: B_t \in C\},$$

where B_t is Brownian motion in \mathbb{R}^n . We are interested in the distribution of L under P_0 . Let $\tilde{D} = S^{n-1} \cap (\mathbb{R}^n \setminus C) = S^{n-1} \setminus C$ and as before set $a_j = a_j^{\tilde{D}} = \sqrt{\lambda_j + \left(\frac{n}{2} - 1\right)^2} - \left(\frac{n}{2} - 1\right)$ where the λ_j satisfy (0.3) with \tilde{D} replacing D. With this notation we have

Theorem 4.

$$P_0\{L \le s\} = \frac{1}{2\pi^{n/2}} \sum_{j=1}^{\infty} \Gamma\left(a_j + \frac{n}{2}\right) B_j^2 s^{a_j/2} {}_2F_1\left(\frac{a_j}{2}, \frac{a_j}{2}; a_j + \frac{n}{2}; s\right)$$

where $_{2}F_{1}(a, b, c; z)$ is the hypergeometric function given by

$$_{2}F_{1}(a,b;c;z) = 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^{2}}{2!} + \dots$$

and the B_j 's are as in Theorem 1 but corresponding to \tilde{D} .

Proof. Consider the generalized cone $\tilde{C} = \mathbb{R}^n \setminus C$. Applying the strong Markov property and Theorem 1 to \tilde{C} we obtain

$$\begin{aligned} &(4.2)\\ P_0\{L \le s\} = \int_{\mathbb{R}^n} \frac{1}{(2\pi s)^{n/2}} e^{-\frac{|y|^2}{2s}} P_y\{\tau_{\tilde{C}} > 1 - s\} dy\\ &= \frac{1}{(2\pi s)^{n/2}} \int_{\tilde{C}} e^{-\frac{|y|^2}{2s}} \sum_{j=1}^{\infty} B_j \left(\frac{|y|^2}{2 - 2s}\right)^{a_j/2} {}_1F_1\left(\frac{a_j}{2}, a_j + \frac{n}{2}, \frac{-|y|^2}{2 - 2s}\right) m_j\left(\frac{y}{|y|}\right) dy\\ &= \frac{1}{(2\pi s)^{n/2}} \int_0^\infty \int_{\tilde{D}} e^{\frac{-r^2}{2s}} \sum_{j=1}^{\infty} B_j \left(\frac{r^2}{2 - 2s}\right)^{a_j/2} {}_1F_1\left(\frac{a_j}{2}, a_j + \frac{n}{2}, \frac{-r^2}{2 - 2s}\right) m_j(\theta) r^{n-1} d\theta dr\\ &= \frac{1}{(2\pi s)^{n/2}} \sum_{j=1}^{\infty} \frac{\Gamma(a_j + n/2)}{\Gamma\left(\frac{a_j + n}{2}\right)} B_j^2 I_j(s), \end{aligned}$$

where we have used our definition of B_j and

$$I_j(s) = \int_0^\infty \left(\frac{r^2}{2-2s}\right)^{a_j/2} {}_1F_1\left(\frac{a_j}{2}, a_j + n/2, \frac{-r^2}{2-2s}\right) e^{\frac{-r^2}{2s}} r^{n-1} dr.$$

By (3.3),

$$_{1}F_{1}\left(\frac{a_{j}}{2},a_{j}+n/2,\frac{-r^{2}}{2-2s}\right) = e^{\frac{-r^{2}}{2-2s}} {}_{1}F_{1}\left(\frac{a_{j}+n}{2},a_{j}+\frac{n}{2},\frac{r^{2}}{2-2s}\right)$$

and we obtain

(4.3)
$$I_j(s) = \int_0^\infty \left(\frac{r^2}{2-2s}\right)^{a_j/2} {}_1F_1\left(\frac{a_j+n}{2}, a_j+n/2, \frac{r^2}{2-2s}\right) e^{\frac{-r^2}{s(2-2s)}} r^{n-1} dr.$$

Making the substitution $t = \frac{r^2}{2-2s}$ the right hand side of (4.3) becomes,

$$(1-s)^{n/2} 2^{\frac{n-2}{2}} \int_0^\infty e^{-t/s} t^{\frac{a_j+n}{2}-1} {}_1F_1\left(\frac{a_j+n}{2}, a_j+\frac{n}{2}, t\right) dt.$$

Applying (2.1) with

(4.4)
$$\mu = \frac{a_j}{2} + \frac{n}{4} - \frac{1}{2} \text{ and } k = \frac{-n}{4}$$

we obtain that

$$I_j(s) = (1-s)^{\frac{n}{2}} 2^{\frac{n-2}{2}} \int_0^\infty e^{\left(\frac{1}{2} - \frac{1}{s}\right)t} t^{\frac{n}{4} - 1} M_{-\frac{n}{4}, \frac{a_j}{2} + \frac{n}{4} - \frac{1}{2}}(t) dt.$$

The last integral can be evaluated using formula (11) of [11, p. 215] with $\nu = \frac{n}{4}$, k and μ as in (4.4), a = 1 $p = \frac{1}{5} - \frac{1}{2}$ to obtain,

(4.5)
$$I_j(s) = (1-s)^{\frac{n}{2}} 2^{\frac{n-2}{2}} s^{\frac{a_j+n}{2}} \Gamma\left(\frac{a_j}{2} + \frac{n}{2}\right) {}_2F_1\left(\frac{a_j+n}{2}, \frac{a_j+n}{2}; a_j + \frac{n}{2}; s\right).$$

Finally,

(4.6)
$$_{2}F_{1}\left(\frac{a_{j}+n}{2},\frac{a_{j}+n}{2},a_{j}+\frac{n}{2};s\right) = (1-s)^{-\frac{n}{2}} {}_{2}F_{1}\left(\frac{a_{j}}{2},\frac{a_{j}}{2};a_{j}+\frac{n}{2};s\right).$$

From (4.5), (4.6) and the right hand side of (4.2), Theorem 4 follows.

Corollary 4.
$$P_0\{L \le s\} \sim \frac{1}{2\pi^{n/2}} \Gamma\left(a_1 + \frac{n}{2}\right) B_1^2 s^{a_1/2} \text{ as } s \to 0.$$

Remark. If n = 2 and C is the upper half space, the cone of angle π in \mathbb{R}^2 , then our formula reduces to

$$P_0\{L \le s\} = \frac{1}{2\pi} \sum_{j+1}^{\infty} \Gamma(j+1) B_j^2 s^{j/2} {}_2F_1\left(\frac{j}{2}, \frac{j}{2}; j+1; s\right).$$

By checking coefficients in the Taylor expansion of $\arcsin(\sqrt{s})$ one sees that

$$P_0\{L \le s\} = \frac{1}{\pi} \arcsin(\sqrt{s})$$

in this case. Notice that this is $\frac{1}{2}P_0\{g_1 \leq s\}$, as it should be by the reflection principle.

5. Finite Cones.

Let $D \subset S^{n-1}$ be a proper, connected open subset and consider the truncated cone $C = \{r\theta, 0 < r < d, \ \theta \in D\} \subset \mathbb{R}^n$. Then

Proposition 1. The Poisson kernel $K(r\theta, 0)$ for C is given by $K(r\theta, 0) = f(r)m_1(\theta)$ where $f(r) = r^{-\beta} - d^{-\beta-\alpha_1}r^{\alpha_1}$, $\beta = \frac{n}{2} - 1 + \alpha_1$, $\alpha_1 = \sqrt{(\frac{n}{2} - 1)^2 + \lambda_1}$ and as before, $m_1(\theta)$ is the first Dirichlet eigenfunction for $L_{s^{n-1}}$ in D.

Proof. One can see that $f(d) = d^{-\beta} - d^{-\beta-\alpha_1}d^{\alpha_1} = 0$ and $f(r) \to \infty$ as $r \to 0$. Moreover, $m_1(\theta) \to 0$ as $\theta \to \partial D$ so $K(r\theta, 0)$ satisfies the proper boundary conditions. It only remains to check that it is harmonic. Again, using the Laplacian is spherical coordinates yields,

$$\Delta K(r\theta, 0) = \frac{f(r)}{r^2} L_{S^{n+1}} m_1(\theta) + m_1(\theta) (f''(r) + \frac{n-1}{r} f'(r))$$

= $m_1(\theta) \left\{ f''(r) + \frac{n-1}{r} f'(r) - \lambda_1 \frac{f(1)}{r^2} \right\}.$

Now an easy calculation and the definition of β and α_1 show that $f''(r) + \frac{n-1}{r}f'(r) - \lambda_1 \frac{f(r)}{r^2} = 0$, finishing the proof.

Corollary 5. Let $C = \{0 < r < d, 0 < \theta < \theta_0\}$ be a truncated cone in \mathbb{R}^2 . If |x| = |y|, then $P_x^0\{\tau_C > t\} = P_y^0\{\tau_C > t\}$.

Proof. Since C is a bounded convex domain we can expand its heat kernel $P_t^C(x, y)$ in terms of eigenfunctions [1] to get

$$P_t^C(x,y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y).$$

Thus

$$P_x^0\{\tau_C > t\} = \frac{1}{K(x,0)} \int_C P_t^C(x,y) K(y,0) dy.$$

Since $K(x,0) = K(r\theta,0) = f(r)m_1(\theta)$, we have, with $y = s\eta$,

$$P_x^0\{\tau_C > t\} = \frac{1}{f(r)m_1(\theta)} \int_0^d \int_0^{\theta_0} \sum_{n=1}^\infty e^{-\lambda_n t} \varphi_n(x)\varphi_n(y)f(s)m_1(\eta)d\eta s ds.$$

We will be done if the integral collapses to a sum only involving $m_1(\theta)$ and some function of r. We first remark that in \mathbb{R}^2 , the $m_j(\eta)$ are just $\sin\left(\frac{j\pi\eta}{\theta_0}\right)$. Now, the eigenfunctions of the truncated cone are all of the form

$$\sin\left(\frac{j\pi\eta}{\theta_0}\right)J_{\frac{j\pi}{\theta_0}}(\sqrt{\lambda_{j,m}}r),$$

where $J_{\frac{j\pi}{\theta_0}}$ are the Bessel functions of order $\frac{j\pi}{\theta_0}$ and $\sqrt{\lambda_{j,k}}$ are the zeros of $J_{\frac{j\pi}{\theta_0}}$ up to a factor of d. By orthogonality of the Bessel functions we then obtain

$$P_x^0\{\tau_c > t\} = \frac{1}{f(r)m_1(\theta)} \int_0^d \sum_{k=1}^\infty e^{-\lambda_{1,k}t} m_1(\theta) J_{\frac{\pi}{\theta_0}}(\sqrt{\lambda_{1,k}} \ r) J_{\frac{\pi}{\theta_0}}(\sqrt{\lambda_{1,k}} \ s) s ds$$
$$= \frac{1}{f(r)} \int_0^d \sum_{k=1}^\infty e^{-\lambda_{1,k}t} J_{\frac{\pi}{\theta_0}}(\sqrt{\lambda_{1,k}} \ r) J_{\frac{\pi}{\theta_0}}(\sqrt{\lambda_{1,k}} \ s) s ds,$$

which is a function of r alone.

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