IN SEARCH OF HOT SPOTS*

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Without a doubt, one of the most fundamental operators in the development of mathematical physics is the Laplace operator (Laplacian)

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and its higher dimensional analogues. This operator, often called the Laplacian after the mathematical astronomer, the Marquis Pierre–Simon De Laplace (1749–1827), lies at the heart of the mathematical descriptions of heat, light, sound, electricity, magnetism, gravitation, and fluid motion. It has been extensively studied by mathematicians and physicists for more than 200 years. These studies have often focused on the geometric properties of its solutions.

One may surmise that anything worth knowing about this operator would already be known. However, it has often occurred in the development of mathematics that simply stated problems require the introduction of new and sophisticated tools to deal with them. Such seems to be the case for the "hot spots conjecture made in 1974 by Jeffrey Rauch of the University of Michigan. In terms of the theory of heat conduction of Joseph Fourier (1768–1830), the conjecture asserts that if one begins with an initial heat distribution on a plate which is insulated around its boundary and waits for the initial transients to settle down, then the hottest and coldest spots will be found on the boundary of the plate. More technically, for "most" initial temperatures $u(x, y)$ if $H(t)$ is a point at which the temperature $T(t, x, y)$ at time t attains its maximum, then the distance from $H(t)$ to the boundary of the region tends to zero as the time t tends to infinity. That is, the "hot spots" move toward the boundary of the region as time evolves. The temperature $T(t, x, y)$ is a solution to the heat equation $\frac{\partial T}{\partial t} = \Delta T$ with initial condition $u(x, y)$ $(T(0, x, y) = u(x, y))$ and zero normal derivative on the boundary of the region. (The normal derivative of a function at a point on the boundary of the region is the derivative of the function in the direction "perpendicular" to the boundary evaluated at the point.) Thus, $T(t, x, y)$ is a solution to the "Neumann problem" in a region (the plate) of the plane.

The "hot spots" conjecture, as it turns out, is a conjecture about the geometry of the first nonconstant eigenfunction of the Laplacian with Neumann boundary conditions. An eigenfunction is a function φ for which $\Delta\varphi$ is a constant multiple

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of the function φ in the region. The constant is called the eigenvalue. The function φ satisfies the Neumann boundary condition if its normal derivative is zero at every point of the boundary of the region. The patterns formed by a thin layer of salt or sand on a vibrating plate give a picture of such eigenfunctions with the salt or sand piling up along the "nodal lines" of the eigenfunction. The "nodal lines" are the points in the region where the eigenfunctions are zero. The first nonconstant eigenfunction is the lowest mode of vibration and the frequency of vibration is the first nonzero eigenvalue.

Much is known about the geometric and analytic properties of eigenfunctions and eigenvalues of the Laplacian. For example, for regions (plates) of finite area there are infinitely many eigenfunctions, and the eigenvalues form a discrete set of real numbers which become arbitrarily large. That is, the sequence of eigenvalues diverges to infinity. The eigenfunctions of the Laplacian provide the building blocks for the solutions of the heat equation. Any solution of the heat equation with initial temperature u can be written as an infinite series involving the eigenvalues, the eigenfunctions, and integrals of the eigenfunctions against the initial temperature u. From this one is able to conclude that the "hot spots" conjecture is "equivalent" to the statement that the first nonconstant eigenfunction for the Neumann problem in any bounded planar region, regardless of its shape, attains its maximum and its minimum on the boundary and only on the boundary of the region.

Despite the efforts of many, the "hot spots" conjecture remained completely open for more than 25 years. In the article, "On the Hot Spots Conjecture of Jeff Rauch," published in the Journal of Functional Analysis in early 1999, Krysztof Burdzy (University of Washington) and I reformulated the conjecture in terms of Brownian motion with reflection on the boundary of the region. Brownian motion is a random process (a stochastic process) of enormous theoretical and practical significance. It originated in the work of the English botanist Robert Brown (1773–1858) who in 1828 wrote an article entitled "A brief account of microscopical observations made in the months of June, July and August, 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies." What Brown observed is that "pollen grains suspended in water undergo a continual swarming motion." A mathematical formulation for Brownian motion was given by Albert Einstein in 1905. Norbert Wiener put this formulation in a rigorous foundation in 1923 by proving the existence of "Wiener measure" on the space of continuous paths.

The connections of Brownian motion to boundary value problems were established in the forties and fifties by Mark Kac, Joseph Doob, Shizuo Kakutani, and others. In particular, it has been known for many years that given an initial temperature u the temperature $T(t,x,y)$ can be obtained by averaging, with respect to Wiener measure, the function u over all the Brownian particles at time t which begin their journeys at the point (x,y) and undergo a reflection perpendicular to the boundary of the region whenever they reach the boundary. Using this formulation and some new techniques on "coupling" of Brownian particles, Burdzy and I proved the "hot spots" conjecture for certain convex and nonconvex regions provided these have some symmetry relative to one of the coordinate axes. The method of proof also works for many regions (plates) which do not have symmetry such as obtuse triangles. In fact, for any obtuse triangle "the first" nonconstant Neumann eigenfunction is monotone on every line segment in the triangle which is parallel to the longest side of the triangle. Hence, its maximum cannot be attained inside the triangle. Prior to this work the conjecture had only been verified for rectangular and circular plates where formulas for the eigenfunctions and eigenvalues are explicitly known.

The results and the Brownian motion techniques used in our article have received considerable attention. In a subsequent paper, "A counterexample to the hot spots conjecture," published in the journal Annals of Mathematics in late 1999, Burdzy and Wendelin Werner (Universit Paris-Sud, France) used these techniques to give an example of a region with two holes where the "hot spots" conjecture fails. More recently, David Jerison (MIT) and Nikolai Nadirashvili (University of Chicago) have given non-Brownian motion proofs but only for convex regions with two axes of symmetry. I believe the "hot spots" conjecture to be true for any bounded planar region with no holes. This modified "hot spots" conjecture remains open even for the case of arbitrary convex regions. An article entitled "Holes and Hot Spots in Nature" describing the "hot spots" conjecture and the progress made up to date was published in the October 1999 issue of the journal Nature. Quoting from this article, "the geometry of the Laplace operator does not reveal its secrets lightly but it undoubtedly remains a hot topic."

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