

# SYMMETRIZATION OF LÉVY PROCESSES AND APPLICATIONS

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ABSTRACT. It is shown that many of the classical generalized isoperimetric inequalities for the Laplacian when viewed in terms of Brownian motion extend to a wide class of Lévy processes. The results are derived from the multiple integral inequalities of Brascamp, Lieb and Luttinger but the probabilistic structure of the processes plays a crucial role in the proofs.

## 1. INTRODUCTION

Let  $D$  be an open connected set in  $\mathbb{R}^d$  of finite Lebesgue measure. Henceforth we shall refer to such sets simply as domains. We will denote by  $D^*$  the open ball in  $\mathbb{R}^d$  centered at the origin 0 with the same Lebesgue measure as  $D$ , and  $|D|$  will denote the Lebesgue measure of  $D$ . There is a large class of quantities which are related to Brownian motion killed upon leaving  $D$  that are maximized, or minimized, by the corresponding quantities for  $D^*$ . Such results often go by the name of *generalized isoperimetric inequalities*. They include the celebrated Rayleigh-Faber-Krahn inequality on the first eigenvalue of the Dirichlet Laplacian, inequalities for transition densities (heat kernels), Green functions, and electrostatic capacities (see [4], [16], [17], [18] and [19]).

Many of these *isoperimetric inequalities* can be beautifully formulated in terms of exit times of the Brownian motion  $B_t$  from the domain  $D$ . For example, if  $\tau_D$  is the first exit time of  $B_t$  from  $D$ , then for all  $x \in D$

$$P^x \{ \tau_D > 0 \} \leq P^0 \{ \tau_{D^*} > 0 \}, \quad (1.1)$$

where 0 is the origin of  $\mathbb{R}^d$ . Inequality (1.1) contains not only the classical Rayleigh-Faber-Krahn inequality but inequalities for heat kernels and Green functions as well. This inequality is now classical and can be found in many places in the literature. For one of its first occurrences, using the Brascamp-Lieb-Luttinger multiple integrals techniques, please see Aizenman and Simon [1]. Similar inequalities can be obtained by these methods for domains of fixed inradius rather than fixed volume. For more on this, we refer the reader to [5] and [12]. Also, versions of some of these results hold

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for Brownian motion on spheres and hyperbolic spaces, see [8] and references therein.

Once these isoperimetric-type inequalities are formulated in terms of exit times of Brownian motion, it is completely natural to enquire as to their validity for other stochastic processes, and particularly for more general Lévy processes whose generators, as pseudo differential operators, are natural extensions of the Laplacian. Such extensions have been obtained in recent years for the so called “symmetric stable processes” in  $\mathbb{R}^d$  and for more general processes obtained from subordination of Brownian motion. We refer the reader to [5], [6], [12], [22].

The purpose of this paper is to show that many of these results continue to hold for very general Lévy processes. At the heart of these extensions are the rearrangement inequalities of Brascamp, Lieb and Luttinger [7]. However, the probabilistic structure of Lévy processes enters in a very crucial way. Of particular importance for our method is the fact, derived from the Lévy-Khintchine formula, that our processes are weak limits of sums of a compound Poisson process and a Gaussian process.

We begin with a general description of Lévy processes. A Lévy process  $X_t$  in  $\mathbb{R}^d$  is a stochastic process with independent and stationary increments which is “stochastically” continuous. That is, for all  $0 < s < t < \infty$ ,  $A \subset \mathbb{R}^d$ ,

$$P^x \{ X_t - X_s \in A \} = P^0 \{ X_{t-s} \in A \},$$

for any given sequence of ordered times  $0 < t_1 < t_2 < \dots < t_m < \infty$ , the random variables  $X_{t_1} - X_0$ ,  $X_{t_2} - X_{t_1}$ ,  $\dots$ ,  $X_{t_m} - X_{t_{m-1}}$  are independent, and for all  $\varepsilon > 0$ ,

$$\lim_{t \rightarrow s} P^x \{ |X_t - X_s| > \varepsilon \} = 0.$$

The celebrated Lévy-Khintchine formula [21] guarantees the existence of a triple  $(b, \mathbb{A}, \nu)$  such that the characteristic function of the process is given by

$$E^x \left[ e^{i\xi \cdot X_t} \right] = e^{-t\Psi(\xi) + i\xi \cdot x}, \quad (1.2)$$

where

$$\Psi(\xi) = -i\langle b, \xi \rangle + \frac{1}{2}\langle \mathbb{A} \cdot \xi, \xi \rangle + \int_{\mathbb{R}^d} \left[ 1 + i\langle \xi, y \rangle \mathbb{I}_B - e^{i\xi \cdot y} \right] d\nu(y).$$

Here,  $b \in \mathbb{R}^d$ ,  $\mathbb{A}$  is a nonnegative  $d \times d$  symmetric matrix,  $\mathbb{I}_B$  is the indicator function of the ball  $B$  centered at the origin of radius 1, and  $\nu$  is a measure on  $\mathbb{R}^d$  such that

$$\int_{\mathbb{R}^d} \frac{|y|^2}{1 + |y|^2} d\nu(y) < \infty \text{ and } \nu(\{0\}) = 0. \quad (1.3)$$

The triple  $(b, \mathbb{A}, \nu)$  is called the characteristics of the process and the measure  $\nu$  is called the Lévy measure of the process. Conversely, given a triple  $(b, \mathbb{A}, \nu)$  with such properties there is Lévy processes corresponding to it. We will use the fact that any Lévy process has a version with paths that are right continuous with left limits, so called “càdlàg” paths.

Next we recall the basic facts on symmetrization needed to state our results, more details on the properties of symmetrization used in this paper can be found in the appendix in Section 6. Given a positive measurable function  $f$ , its symmetric decreasing rearrangement  $f^*$  is the unique function satisfying

$$\begin{aligned} f^*(x) &= f^*(y), \text{ if } |x| = |y|, \\ f^*(x) &\leq f^*(y), \text{ if } |x| \geq |y|, \\ \lim_{|x| \rightarrow |y|^+} f^*(x) &= f^*(y), \end{aligned}$$

and

$$m \{f > t\} = m \{f^* > t\}, \tag{1.4}$$

for all  $t \geq 0$ . Following [14], under the assumption that  $f$  vanishes at infinity, an explicit expression for this function is:

$$f^*(x) = \int_0^\infty \chi_{\{|f|>t\}}^*(x) dt.$$

This explicit representation is used only at the end of section §4 in the case that  $f$  is the indicator function of an open set of finite area .

For symmetrization purposes, in this paper we will only consider Lévy measures  $\nu$  that are absolutely continuous with respect to the Lebesgue measure  $m$ . It may be that some of the results in this paper hold for more general Lévy processes but at this stage we are not able to go beyond the absolute continuity case. Let  $\phi$  be the density of  $\nu$  and  $\phi^*$  be its symmetric decreasing rearrangement. Since the function

$$\psi(y) = 1 - \frac{|y|^2}{1 + |y|^2}$$

is a positive, decreasing and radially symmetric, that is,  $\psi^* = \psi$ , it follows that (see Theorem 3.4 in [14])

$$\int_{\mathbb{R}^d} \frac{|y|^2}{1 + |y|^2} \phi^*(y) dy \leq \int_{\mathbb{R}^d} \frac{|y|^2}{1 + |y|^2} \phi(y) dy < \infty. \tag{1.5}$$

Hence the measure  $\phi^*(y) dy$  satisfies (1.3) and it is also a Lévy measure.

We denote the  $d \times d$  identity matrix by  $I_d$  and the determinant of  $\mathbb{A}$  by  $\det \mathbb{A}$ . Set  $\mathbb{A}^* = (\det \mathbb{A})^{1/d} I_d$  and define  $X_t^*$  to be the rotationally invariant Lévy process in  $\mathbb{R}^d$  associated to the triple  $(0, \mathbb{A}^*, \phi^*(y) dy)$ . We will often refer to  $X_t^*$  as the symmetrization of  $X_t$ .

Notice that

$$E^x \left[ e^{i\xi \cdot X_t^*} \right] = e^{-t\Psi^*(\xi) + i\xi \cdot x}, \tag{1.6}$$

where

$$\begin{aligned} \Psi^*(\xi) &= \frac{1}{2} \langle \mathbb{A}^* \cdot \xi, \xi \rangle + \int_{\mathbb{R}^d} \left[ 1 - e^{i\xi \cdot y} \right] \phi^*(y) dy \\ &= \frac{1}{2} \langle \mathbb{A}^* \cdot \xi, \xi \rangle + \int_{\mathbb{R}^d} [1 - \cos(\xi \cdot y)] \phi^*(y) dy, \end{aligned}$$

where the last inequality follows from the fact that  $\phi^*$  is symmetric and  $y \rightarrow \sin(\xi \cdot y)$  is antisymmetric.

The next two theorems are the main results of this paper.

**Theorem 1.1.** *Suppose  $X_t$  is a Lévy process with Lévy measure absolutely continuous with respect to the Lebesgue measure and let  $X_t^*$  be the symmetrization of  $X_t$  constructed as above. Let  $f_1, \dots, f_m$  be nonnegative continuous functions and let  $D_1, \dots, D_m$  be domains in  $\mathbb{R}^d$ . Then for all  $z \in \mathbb{R}^d$ ,*

$$E^z \left[ \prod_{i=1}^m f_i(X_{t_i}) \mathbb{1}_{D_i}(X_{t_i}) \right] \leq E^0 \left[ \prod_{i=1}^m f_i^*(X_{t_i}^*) \mathbb{1}_{D_i^*}(X_{t_i}^*) \right], \quad (1.7)$$

for all  $0 \leq t_1 \leq \dots \leq t_m$ .

One easily proves that this result is not valid when the functions  $f_1, \dots, f_m$  are not continuous. However, if we assume further that the distributions of  $X_t$  and  $X_t^*$  are absolutely continuous with respect to the Lebesgue measure, we can extend Theorem 1.1 to measurable functions.

**Theorem 1.2.** *Suppose  $X_t$  is a Lévy process with Lévy measure absolutely continuous with respect to the Lebesgue measure and let  $X_t^*$  be the symmetrization of  $X_t$  as constructed above. Assume further that for all  $t > 0$  the distributions of  $X_t$  and  $X_t^*$  are absolutely continuous with respect to the Lebesgue measure. That is, for all  $t > 0$ ,*

$$P^x \{X_t \in A\} = \int_A p(t, x, y) dy$$

and

$$P^x \{X_t^* \in A\} = \int_A p^*(t, x, y) dy,$$

for any Borel set  $A \subset \mathbb{R}^d$ . Let  $f_1, \dots, f_m$ ,  $m \geq 1$ , be nonnegative measurable functions. Then for all  $z \in \mathbb{R}^d$ ,

$$E^z \left[ \prod_{i=1}^m f_i(X_{t_i}) \right] \leq E^0 \left[ \prod_{i=1}^m f_i^*(X_{t_i}^*) \right],$$

for all  $0 \leq t_1 \leq \dots \leq t_m$ .

**Remark 1.3.** *A sufficient condition for the absolute continuity of the law of a Lévy process is given in [21], page 177. In our case this is satisfied by both  $X_t$  and  $X_t^*$  whenever  $\det(\mathbb{A}) > 0$  or  $\phi \notin L^1(\mathbb{R}^d)$ .*

As we shall see below, Theorem 1.1 implies a generalization of (1.1) to Lévy processes whose Lévy measure is absolutely continuous with respect to the Lebesgue measure. In fact, we will obtain a more general result which applies to Schrödinger perturbations of Lévy semigroups. Let  $D \subset \mathbb{R}^d$  be a domain of finite measure, and consider

$$\tau_D^X = \inf \{t > 0 : X_t \notin D\},$$

the first exit time of  $X_t$  from  $D$ . We also have the corresponding quantity  $\tau_{D^*}^{X^*}$  for  $X_t^*$  in  $D^*$ . As explained in §5, the following isoperimetric-type inequality is a consequence of Theorem 1.1.

**Theorem 1.4.** *Let  $D$  be a domain in  $\mathbb{R}^d$  of finite measure and  $f$  and  $V$  be nonnegative continuous functions. Suppose  $X_t$  is a Lévy process with Lévy measure absolutely continuous with respect to the Lebesgue measure and  $X_t^*$  is the symmetrization of  $X_t$ . Then for all  $z \in \mathbb{R}^d$  and all  $t > 0$ ,*

$$\begin{aligned} & E^z \left\{ f(X_t) \exp \left( - \int_0^t V(X_s) ds \right); \tau_D^X > t \right\} \\ & \leq E^0 \left\{ f^*(X_t^*) \exp \left( - \int_0^t V^*(X_s^*) ds \right); \tau_{D^*}^{X^*} > t \right\}. \end{aligned} \tag{1.8}$$

Our symmetrization results are based on the following now classical rearrangement inequality of Brascamp, Lieb and Luttinger [7].

**Theorem 1.5.** *Let  $f_1, \dots, f_m$  be nonnegative functions in  $\mathbb{R}^d$  and denote by  $f_1^*, \dots, f_m^*$  be their symmetric decreasing rearrangements. Then*

$$\begin{aligned} & \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{j=1}^m f_j \left( \sum_{i=1}^k b_{ji} x_i \right) dx_1 \cdots dx_k \leq \\ & \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{j=1}^m f_j^* \left( \sum_{i=1}^k b_{ji} x_i \right) dx_1 \cdots dx_k, \end{aligned}$$

for all positive integers  $k, m$ , and any  $m \times k$  matrix  $B = [b_{ji}]$ .

As explained in [5] and [12], if we additionally assume that the process  $X_t$  is isotropic unimodal, Theorem 1.1 is an immediate consequence of Theorem 1.5. Recall that  $X_t$  is isotropic unimodal if it has transition densities  $p(t, x, y)$  of the form

$$p(t, x, y) = q_t(|x - y|), \tag{1.9}$$

where  $q_t$  is a function such that

$$q_t(r_1) \leq q_t(r_2),$$

for all  $r_1 \geq r_2$  and all  $t > 0$ . Thus for such Lévy processes (with  $y$  fixed)

$$[p(t, \cdot, y)]^* = p(t, \cdot, 0),$$

and  $X_t = X_t^*$ . This class of Lévy processes includes the Brownian motion, rotational invariant symmetric  $\alpha$ -stable processes, relativistic stable processes and any other subordinations of the Brownian motion. Notice that in our more general setting, and under the assumption that the distribution of  $X_t$  is absolutely continuous relative to the Lebesgue measure, we cannot even ensure that  $[p(t, \cdot, y)]^*$  is the transition density of a Lévy processes.

The rest of the paper is organized as follows. In §2 we will prove Theorem 1.1 for Compound Poisson processes. We will consider the case of Gaussian Lévy processes in §3. Theorem 1.1 and Theorem 1.2 are proved in §4, using

a weak approximation of  $X_t$  and  $X_t^*$  by Lévy processes of the form  $G_t + C_t$ , where  $G_t$  is a nondegenerate Gaussian process and  $C_t$  is an independent compound process. We will then show some of the applications in §5. For the convenience of the reader, and for completeness, we include an appendix in §6 with various facts on symmetrization used in the proofs.

## 2. SYMMETRIZATION OF COMPOUND POISSON PROCESSES

In this section we prove a version of the inequality (1.7) for compound Poisson processes, in the case that  $D_i = \mathbb{R}^d$  for all  $1 \leq i \leq m$ . This result, combined with the results in §3, will lead to a proof of Theorem 1.1.

We start by recalling the structure of compound Poisson processes in terms of random walks. If  $C_t$  is a compound Poisson process, starting at  $x$ , then its characteristic function is given by

$$E^x \left( e^{i\xi \cdot C_t} \right) = e^{ix \cdot \xi - t\Psi_C(\xi)}, \quad (2.1)$$

where

$$\Psi_C(\xi) = c \int_{\mathbb{R}^d} \left[ 1 - e^{i\xi \cdot y} \right] \phi(y) dy,$$

and  $\phi$  is a probability density. We now use the fact that  $C_t$  can be written in terms of sums of independent random variables. That is, by Theorem 4.3 [21] there exist a Poisson process  $N_t$  with parameter  $c > 0$ , and a sequence of i.i.d. random variables  $\{X_n\}_{n=1}^{\infty}$  such that

- (1)  $\{N_t\}_{t>0}$  and  $\{X_n\}_{n=1}^{\infty}$  are independent,
- (2)  $\phi(y)$  is the density of the distribution of  $X_i$ ,  $i \geq 1$ ,
- (3)  $C_t = S_{N_t} + x$ , where  $S_n = X_1 + \dots + X_n$  and  $S_0 = 0$ .

Hence if  $f$  is a nonnegative Borel function, then

$$\begin{aligned} E^x [f(C_t)] &= E^x [f(S_{N_t})] \\ &= \sum_{n=0}^{\infty} P[N_t = n] E[f(x + S_n)]. \end{aligned} \quad (2.2)$$

Let  $\phi^*$  be the symmetric decreasing rearrangement of  $\phi$ . Since

$$\int_{\mathbb{R}^d} \phi^*(y) dy = \int_{\mathbb{R}^d} \phi(y) dy = 1,$$

we can consider a new sequence of i.i.d. random variables  $\{X_n^*\}_{n=1}^{\infty}$  independent of  $N_t$  such that  $\phi^*(y)$  is the density of  $X_n^*$ . Define  $S_n^* = X_1^* + \dots + X_n^*$  to be the corresponding random walk and  $C_t^*$  the compound Poisson process given by

$$C_t^* = S_{N_t}^*.$$

Notice that the distribution  $\mu_t$  of  $C_t$  is not absolutely continuous with respect to Lebesgue measure. However, if  $C_0 = x$  we have the following representation

$$\mu_t = P[N_t = 0] \delta_x + \sum_{k=1}^{\infty} P[N_t = k] \mu_k(x), \quad (2.3)$$

with  $\mu_k$  the distribution of  $S_k$ . That is,

$$\begin{aligned} E^x [f(S_k)] &= \int_{\mathbb{R}^d} f(x+y) d\mu_k(y) \\ &= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} f\left(\sum_{j=0}^k x_j\right) \prod_{i=1}^k \phi(x_i) dx_1 \cdots dx_k. \end{aligned}$$

Thus if  $f$  is a bounded measurable function we have that

$$f^*(S_0^*) = f^*(0) = \|f\|_{L^\infty}$$

and the inequality

$$f(S_0 + x) = f(x) \leq f^*(S_0^*),$$

can only be asserted to hold almost everywhere.

The next result is a version of inequality (1.7) for random walks where the functions are only assumed to be measurable but the conclusion is only a.e. with respect to the Lebesgue measure. We label it as ‘‘Theorem’’ because it may be of some independent interest.

**Theorem 2.1.** *Let  $f_1, \dots, f_m$  nonnegative functions and  $k_1 \leq \dots \leq k_m$  nonnegative integers. Then*

$$E \left[ \prod_{i=1}^m f_i(x_0 + S_{k_i}) \right] \leq E \left[ \prod_{i=1}^m f_i^*(S_{k_i}^*) \right], \quad (2.4)$$

*almost everywhere in  $x_0$ , with respect to Lebesgue measure. In the case that  $f_1, \dots, f_m$  are continuous, (2.4) holds pointwise.*

*Proof.* Given that  $X_1, \dots, X_{k_m}$  are i.i.d we can apply Theorem 1.5 to obtain that

$$\begin{aligned}
& E \left[ \prod_{i=1}^m f_i(x_0 + S_{k_i}) \right] \\
&= E \left[ \prod_{i=1}^m f_i(x_0 + X_1 + \dots + X_{k_i}) \right] \\
&= \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \left[ \prod_{i=1}^m f_i \left( \sum_{j=0}^{k_i} x_j \right) \right] \prod_{i=1}^{k_m} \phi(x_i) dx_1 \dots dx_m \\
&\leq \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \left[ \prod_{i=1}^m f_i^* \left( \sum_{j=1}^{k_i} x_j \right) \right] \prod_{i=1}^{k_m} \phi^*(x_i) dx_1 \dots dx_m \\
&= E \left[ \prod_{i=1}^m f_i^*(S_{k_i}^*) \right]. \tag{2.5}
\end{aligned}$$

□

We can now prove the inequality (1.7) for the compound Poisson process  $C_t$  under the assumption that all the domains are  $\mathbb{R}^d$ . Let  $f_1, \dots, f_m$  be nonnegative continuous functions. Since  $N_t$  is independent of  $S_k$  and  $S_k^*$ , we can combine (2.2) and Theorem 2.1 to obtain

$$\begin{aligned}
& E^x \left[ \prod_{i=1}^m f_i(S_{N_{t_i}}) \right] \\
&= \sum_{k_1 \leq k_2 \leq \dots \leq k_m}^{\infty} P[N_{t_1} = k_1, \dots, N_{t_m} = k_m] E \left[ \prod_{i=1}^m f_i(x + S_{k_i}) \right] \\
&\leq \sum_{k_1 \leq k_2 \leq \dots \leq k_m}^{\infty} P[N_{t_1} = k_1, \dots, N_{t_m} = k_m] E \left[ \prod_{i=1}^m f_i^*(S_{k_i}^*) \right] \\
&= E^0 \left[ \prod_{i=1}^m f_i^*(S_{N_{t_i}}^*) \right]. \tag{2.6}
\end{aligned}$$

Thus

$$E^x \left[ \prod_{i=1}^m f_i(C_{t_i}) \right] \leq E^0 \left[ \prod_{i=1}^m f_i^*(C_{t_i}^*) \right], \tag{2.7}$$

which is desired result.

### 3. SYMMETRIZATION OF GAUSSIAN PROCESSES

Let  $G_t$  be a nondegenerate Gaussian process. Then there exist  $b \in \mathbb{R}^d$  and a strictly positive definite symmetric  $d \times d$  matrix  $\mathbb{A}$  such that the density

of  $G_t$  is given by

$$f_{\mathbb{A},b}(t,x) = \frac{1}{[2t\pi]^{d/2} \sqrt{\det \mathbb{A}}} \exp \left[ -\frac{1}{2t} \langle (x - tb), \mathbb{A}^{-1} \cdot (x - tb) \rangle \right],$$

for all  $x \in \mathbb{R}^d$  and all  $t > 0$ .

Let us first assume that  $b = 0$ . Let  $u > 0$ , then

$$\begin{aligned} & \left\{ x \in \mathbb{R}^d : f_{\mathbb{A},0}(t,x) > u \right\} \\ &= \left\{ x \in \mathbb{R}^d : \langle x, \mathbb{A}^{-1} \cdot x \rangle < t \ln \left[ \frac{1}{(2t\pi)^d u^2 \det \mathbb{A}} \right] \right\} \\ &= \left\{ x \in \mathbb{R}^d : \langle \mathbb{A}^{-1/2} \cdot x, \mathbb{A}^{-1/2} \cdot x \rangle < t \ln \left[ \frac{1}{(2t\pi)^d u^2 \det \mathbb{A}} \right] \right\}. \end{aligned}$$

A change of variables implies that

$$m \left\{ x \in \mathbb{R}^d : f_{\mathbb{A},0}(t,x) > u \right\} = \frac{1}{[\det \mathbb{A}]^{1/2}} m \left\{ B(r_{\mathbb{A},d,u,t}) \right\},$$

where

$$r_{\mathbb{A},d,u,t} = t \ln \left[ \frac{1}{(2t\pi)^d u^2 \det \mathbb{A}} \right].$$

Consider the diagonal matrix

$$\mathbb{A}^* = (\det \mathbb{A})^{\frac{1}{d}} I_d.$$

Then

$$m \left\{ x \in \mathbb{R}^d : f_{\mathbb{A},0}(t,x) > u \right\} = m \left\{ x \in \mathbb{R}^d : f_{\mathbb{A}^*,0}(t,x) > u \right\},$$

for all  $u > 0$ . Given that  $f_{\mathbb{A}^*,0}(t,x)$  is rotational invariant and radially decreasing, we conclude that

$$[f_{\mathbb{A},b}(t,x)]^* = [f_{\mathbb{A},0}(t,x - tb)]^* = f_{\mathbb{A}^*,0}(t,x). \quad (3.1)$$

If  $G_t$  is a degenerate Gaussian process, then

$$E \left( e^{i\xi \cdot G_t} \right) = \exp \left( itb \cdot \xi - i\frac{t}{2} \langle \mathbb{A} \cdot \xi, \xi \rangle \right), \quad (3.2)$$

where  $\mathbb{A}$  is a positive definite  $d \times d$  matrix such that  $\det \mathbb{A} = 0$ .

Let  $\{v_1, \dots, v_d\}$  be the orthonormal eigenvectors of  $\mathbb{A}$  with eigenvalues  $\lambda_1, \dots, \lambda_d$ . We can assume that  $\{\lambda_1, \dots, \lambda_k\}$ ,  $1 \leq k < d$ , are the nonzero eigenvalues of  $\mathbb{A}$ . Let  $W$  be the subspace spanned by  $v_1, \dots, v_k$ . Then  $G_t$  can be identified with a non degenerate Gaussian process in the lower dimension space  $W$  and

$$P^z [G_t \in D] = P^z [G_t \in P_W(D)],$$

where  $P_W(D)$  is the projection of  $D$  on the space  $W$ .

Define  $\mathbb{A}^*$  to be the symmetric positive defined matrix with eigenvectors  $v_1, \dots, v_d$  such that

$$\mathbb{A}^* v_i = 0, \quad k < i \leq d,$$

and

$$\mathbb{A}^* v_i = \lambda v_i, \quad 1 \leq i \leq k,$$

where

$$\lambda = (\lambda_1 \cdots \lambda_k)^{1/k}.$$

The arguments of this section imply that

$$\begin{aligned} P^z [G_t \in D] &= P^z [G_t \in P_W(D)] \\ &= P^0 [G_t^* \in D_W^*]. \end{aligned}$$

where  $D_W^*$  is the ball in  $W$ , centered at the origin, with the same  $k$ -dimension measure as  $P_W(D)$ . Hence the corresponding symmetrization for this processes should be done in lower dimensions.

#### 4. SYMMETRIZATION OF LÉVY PROCESSES: PROOF OF THEOREM 1.1

We will now consider general Lévy processes whose Lévy measures are absolutely continuous with respect to the Lebesgue measure. Our proof requires two basic results on symmetrization of functions that are included in the Appendix in §6.

Recall that under our assumptions

$$E^x \left[ e^{i\xi \cdot X_t} \right] = e^{-t\Psi(\xi) + i\xi \cdot x},$$

where

$$\Psi(\xi) = -i\langle b, \xi \rangle + \frac{1}{2} \langle \mathbb{A} \cdot \xi, \xi \rangle + \int_{\mathbb{R}^d} \left[ 1 + i\langle \xi, y \rangle \mathbb{I}_B - e^{i\xi \cdot y} \right] \phi(y) dy,$$

$B$  is the unit ball centered at the origin and  $\phi$  is such that

$$\int_{\mathbb{R}^d} \frac{|y|^2}{1 + |y|^2} \phi(y) dy < \infty. \quad (4.1)$$

Consider the sequence

$$\phi_n(y) = \phi(y) \mathbb{I}_{\{t \in \mathbb{R}: \frac{1}{n} < t\}}(|y|),$$

and let  $\phi_n^*(y)$  be its symmetric decreasing rearrangement. Thanks to (4.1),

$$c_n = \int_{\mathbb{R}^d} \phi_n(y) dy < \infty,$$

and

$$\int_B |y_i| \phi_n(y) dy < \infty, \quad 1 \leq i \leq d,$$

where again  $B$  is the unit ball.

Consider  $C_{n,t}$  a compound Poisson process with characteristic function

$$E \left( e^{i\xi \cdot C_{n,t}} \right) = e^{-t\Psi_{C,n}(\xi)}, \quad (4.2)$$

where

$$\Psi_{C,n}(\xi) = c_n \int_{\mathbb{R}^d} \left[ 1 - e^{i\xi \cdot y} \right] \frac{\phi_n(y)}{c_n} dy.$$

Given that all the eigenvalues of  $\mathbb{A}$  are nonnegative, if  $\{\epsilon_n\}_{n=1}^\infty$  is a sequence of positive numbers converging to zero, then  $\mathbb{A}_n = \mathbb{A} + \epsilon_n I_d$  is a sequence of nonnegative nonsingular matrices. Let  $G_{n,t}$  be a Gaussian process starting at  $x$ , independent of  $C_{n,t}$ , and associated with the matrix  $\mathbb{A}_n$  and the vector  $b_n = b - \int_B y \phi_n(y) dy$ . Set  $X_{n,t} = C_{n,t} + G_{n,t}$ . Since  $C_{n,t}$  and  $G_{n,t}$  are independent,

$$E^x \left[ e^{i\xi \cdot X_{n,t}} \right] = e^{-t\Psi_n(\xi) + i\xi \cdot x},$$

where

$$\begin{aligned} \Psi_n(\xi) &= -i\langle b_n, \xi \rangle + \frac{1}{2} \langle \mathbb{A}_n \cdot \xi, \xi \rangle + \int_{\mathbb{R}^d} \left[ 1 - e^{i\xi \cdot y} \right] \phi_n(y) dy \\ &= -i\langle b, \xi \rangle + \frac{1}{2} \langle \mathbb{A}_n \cdot \xi, \xi \rangle + \int_{\mathbb{R}^d} \left[ 1 + i\langle \xi, y \rangle \mathbb{I}_B - e^{i\xi \cdot y} \right] \phi_n(y) dy. \end{aligned} \quad (4.3)$$

Let  $S_{n,k} = X_1^n + \dots + X_k^n$  be the random walk associated to  $C_{n,t}$ . If  $f_1, \dots, f_m$  are nonnegative continuous functions and  $t_1 \leq \dots \leq t_m$ , then

$$\begin{aligned} & E^x \left[ \prod_{i=1}^m f_i(X_{n,t_i}) \right] \\ &= E^x \left[ \prod_{i=1}^m f_i(C_{n,t_i} + G_{n,t_i}) \right] \\ &= \sum_{k_1 \leq k_2 \leq \dots \leq k_m} P[N_{t_1} = k_1, \dots, N_{t_m} = k_m] E^x \left[ \prod_{i=1}^m f_i(S_{n,k_i} + G_{n,t_i}) \right]. \end{aligned} \quad (4.4)$$

Now Theorem 1.5 and equality (3.1) imply that

$$\begin{aligned} & E^x \left[ \prod_{i=1}^m f_i \left( G_{n,t_i} + \sum_{j=1}^{k_i} X_j^n \right) \right] \\ &= \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \prod_{i=1}^m f_i \left( \sum_{j=0}^{k_i} x_j \right) f_{\mathbb{A}_n, b_n}(t, x_0 - x) \prod_{j=1}^{k_m} \phi(x_j) dx_0 \dots dx_{k_m} \\ &\leq \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \prod_{i=1}^m f_i^* \left( \sum_{j=0}^{k_i} x_j \right) f_{\mathbb{A}_n, 0}^*(t, x_0) \prod_{j=1}^{k_m} \phi^*(x_j) dx_0 \dots dx_{k_m} \\ &= \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \prod_{i=1}^m f_i^* \left( \sum_{j=0}^{k_i} x_j \right) f_{\mathbb{A}_n^*, 0}(t, x_0) \prod_{j=1}^{k_m} \phi^*(x_j) dx_0 \dots dx_{k_m} \\ &= E^x \left[ \prod_{i=1}^m f_i^* (G_{n,t_i}^* + S_{n,k}^*) \right]. \end{aligned} \quad (4.5)$$

This implies that

$$E^x \left[ \prod_{i=1}^m f_i(X_{n,t_i}) \right] \leq E^0 \left[ \prod_{i=1}^m f_i^*(X_{n,t_i}^*) \right]. \quad (4.6)$$

Theorem 1.1 will be a consequence of (4.6) and the following result on weak convergence.

**Theorem 4.1.** *Let  $f_1, \dots, f_k$  be nonnegative bounded continuous functions, and  $0 < t_1 < \dots < t_m$ . Then for all  $x \in \mathbb{R}^d$ ,*

$$\lim_{n \rightarrow \infty} E^x \left[ \prod_{i=1}^k f_i(X_{n,t_i}) \right] = E^x \left[ \prod_{i=1}^k f_i(X_{t_i}) \right], \quad (4.7)$$

and

$$\lim_{n \rightarrow \infty} E^x \left[ \prod_{i=1}^k f_i(X_{n,t_i}^*) \right] = E^x \left[ \prod_{i=1}^k f_i(X_{t_i}^*) \right].$$

*Proof.* Notice that for all  $\xi \in \mathbb{R}^d$ ,

$$\lim_{n \rightarrow \infty} \langle \mathbb{A}_n \cdot \xi, \xi \rangle = \langle \mathbb{A} \cdot \xi, \xi \rangle.$$

Given that there exists  $C \in \mathbb{R}^+$  such that,

$$\left| 1 + i \langle \xi, y \rangle - e^{i \xi \cdot y} \right| \phi_n(y) \leq C |\xi|^2 |y|^2 \phi(y) < \infty, \quad (4.8)$$

for all  $y \in B$ , and

$$\left| 1 - e^{i \xi \cdot y} \right| \phi_n(y) \leq 2 \phi(y) < \infty, \quad (4.9)$$

for all  $y \in \mathbb{R}^d \setminus B$ , it follows from the Dominated Convergence Theorem that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \Psi_n(\xi) \quad (4.10) \\ &= \lim_{n \rightarrow \infty} \left( -i \langle b, \xi \rangle + \frac{1}{2} \langle \mathbb{A}_n \cdot \xi, \xi \rangle + \int_{\mathbb{R}^d} \left[ 1 + i \langle \xi, y \rangle \mathbb{I}_B - e^{i \xi \cdot y} \right] \phi_n(y) dy \right) \\ &= \left( -i \langle b, \xi \rangle + \frac{1}{2} \langle \mathbb{A} \cdot \xi, \xi \rangle + \int_{\mathbb{R}^d} \left[ 1 + i \langle \xi, y \rangle \mathbb{I}_B - e^{i \xi \cdot y} \right] \phi(y) dy \right). \end{aligned}$$

We conclude that

$$\lim_{n \rightarrow \infty} E^x \left[ e^{i \xi \cdot X_{n,t}} \right] = E^x \left[ e^{i \xi \cdot X_t} \right]. \quad (4.11)$$

On the other hand, using the last two inequalities and the fact that

$$\lim_{n \rightarrow \infty} \det \mathbb{A}_n = \det \mathbb{A},$$

we can easily prove that

$$\lim_{n \rightarrow \infty} \langle \mathbb{A}_n^* \cdot \xi, \xi \rangle = \langle \mathbb{A}^* \cdot \xi, \xi \rangle.$$

Lemma 6.1 implies that

$$\phi_n^*(x) \leq \phi^*(x),$$

for all  $x \in \mathbb{R}$ , and all  $n \geq 1$ . In addition, Proposition 6.2 gives that

$$\lim_{n \rightarrow \infty} \phi_n^* = \phi^*, \text{ a.e. .}$$

Thus the same argument used to prove (4.11) yields

$$\lim_{n \rightarrow \infty} E^x \left[ e^{i\xi \cdot X_{n,t}^*} \right] = E^x \left[ e^{i\xi \cdot X_t^*} \right]. \quad (4.12)$$

Now, if  $\xi_1, \dots, \xi_m \in \mathbb{R}^d$ , then

$$\begin{aligned} \sum_{j=1}^m \xi_j \cdot X_{n,t_j} &= (\xi_1 + \dots + \xi_m) \cdot X_{n,t_1} \\ &+ \sum_{j=2}^m (\xi_m + \dots + \xi_j) \cdot (X_{n,t_j} - X_{n,t_{j-1}}). \end{aligned} \quad (4.13)$$

Since  $t_1 < \dots < t_m$  we have that

$$\begin{aligned} &E^x \left\{ \exp \left[ i \sum_{j=1}^m \xi_j \cdot X_{n,t_j} \right] \right\} \\ &= E^x \left\{ \exp [i(\xi_1 + \dots + \xi_m) \cdot X_{n,t_1}] \right\} \\ &\times \prod_{j=2}^m E^0 \left\{ \exp [i(\xi_m + \dots + \xi_j) \cdot (X_{n,t_j} - X_{n,t_{j-1}})] \right\}. \end{aligned} \quad (4.14)$$

The desired result immediately follows from (4.11), (4.12) and the fact that our characteristic functions are continuous at 0. This last observation follows from the Lévy-Khintchine formula.  $\square$

Combining (4.6) and Theorem 4.1, we obtain

$$E^x \left[ \prod_{i=1}^m f_i(X_{t_i}) \right] \leq E^0 \left[ \prod_{i=1}^m f_i^*(X_{t_i}^*) \right]. \quad (4.15)$$

for all nonnegative bounded continuous functions  $f_1, \dots, f_m$ .

Let  $f$  be a nonnegative continuous functions, and consider the sequence

$$f_n = \max\{f, n\}.$$

Then

$$0 \leq f_n(x) \leq f_{n+1}(x) \leq f(x), \text{ for all } x \in \mathbb{R}^d.$$

Thus Proposition 6.1 implies that

$$0 \leq f_n^*(x) \leq f_{n+1}^*(x), \text{ for all } x \in \mathbb{R}^d.$$

Since  $f^*$  is continuous, Proposition 6.2 yield

$$\lim_{n \rightarrow \infty} f_n^*(x) = f^*(x), \text{ for all } x \in \mathbb{R}^d.$$

The Monotone Convergence Theorem for the laws of  $X_t$  and  $X_t^*$  imply (4.15) for all nonnegative continuous functions.

To finish the proof of Theorem 1.1, we must show that we may replace a continuous function  $f_i$  by the indicator (characteristic) function of a domain of finite volume. Let  $O$  be an open set of finite volume and consider

$$\psi_n(x) = 1 - (1 - nd(x, F))_+, \text{ where } F = \mathbb{R}^d \setminus O.$$

Notice that  $\psi_n(x) = 0$  if  $x \in F$ , and  $\psi_n(x) = 1$ , if  $d(x, \mathbb{R}^d \setminus O) \geq \frac{1}{n}$ . In addition,

$$0 \leq \psi_n(x) \leq \psi_{n+1}(x) \leq 1, \text{ for all } x \in \mathbb{R}^d.$$

By Proposition 6.1,

$$0 \leq \psi_n^*(x) \leq \psi_{n+1}^*(x) \leq 1, \text{ for all } x \in \mathbb{R}^d.$$

Therefore

$$\psi_n^*(x) = \int_0^\infty \mathbb{I}_{\{\psi_n^* > t\}}(x) dt = \int_0^1 \mathbb{I}_{\{\psi_n > t\}}^*(x) dt.$$

Let  $O_n = \{x : d(x, \mathbb{R}^d \setminus O) > \frac{1}{n}\}$ ,  $O^* = B(0, r)$ , and  $O_n^* = B(0, r_n)$ . If  $0 < t < 1$ , then

$$m\{\psi_n > t\}^* = m\{\psi_n > t\} > m\{O_n^*\}.$$

Hence, for all  $x$ ,

$$\mathbb{I}_{B(0, r_n)}(x) < \mathbb{I}_{\{\psi_n > t\}}^*(x) < \mathbb{I}_{B(0, r)}(x).$$

Integrating in  $t$  we obtain

$$\mathbb{I}_{B(0, r_n)}(x) \leq \psi_n^*(x) \leq \mathbb{I}_{B(0, r)}(x).$$

We conclude that

$$\lim_{n \rightarrow \infty} \psi_n^*(x) = \mathbb{I}_{B(0, r)}(x), \text{ for all } x \in \mathbb{R},$$

and Theorem 1.1 follows from the Monotone Convergence Theorem.  $\square$

Now we will prove Theorem 1.2. Without loss of generality we can assume that the functions  $f_1, \dots, f_m$  are finite almost everywhere with respect to the Lebesgue measure. Let  $1 \leq i \leq m$ . We first assume that there exists a constant  $M_i$

$$f_i(x) \leq M_i,$$

for all  $x \in \mathbb{R}$ . Then there exists a sequence of nonnegative continuous functions  $\{\phi_n\}_{n=1}^\infty$  such that

$$\lim_{n \rightarrow \infty} \phi_n = f_i,$$

almost everywhere with respect to the Lebesgue measure, and

$$\phi_n(x) \leq M_i,$$

for all  $x \in \mathbb{R}$ . By proposition 6.2

$$\lim_{n \rightarrow \infty} \phi_n^* = f_i^*,$$

almost everywhere with respect to the Lebesgue measure. Thus the absolute continuity of the laws of  $X_{t_i}$  and  $X_{t_i}^*$  with respect to the Lebesgue measure and the Dominated Convergence Theorem yield (4.15) if the functions  $f_1, \dots, f_m$  are bounded.

Finally, if  $f_i$  is not bounded, consider the sequence

$$f_n = \max\{f_i, n\}.$$

As before, Proposition 6.1 and Proposition 6.2 imply

$$f_n^* \leq f_{n+1}^* \leq f_i^* \text{ for all } x \in \mathbb{R},$$

and

$$\lim_{n \rightarrow \infty} f_n^* = f_i^*,$$

almost everywhere with respect to the Lebesgue measure. As before, Theorem 1.2 follows from the absolute continuity of the laws of  $X_{t_i}$  and  $X_{t_i}^*$  with respect to the Lebesgue measure and the Monotone Convergence Theorem.

## 5. SOME APPLICATIONS

In this section we give several applications of Theorem 1.1, we begin with the proof of Theorem 1.4. Recall that

$$\tau_D^X = \inf \{t > 0 : X_t \notin D\}$$

is the first exit time of  $X_t$  from a domain  $D$ . Let  $D_k$  be a sequence of bounded domains with smooth boundaries such that  $\overline{D_k} \subset D_{k+1}$ , and  $\cup_{k=1}^{\infty} D_k = D$ . Since any Lévy process has a version with right continuous paths, we have

$$\begin{aligned} & E^{z_0} \left\{ f(X_t) \exp \left( - \int_0^t V(X_s) ds \right) ; \tau_D^X > t \right\} \\ &= E^{z_0} \left\{ f(X_t) \exp \left( - \int_0^t V(X_s) ds \right) ; X_s \in D, \forall s \in [0, t] \right\} \\ &= \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} E^{z_0} \left\{ f(X_t) \exp \left( - \frac{t}{m} \sum_{i=1}^m V(X_{\frac{it}{m}}) \right) ; X_{\frac{it}{m}} \in D_k, i = 1, \dots, m \right\} \\ &= \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} E^{z_0} \left\{ f(X_t) \prod_{i=1}^m \exp \left( - \frac{t}{m} V(X_{\frac{it}{m}}) \right) \mathbb{I}_{D_k} \left( X_{\frac{it}{m}} \right) \right\}. \end{aligned} \tag{5.1}$$

Since

$$[\exp(-sV(x))]^* = \exp(-sV^*(x)),$$

for all  $s > 0$  and all  $x \in \mathbb{R}^d$ , Theorem 1.1 implies that

$$\begin{aligned} & E^{z_0} \left\{ f(X_t) \prod_{i=1}^m \exp \left( - \frac{t}{m} V(X_{\frac{it}{m}}) \right) \mathbb{I}_{D_k} \left( X_{\frac{it}{m}} \right) \right\} \\ &\leq E^0 \left\{ f^*(X_t^*) \prod_{i=1}^m \exp \left( - \frac{t}{m} V^*(X_{\frac{it}{m}}^*) \right) \mathbb{I}_{D_k^*} \left( X_{\frac{it}{m}}^* \right) \right\}. \end{aligned}$$

Hence we have the following

$$\begin{aligned} & E^z \left\{ f(X_t) \exp \left( - \int_0^t V(X_s) ds \right); \tau_D^X > t \right\} \\ & \leq E^0 \left\{ f^*(X_t^*) \exp \left( - \int_0^t V^*(X_s^*) ds \right); \tau_{D^*}^{X^*} > t \right\}, \end{aligned} \quad (5.2)$$

which is Theorem (1.4). Taking  $V = 0$  and  $f = 1$ , gives

$$P^z \left\{ \tau_D^X > t \right\} \leq P^0 \left\{ \tau_{D^*}^{X^*} > t \right\}, \quad (5.3)$$

which is a generalization of inequality (1.1). Integrating this inequality with respect to  $t$  gives the following result.

**Corollary 5.1.** *If  $\psi$  is a nonnegative increasing function, then*

$$E^z \left[ \psi \left( \tau_D^X \right) \right] \leq E^0 \left[ \psi \left( \tau_{D^*}^{X^*} \right) \right], \quad (5.4)$$

for all  $z \in D$ . In particular

$$E^z \left[ \left( \tau_D^X \right)^p \right] \leq E^0 \left[ \left( \tau_{D^*}^{X^*} \right)^p \right], \quad (5.5)$$

for all  $0 < p < \infty$ .

Our results imply many isoperimetric inequalities for the potentials and the eigenvalues of Schrödinger operators of the form

$$H_{D,V}^X = H_D^X + V,$$

where  $H_D^X$  is the pseudo differential operator associated to  $X_t$  with Dirichlet Boundary conditions on  $D$ . For the convenience of the reader we will give a brief description of the operators and semigroups associated to Lévy processes.

For purposes of our formulae below we define the Fourier transform of an  $L^2(\mathbb{R}^d)$  function as

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx,$$

with

$$f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \widehat{f}(\xi) d\xi.$$

We define the semigroup associated to the Lévy process  $X_t$  by

$$\begin{aligned} T_t f(x) &= E^x [ f(X_t) ] \\ &= \frac{1}{(2\pi)^{d/2}} E^0 \left[ \int_{\mathbb{R}^d} e^{i(X_t+x) \cdot \xi} \widehat{f}(\xi) d\xi \right] \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix \cdot \xi} E^0 \left[ e^{iX_t \cdot \xi} \right] \widehat{f}(\xi) d\xi \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{-t\Psi(\xi)} \widehat{f}(\xi) d\xi. \end{aligned}$$

This semigroup takes  $C_0(\mathbb{R}^d)$  into itself. That is, it is a Feller semigroup. From this we see that, at least formally for  $f \in \mathcal{S}(\mathbb{R}^d)$ , the infinitesimal generator is

$$H^X f(x) = -\frac{\partial T_t f(x)}{\partial t} \Big|_{t=0} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \Psi(\xi) \hat{f}(\xi) d\xi.$$

Then the Lévy-Khintchine formula implies that the operator associated to  $X_t$  is given by

$$\begin{aligned} H^X f(x) &= \sum_{j=1}^d b_j \partial_j f(x) - \frac{1}{2} \sum_{j,k=1}^d a_{jk} \partial_j \partial_k f(x) \\ &+ \int_{\mathbb{R}^d} \left[ f(x+y) - f(x) - y \cdot \nabla f(x) \mathbb{I}_{\{|y|<1\}} \right] d\nu(y), \end{aligned} \quad (5.6)$$

where  $a_{jk}$  are the entries of the matrix  $\mathbb{A}$ . For instance:

- (1) If  $X_t$  is a standard Brownian motion:

$$H^X f = -\frac{1}{2} \Delta f.$$

- (2) If  $X_t$  is a symmetric stable processes of order  $0 < \alpha < 2$ :

$$H^X f = -\left(-\frac{1}{2} \Delta\right)^{\alpha/2} f.$$

- (3) If  $X_t$  is a Poisson process of intensity  $c$ :

$$H^X f(x) = c \left[ f(x+1) - f(x) \right].$$

- (4) If  $X_t$  is a compound Poisson process with measure  $\nu$  and  $c = 1$ :

$$H^X f(x) = \int [f(x+y) - f(x)] d\nu(y).$$

In this paper we are interested not on the “free” semigroup for  $X_t$  but rather on its “killed” semigroup and its perturbation by the potential  $V$ . That is, we want properties of the semigroup

$$T_t^{D,V} f(z) = E^z \left\{ f(X_t) \exp \left( - \int_0^t V(X_s) ds \right); \tau_D^X > t \right\}, \quad (5.7)$$

defined for  $t > 0$ ,  $z \in D$ , and  $f \in L^2(D)$ . Recall our assumption that  $V$  is nonnegative and continuous. Thus,

$$\begin{aligned} |T_t^{D,V} f(z)| &= \left| E^z \left\{ f(X_t) \exp \left( - \int_0^t V(X_s) ds \right); \tau_D^X > t \right\} \right| \\ &\leq E^z \left\{ |f(X_t)|; \tau_D^X > t \right\} = T_t^D |f|(z). \end{aligned} \quad (5.8)$$

For the rest of the paper we shall assume that the distributions of  $X_t$  and  $X_t^*$  have densities  $p^X(t, z, w)$  and  $p^{X^*}(t, z, w)$ , respectively, which are

continuous in both  $z$  and  $w$  for all  $t > 0$ . The killed semigroup has a heat kernel  $p_{D,V}^X(t, z, w)$  satisfying

$$T_t^{D,V} f(z) = \int_D p_{D,V}^X(t, z, w) f(w) dw. \quad (5.9)$$

Inequality (5.2) is equivalent to

$$\int_D f(w) p_{D,V}^X(t, z, w) dw \leq \int_{D^*} f^*(w) p_{D^*,V^*}^{X^*}(t, 0, w) dw, \quad (5.10)$$

for all  $z \in D$  and all  $t > 0$ , and this in fact holds for all nonnegative Borel functions  $f$  by Theorem 1.2. Since  $f$  is arbitrary, the continuity assumption of the kernels together with (5.8) gives that for all  $z, w \in D$ ,

$$p_{D,V}^X(t, z, w) \leq p_{D^*,V^*}^{X^*}(t, 0, 0) \leq p_{D^*}^{X^*}(t, 0, 0) < \infty. \quad (5.11)$$

If in addition  $X_t$  is transient, we can integrate (5.10) in time to obtain the following isoperimetric inequality for the potentials associated to  $X_t$  and  $X_t^*$ .

**Corollary 5.2.** *Suppose both  $X_t$  and  $X_t^*$  are transient and have continuous densities for all  $t > 0$ . Then for all  $z \in D$ ,*

$$\int_D f(w) G_{D,V}^X(z, w) dw \leq \int_{D^*} f^*(w) G_{D^*,V^*}^{X^*}(0, w) dw, \quad (5.12)$$

where  $G_{D,V}^X(z, w)$  and  $G_{D^*,V^*}^{X^*}(0, w)$  are the Green's functions corresponding to  $X_t$  and  $X_t^*$ , respectively.

By inequalities (5.10), (5.12), and Proposition 2.1 in [2] (see also page 671 of [8]), we have

**Corollary 5.3.** *Suppose both  $X_t$  and  $X_t^*$  are symmetric, transient and have continuous densities for all  $t > 0$ . Then for all increasing convex functions  $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ ,*

$$\int_D \Phi(p_{D,V}^X(t, z, w)) dw \leq \int_{D^*} \Phi(p_{D^*,V^*}^{X^*}(t, w, 0)) dw, \quad (5.13)$$

and

$$\int_D \Phi(G_{D,V}^X(z, w)) dw \leq \int_{D^*} \Phi(G_{D^*,V^*}^{X^*}(w, 0)) dw, \quad (5.14)$$

for all  $z \in D$ ,  $t > 0$ .

These Corollaries extend several results in C. Bandle [4], see for example page 214.

The heat kernel  $p_{D,V}^X(t, z, w)$  can also be represented in terms of the multidimensional distributions. One easily proves, see [12], that

$$\begin{aligned} & p_{D,V}^X(t, z, w) \\ &= p^X(t, z, w) E^z \left\{ \exp \left[ - \int_0^t V(X_s) ds \right] ; \tau_D^X > t \mid X_t = w \right\}. \end{aligned} \quad (5.15)$$

If  $0 = t_0 < t_1 < \dots < t_m < t$ , the conditional finite dimensional distribution

$$P^{z_0} \left\{ X_{t_1} \in dz_1, \dots, X_{t_m} \in dz_m \mid X_t = w \right\},$$

is given by

$$\frac{p^X(t - t_m, z_m, w)}{p^X(t, z_0, w)} \prod_{i=1}^m p^X(t_i - t_{i-1}, z_i, z_{i-1}) dz_1 \dots dz_m.$$

Combining (5.15) with the arguments used in (5.1) we have that

$$\begin{aligned} & p_{D,V}^X(t, z, w) \\ &= \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \int_{D_k} \dots \int_{D_k} e^{-\frac{t}{m} \sum_{i=1}^m V(X_{\frac{it}{m}})} \prod_{i=1}^{m+1} p^X\left(\frac{t}{m}, z_i, z_{i-1}\right) dz_1 \dots dz_m, \end{aligned} \quad (5.16)$$

where  $z_0 = z$  and  $z_{m+1} = w$ .

The proof of Theorem 1.1 can be adapted to obtain

$$\int_D p_{D,V}^X(t, w, w) dw \leq \int_{D^*} p_{D^*,V^*}^{X^*}(t, w, w) dw < \infty, \quad (5.17)$$

where the last inequality follows from (5.11) and the fact that  $|D^*| < \infty$ . That is, the trace of the Schrödinger semigroup for  $H_{D,V}^X$  is maximized by the trace of the Schrödinger semigroup  $H_{D^*,V^*}^{X^*}$ .

As explain in [23], the amount of heat contained in the domain  $D$  at time  $t$ , when  $D$  has temperature 1 at  $t = 0$  and the boundary of  $D$  is kept at temperature 0 at all times, is given by

$$Q_t(D) = \int_D \int_D p_D^B(t, z, w) dz dw,$$

where  $B$  is a Brownian motion. Also the torsional rigidity of  $D$  is given by

$$\int_0^\infty Q_t(D) dt = \int_D \int_D G_D^B(z, w) dz dw.$$

Using the representation (5.16), we obtain the following results for the heat content and torsional rigidity of Lévy processes.

**Corollary 5.4.** *Suppose both  $X_t$  and  $X_t^*$  are transient and have continuous densities for all  $t > 0$ . Then for all  $z \in D$  and  $t > 0$ ,*

$$\int_D \int_D p_{D,V}^X(t, z, w) dz dw \leq \int_{D^*} \int_{D^*} p_{D^*,V^*}^{X^*}(t, z, w) dz dw, \quad (5.18)$$

and

$$\int_D \int_D G_{D,V}^X(z, w) dz dw \leq \int_{D^*} \int_{D^*} G_{D^*,V^*}^{X^*}(z, w) dz dw. \quad (5.19)$$

We recall that the semigroup of the process  $X_t$  is self-adjoint in  $L^2$  if and only if the process  $X_t$  is symmetric. That is, for any Borel set  $A \subset \mathbb{R}^d$ ,

$$P^0\{X_t \in A\} = P^0\{X_t \in -A\}.$$

In terms of the exponent in the Lévy-Khintchine formula this leads to the representation (see [3])

$$\Psi(\xi) = \frac{1}{2} \langle \mathbb{A} \cdot \xi, \xi \rangle - \int_{\mathbb{R}^d} [\cos(x \cdot \xi) - 1] d\nu(x),$$

where  $\mathbb{A}$  is a symmetric matrix and  $\nu$  is a symmetric Lévy measure. That is,  $[\nu(A) = \nu(-A)]$  for all Borel sets  $A$ . In this case the general theory of Dirichlet forms (see [10]) guarantees that the Markovian semigroup generated by  $X_t$  gives rise to the self-adjoint generator  $H^X$ . Recall that  $H_{V,D}^X$  is the operator obtained by imposing Dirichlet boundary conditions on  $D$  to the Schrödinger operator  $H^X + V$ . That is, the generator of the killed semigroup  $\{T_t^{D,V}\}_{t \geq 0}$ . By (5.11) we have that

$$\begin{aligned} \int_D p_{D,V}^X(t, w, w) dw &\leq \int_{D^*} p_{D^*,V^*}^{X^*}(t, 0, 0) dw & (5.20) \\ &= p_{D^*,V^*}^{X^*}(t, 0, 0) |D^*| < \infty. \end{aligned}$$

That is, the semigroup of the killed process has finite trace.

Whenever  $D$  is of finite volume, the operator  $T_t^{D,V}$  maps  $L^2(D)$  into  $L^\infty(D)$  for every  $t > 0$ . This follows from (5.11) and the general theory of heat semigroups as described on page 59 of [10]. In fact, under these assumptions it follows from [10] that there exists an orthonormal basis of eigenfunctions  $\{\varphi_{D,V,X}^n\}_{n=1}^\infty$  for  $L^2(D)$  and corresponding eigenvalues  $\{\lambda_n(D, V, X)\}_{n=1}^\infty$  for the semigroup  $\{T_t^{D,V}\}_{t \geq 0}$  satisfying

$$0 < \lambda_1(D, V, X) < \lambda_2(D, V, X) \leq \lambda_3(D, V, X) \leq \dots$$

with  $\lambda_n(D, V, X) \rightarrow \infty$  as  $n \rightarrow \infty$ . That is, the pair

$$\{\varphi_{D,V,X}^n, \lambda_n(D, V, X)\}$$

satisfies

$$T_t^D \varphi_{D,V,X}^n(z) = e^{-\lambda_n(D,V,X)t} \varphi_{D,V,X}^n(z), \quad z \in D, \quad t > 0.$$

Notice that  $\lambda_n(D, V, X)$  is a Dirichlet eigenvalue of  $H^X + V$  on  $D$  with eigenfunction  $\varphi_{D,V,X}^n(z)$ . Under such assumptions we have

$$p_{D,V}^X(t, z, w) = \sum_{n=1}^{\infty} e^{-\lambda_n(D,V,X)t} \varphi_{D,V,X}^n(z) \varphi_{D,V,X}^n(w). \quad (5.21)$$

This eigenfunction expansion for  $p_{D,V}^X(t, z, w)$  implies that

$$-\lambda_1(D, V, X) = \lim_{t \rightarrow \infty} \frac{1}{t} \log E^z \left\{ \exp \left( - \int_0^t V(X_s) ds \right) ; \tau_D^X > t \right\}, \quad (5.22)$$

for all domains  $D$  of finite volume. This gives the following corollary.

**Corollary 5.5** (Faber-Krahn inequality for Lévy Processes). *Suppose both  $X_t$  and  $X_t^*$  are symmetric, transient and have continuous densities for all  $t > 0$ . Then*

$$\lambda_1(D^*, V^*, X^*) \leq \lambda_1(D, V, X). \quad (5.23)$$

More generally, we also have the trace inequality

$$\sum_{n=1}^{\infty} e^{-t\lambda_n(D, X, V)} \leq \sum_{n=1}^{\infty} e^{-t\lambda_n(D^*, X^*, V^*)},$$

valid for all  $t > 0$ .

Finally, denote by  $C_X(A)$  the capacity of the set  $A$  for the process  $X_t$ . In [24], T. Watanabe proved that

$$C_X(A) \geq C_{X^*}(A^*). \quad (5.24)$$

(This question, for Riesz capacities of all orders was raised by P. Mattila in [11].) As explained in [13], this inequality can be obtained from the existing rearrangement inequalities of multiple integrals only in the case that  $X_t$  is isotropic unimodal. For general Lévy processes we have the following representation of the capacity due to Port and Stone [20]

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int P^{z_0} (\tau_{A^c}^X \leq t) dz_0 = C_X(A). \quad (5.25)$$

Since

$$\begin{aligned} & \int P^{z_0} (\tau_{A^c}^X \leq t) dz_0 \\ &= \lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \int \dots \int \left[ 1 - \prod_{j=1}^m I_{A_k^c}(z_j) \right] \prod_{j=1}^m p^X \left( \frac{t}{m}, z_j, z_{j-1} \right) dz_0 \dots dz_m, \end{aligned} \quad (5.26)$$

where  $A_k$  is a decreasing sequence of compact sets such that the interior of  $A_k$  contains  $A$  for all  $k$  and  $\bigcap_{k=1}^{\infty} A_k = A$ . We would expect to obtain (5.24) using a result similar to Theorem 1.5 for more general Lévy processes. However, the corresponding rearrangement inequality for this type of multiple integrals is only known for radially symmetric decreasing functions. That is, only when  $X_t$  is an isotropic unimodal Lévy process.

## 6. APPENDIX: SOME SYMMETRIZATION FACTS

We recall once again that the symmetric decreasing rearrangement  $f^*$  of  $f$  is the function satisfying

$$\begin{aligned} f^*(x) &= f^*(y), \text{ if } |x| = |y|, \\ f^*(x) &\leq f^*(y), \text{ if } |x| \geq |y|, \\ \lim_{|x| \rightarrow |y|^+} f^*(x) &= f^*(y), \end{aligned}$$

and

$$m \{f > t\} = m \{f^* > t\}, \quad (6.1)$$

for all  $t \geq 0$ . Define  $r(f, t)$  as

$$m\{f > t\} = m\{B(0, r(f, t))\}. \quad (6.2)$$

Whenever  $f$  is a radially symmetric nonincreasing function such that  $f$  is right continuous at  $|x_0|$ , we have that

$$r(f, f(x_0)) = \sup\{r > 0 : f(r) > f(x_0)\} = |x_0|. \quad (6.3)$$

In particular, given that  $f^*$  is right continuous as a function of the radius, we have

$$r(f^*, t) = \sup\{r > 0 : f^*(r) > t\},$$

for all  $t > 0$ .

Our first result states that symmetrization preserves continuity and order, see page 81 of [14].

**Proposition 6.1.** *Let  $f$  be a nonnegative function. If  $f$  is continuous, then  $f^*$  is continuous. In addition, if  $g$  is a nonnegative function such that  $g(x) \leq f(x)$  almost everywhere with respect to the Lebesgue measure, then*

$$g^*(x) \leq f^*(x), \quad (6.4)$$

for all  $x \in \mathbb{R}$ .

*Proof.* Let us assume that  $f^*(x)$  is not continuous at  $x_0$ . Given that  $f^*$  is radially symmetric decreasing and right continuous as a function of the radius,  $x_0 \neq 0$  and there exist  $t_1$  such that

$$m\{f^* > s\} = m\{f^* > f^*(x_0)\} \neq 0,$$

for all  $s \in [f^*(x_0), t_1)$ . However the continuity of  $f$  implies that the set

$$\left\{x \in \mathbb{R}^d : f^*(x_0) < f(x) < s\right\}$$

is nonempty and open. Therefore

$$m\{f > s\} < m\{f > f^*(x_0)\},$$

which is a contradiction.

Now if  $g(x) \leq f(x)$ , almost everywhere with respect to the Lebesgue measure, then

$$\begin{aligned} m\{B(0, r(g^*, t))\} &= m\{g > t\} \\ &\leq m\{f > t\} \\ &= m\{B(0, r(f^*, t))\}. \end{aligned} \quad (6.5)$$

That is,

$$r(g^*, t) \leq r(f^*, t), \quad (6.6)$$

for all  $t > 0$ . Let us assume that there exists  $x \in \mathbb{R}^d$  such that  $f^*(|x|) < g^*(|x|)$ . Since  $g^*$  is decreasing and right continuous as a function of the radius, we have

$$r(g^*, g^*(|x|)) < r(g^*, f^*(|x|)).$$

On the other hand, by (6.5) and (6.6),

$$|x| = r(g^*, g^*(|x|)) < r(g^*, f^*(|x|)) \leq r(f^*, f^*(|x|)) = |x|.$$

□

Finally, we prove that symmetrization preserves almost everywhere convergence.

**Proposition 6.2.** *Let  $\{\phi_n\}_{n=1}^\infty$  be a sequence of bounded functions such that*

$$\lim_{n \rightarrow \infty} \phi_n = \phi,$$

*almost everywhere with respect to the Lebesgue measure. If  $x_0$  is a point of continuity of  $\phi^*$  then*

$$\lim_{n \rightarrow \infty} \phi_n^*(x_0) = \phi^*(x_0).$$

*In particular*

$$\lim_{n \rightarrow \infty} \phi_n^* = \phi^*, \tag{6.7}$$

*almost everywhere with respect to the Lebesgue measure.*

*Proof.* Assume there exists  $x_0$  a continuity point of  $\phi^*$  such that

$$\lim_{n \rightarrow \infty} \phi_n^*(x_0) \neq \phi^*(x_0).$$

Then there exists  $\epsilon > 0$  and a subsequence  $n_k$  such that either

$$\phi_{n_k}^*(x_0) > \phi^*(x_0) + \epsilon, \tag{6.8}$$

or

$$\phi_{n_k}^*(x_0) < \phi^*(x_0) - \epsilon. \tag{6.9}$$

Let us assume that (6.8) holds. Since  $x_0$  is a continuity point of  $\phi^*$ , there exists  $0 < \delta < \epsilon$  and  $y_0$  a continuity point of  $\phi^*$  such that

$$\phi^*(x_0) + \delta = \phi^*(y_0), \text{ and } |y_0| < |x_0|.$$

However, thanks to (6.1) and (6.3),

$$\begin{aligned} m\{B(0, |x_0|)\} &= \limsup_{n_k \rightarrow \infty} m\{\phi_{n_k}^* > \phi_{n_k}^*(x_0)\} \\ &\leq \limsup_{n_k \rightarrow \infty} m\{\phi_{n_k}^* > \phi^*(x_0) + \delta\} \\ &= m\{\phi^* > \phi^*(y_0)\} \\ &= m\{B(0, |y_0|)\}, \end{aligned}$$

which is a contradiction. On the other hand, if

$$\phi_{n_k}^*(x_0) < \phi^*(x_0) - \epsilon.$$

There exists  $0 < \delta < \epsilon$  and  $y_0$  a continuity point of  $\phi^*$  such that

$$\phi^*(x_0) - \delta = \phi^*(y_0), \text{ and } |y_0| > |x_0|.$$

Since

$$\begin{aligned}
 m \{B(0, |x_0|)\} &= \limsup_{n_k \rightarrow \infty} m \{\phi_{n_k}^* > \phi_{n_k}^*(x_0)\} \\
 &\geq \limsup_{n_k \rightarrow \infty} m \{\phi_{n_k}^* > \phi^*(x_0) - \delta\} \\
 &= m \{\phi^* > \phi^*(y_0)\} \\
 &= m \{B(0, |y_0|)\},
 \end{aligned}$$

we also obtain a contradiction and this proves the Proposition.  $\square$

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