Finite Dimensional Distributions. What can you do with them?

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1. **Lecture 1**
   - **Question: Is the lowest eigenvalue for stable processes in an interval "knowable" (as in the case of Brownian motion)?**
   - **Best known bounds–irrelevant**
   - **K.L. Chung's (and Taylor's) LIL–also irrelevant**
   - **Quick review of Lévy Processes (probably not necessary nor sufficient)**
   - **"Heat" semigroup for Brownian motion and stable processes and their eigenvalues and eigenfunctions**

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2. **Lecture 2**
   - Eigenvalues and eigenfunctions as survival time (lifetimes) probabilities
   - My long and twisted road to Finite Dimensional Distributions (F.D.D.)
   - Isoperimetric and Isoperimetric–type inequalities: Fixed volume, inradius, diameter, problems, questions, etc...

To study properties of the functions:

$$\Phi_m(x,D) = P_x\{B_{t_1} \in D, B_{t_2} \in D, \ldots, B_{t_m} \in D\}$$

B_t = Brownian motion (twice the speed) in $\mathbb{R}^d$, $D \subset \mathbb{R}^d$ open connected (referred to as "domains"), $x \in D$,

$$0 < t_1 < t_2 \cdots < t_m$$

Same as studying Multiple Integrals:

$$\Phi_m(x,D) = \int_D \cdots \int_D \Pi_{j=1}^m p^{(2)}_{t_j - t_{j-1}}(x_j - x_{j-1}) dx_1 \ldots dx_m,$$

$$x_0 = x \quad \text{and} \quad p_t^{(2)}(y) = \frac{1}{(4\pi t)^{d/2}} e^{-|y|^2/4t}$$

More general, study for any times:

$$\Phi_m(x,D) = \int_D \cdots \int_D \Pi_{j=1}^m p^{(2)}_{t_j}(x_j - x_{j-1}) dx_1 \ldots dx_m, \; 0 < t_j < \infty$$

But Why? Not Terribly Exciting, you may say. I agree!

Question

What is the smallest Dirichelt eigenvalue $\lambda_{1,\alpha}$ for the rotationally invariant stable processes of order $0 < \alpha < 2$ for the interval $(-1, 1)$?

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Question

What is the smallest Dirichelt eigenvalue $\lambda_{1,\alpha}$ for the rotationally invariant stable processes of order $0 < \alpha < 2$ for the interval $(-1, 1)$?

Note: I learned this from Davar Khoshnevisan about 8 years ago.
Has been investigated by

- Investigated by: M.Kac-H. Pollar (1950). H. Widom (1961), J. Taylor (1967), B. Fristedt (1974), J. Bertoin (1996), Khosnevisan–Z. Shi (1998).

- I don't know the answer and, to be perfectly honest, don't care.
- In the process of investigating this "simple" question we "discovered" that little is known about the "fine" spectral theoretic properties of stables.
- More Exciting: The techniques give new Theorem for the Laplacian (BM).

R.B.and R. Latala and P. Méndez (2001) and R.B.and T. Kulczycki (2004)

$$C_{\alpha,d} = \frac{\Gamma(\frac{d}{2})}{2^{\alpha}\Gamma(1+\frac{d}{2})\Gamma(\frac{d+\alpha}{2})}$$

$B(0,1) =$ unit ball in $\mathbb{R}^d$.

$$\frac{1}{C_{\alpha,d}} \leq \lambda_{1,\alpha}(B(0,1)) \leq \frac{1}{C_{\alpha,d}} \frac{B(d/2, \alpha/2+1)}{B(\alpha/2, \alpha+1)}$$

For $\alpha = 1$ (Cauchy processes), $B(0,1) = (-1,1)$ (as in Davar's question)

$$1 \leq \lambda_{1,1} \leq \frac{3\pi}{8} \approx 1.178$$

Note:

$$\frac{3\pi}{8} < \frac{\pi}{2} = \sqrt{\frac{\pi^2}{4}}$$

That is, eigenvalue for Cauchy is not the square root of the one for Brownian motion!

Variational Formula: For any $D \subset \mathbb{R}^d$

The Dirichlet form, $(\mathcal{E}, \mathcal{F})$, for stables processes, $0 < \alpha < 2$, in $\mathbb{R}^d$ is:

$$\mathcal{E}(f,g) = A_{\alpha,d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(f(x) - f(y))(g(x) - g(y))}{|x-y|^{\alpha+d}}\, dx\, dy$$

and

$$\mathcal{F} = \left\{ f \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^2}{|x-y|^{\alpha+d}}\, dx\, dy < \infty \right\}$$

with

$$A_{\alpha,d} = \frac{\alpha \Gamma\left(\frac{d+\alpha}{2}\right)}{2^{1-\alpha} \pi^{d/2} \Gamma\left(1 - \frac{\alpha}{2}\right)}$$

From this we have for any region $D \subset \mathbb{R}^d$:

$$\lambda_{1,\alpha}(D) = \inf \left\{ A_{\alpha,d} \int_D \int_D \frac{|u(x) - u(y)|^2}{|x-y|^{\alpha+d}}\, dx\, dy + 2A_{\alpha,d} \int_D |u(x)|^2 k_D(x) dx \right\}$$

where "inf" is over all $u \in C_0^\infty$ with

$$\int_D |u(y)|^2 dy = 1.$$

$$K_D(x) = \int_{D^c} \frac{dy}{|x-y|^{\alpha+d}}$$

Reference: "Potential Theory of subbordinate Killed Brownian motion" R.Song-Z.Vondracek, 2003.

Theorem (Chungs's LIL. Set $B_t^* = \sup_{0\le s\le t} |B_s|$)

$$\liminf_{t\to\infty} \left(\frac{\log\, \log t}{t}\right)^{1/2} B_t^* = \frac{\pi}{2}, \quad a.s. \tag{1}$$

But, is $\frac{\pi}{2}$ really just our good-old-friend $\frac{\pi}{2}$ or is it something else?

(1) comes from Borel–Cantelli arguments and the "small balls" probability estimate: (See Bertoin, "Lévy Processes" page 224, Bañuelos–Moore, "Probabilistic behavior of harmonic functions," Chapter 4.)

$$P_0\left\{B_1^* < \varepsilon\right\} \approx e^{-\frac{\pi^2}{4\varepsilon^2}}, \ \varepsilon \to 0$$

$$P_0\left\{B_1^* < \varepsilon\right\} = P_0\left\{\frac{1}{\varepsilon}B_t^* < 1\right\} = P_0\left\{B_{\frac{1}{\varepsilon^2}}^* < 1\right\} = P_0\left\{\tau_{(-1,1)} > \frac{1}{\epsilon^2}\right\}$$

$$\tau_{(-1,1)} = \inf\{t > 0 : B_t \notin (-1,1)\} = \text{ first exit time from the interval}$$

As we shall see,

$$P_0 \left\{ \tau_{(-1,1)} > t \right\} \approx e^{-\lambda_1 t} \varphi_1(0) \int_1^1 \varphi_1(y)\, dy, \quad t \to \infty,$$

where λ_1 is the smallest eigenvalue for one half of the Laplacian in the interval $(-1, -1)$ with Dirichlet boundary conditions and φ_1 is the corresponding eigenfunction. That is, $\pi^2/4$ and the "sin" function.

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For any $0 < \alpha < 2$, let X_t^α be the rotationally invariant stable process of order α. A similar statement holds for the "small ball" probabilities and there is

Theorem (J. Taylor 1967)

$$\liminf_{t \to \infty} \left(\frac{\log \, \log t}{t} \right)^{1/\alpha} X_t^* = (\lambda_{1,\alpha})^{1/\alpha}, \quad a.s. \tag{2}$$

For several other occurrences of the eigenvalue in "sample path behavior," see **Erkan Nane:** "Higher order PDE's and iterated Processes" and "Iterated Brownian motion in bounded domains in $\mathbb{R}^n$ "

Constructed by **Paul Lévy** in the 30's (shortly after Wiener constructed Brownian motion). Other names: **de Finetti, Kolmogorov, Khintchine, Itô.**

- Rich stochastic processes, generalizing several basic processes in probability: Brownian motion, Poisson processes, stable processes, subordinators, ...
- Regular enough for interesting analysis and applications. Their paths consist of continuous pieces intermingled with jump discontinuities at random times. Probabilistic and analytic properties studied by many.
- Many Developments in Recent Years:
  - **Applied:** Queueing Theory, Math Finance, Control Theory, Porous Media ...
  - **Pure:** Investigations on the "fine" potential and spectral theoretic properties for subclasses of Lévy processes

- **General Theory**

  1. J. Bertoin, **Lévy Processes**, Cambridge University Press, 1996.
  2. D. Applebaum. **Lévy Processes and Stochastic Calculus,** Cambridge University Press, 2004
  [3. K. Sato. **Lévy Processes and infinitely divisible distributions,** Cambridge Studies in Advanced Mathematics, 1999

- **Applications**

  1. B. Øksendal and A. Sulem. **Applied Stochastic Control of Jump Diffusions,** Springer, 2004
  2. R. Cont and P. Tankov, **Financial Modelling with Jump Processes**, Chapman & Hall/CRC, 2004

- **Recent analytical, probabilistic, geometric, developments:** Work of R. Bass, K. Burdzy, K. Bogdan, Z.Q. Chen, T. Kulczycki, R. Song, J-M. Wu, Z. Vondracek, . . .
  - http://www.math.purdue.edu/∼banuelos (for spectral theoretic stuff)

A Lévy Process is a stochastic process $X = (X_t), t \geq 0$ with

- X has independent and stationary increments
- $X_0 = 0$ (with probability 1)
- X is *stochastically continuous*: For all $\varepsilon > 0$,

$$\lim_{t \to s} P\{|X_t - X_s| > \varepsilon\} = 0$$

Note: Not the same as a.s. continuous paths. However, it gives "cadlag" paths: Right continuous with left limits.

- **Stationary increments:** $0 < s < t < \infty$, $A \in \mathbb{R}^d$ Borel

$$P\{X_t - X_s \in A\} = P\{X_{t-s} \in A\}$$

- **Independent increments:** For any given sequence of ordered times

$$0 < t_1 < t_2 < \cdots < t_m < \infty,$$

the random variables

$$X_{t_1} - X_0,\ X_{t_2} - X_{t_1}, \ldots, X_{t_m} - X_{t_{m-1}}$$

are independent.

The characteristic function of X_t is

$$\varphi_t(\xi) = E\left(e^{i\xi\cdot X_t}\right) = \int_{\mathbb{R}^d} e^{i\xi\cdot x} p_t(dx) = (2\pi)^{d/2}\widehat{p_t}(\xi)$$

where p_t is the distribution of X_t. Notation (same with measures)

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(x)dx, \ \ f(x = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(\xi)d\xi$$

The characteristic function has the form $\varphi_t(\xi) = e^{t\rho(\xi)}$, where

$$\rho(\xi) = ib \cdot \xi - \frac{1}{2}\xi \cdot A\xi + \int_{\mathbb{R}^d} \left(e^{i\xi \cdot x} - 1 - i\xi \cdot x1_{\{|x|<1\}}(x)\right)\nu(dx)$$

for some $b \in \mathbb{R}^d$, a non–negative definite symmetric $n \times n$ matrix A and a Borel measure ν on $\mathbb{R}^d$ with $\nu\{0\} = 0$ and

$$\int_{\mathbb{R}^d} \min\left(|x|^2, 1\right)\nu(dx) < \infty$$

$\rho(\xi)$ is called the **symbol** of the process or the **characteristic exponent.** The triple (b, A, ν) is called the **characteristics of the process.**

Converse also true. Given such a triple we can construct a Lévy process.

Examples

1. **Standard Brownian motion:**
   With $(0, I, 0)$, I the identity matrix,
   $$X_t = B_t, \qquad \text{Standard Brownian motion}$$

2. **Gaussian Processes, "General Brownian motion":**
   $(0, A, 0)$, X_t is "generalized" Brownian motion, mean zero, covariance
   $$E(X_s^j X_t^i) = a_{ij} \min(s, t)$$
   X_t has the normal distribution (assume here that $det(A) > 0$)
   $$\frac{1}{(2\pi t)^{d/2}\sqrt{\det(A)}} \exp\left(-\frac{1}{2t} x \cdot A^{-1} x\right)$$

3. **"Brownian motion" plus drift:** With $(b, A, 0)$ get Brownian motion with a drift:
   $$X_t = bt + B_t$$

4. **Poisson Process:** The Poisson Process $X_t = N_\lambda(t)$ of intensity $\lambda > 0$ is a Lévy process with $(0, 0, \lambda\delta_1)$ where δ_1 is the Dirac delta at 1.

$$P\{N_\lambda(t) = m\} = \frac{e^{-\lambda t}(\lambda t)^m}{m!}, \ m = 1, 2, \ldots$$

$N_\lambda(t)$ has continuous paths except for jumps of size 1 at the random times

$$\tau_m = \inf\{t > 0 : N_\lambda(t) = m\}$$

5. **Compound Poisson Process** obtained by summing iid random variables up to a Poisson Process.

6. **Relativistic Brownian motion** According to quantum mechanics, a particle of rest mass m moving with momentum p has kinetic energy

$$E(p) = \sqrt{m^2c^4 + c^2|p|^2} - mc^2$$

where c is speed of light. Then $\rho(p) = -E(p)$ is the symbol of a Lévy process, called *"relativistic Brownian motion."*

7. **The zeta process:** Consider the Riemann zeta function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-z}}, \qquad z = x + iy \in \mathbb{C}$$

Khintchine: For every fix $x > 1$,

$$\rho_x(y) = \log\left(\frac{\zeta(x+iy)}{\zeta(y)}\right)$$

is the symbol of a Lévy process–in fact, limits of Poissons. (See Applebaum page 39.)

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Biane–Pitman-Yor: "Probability laws related to the Jacobi theta and Riemann Zeta functions and Brownian excursions, Bull. Amer math. Soc., 2001.

M Yor: A note about Selberg's integrals with relation with the beta–gamma algebra, 2006.

8. **The rotationally invariant stable processes:** These are self–similar processes, denoted by X_t^α, in $\mathbb{R}^d$ with symbol

$$\rho(\xi) = -|\xi|^\alpha, \qquad 0 < \alpha \leq 2.$$

That is,

$$\varphi_t(\xi) = E\left(e^{i\xi \cdot X_t^\alpha}\right) = e^{-t|\xi|^\alpha}$$

$\alpha = 2$ is **Brownian motion.** $\alpha = 1$ is the **Cauchy processes.**

$\alpha = 3/2$ is called the **Haltsmark distribution** used to model gravitational fields of stars. (See V.M. Zolotarev (1986) "One dimensional Stable Distributions".)

Transition probabilities:

$$P_x\{X_t^\alpha \in A\} = \int_A p_t^\alpha(x-y)dy, \qquad \text{any Borel} \qquad A \subset \mathbb{R}^d$$

$$p_t^\alpha(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} e^{-t|\xi|^\alpha} d\xi$$

$$p_t^2(x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}, \qquad \alpha = 2, \qquad \textbf{Brownian motion}$$

$$p_t^1(x) = \frac{C_d\, t}{(|x|^2 + t^2)^{\frac{d+1}{2}}}, \qquad \alpha = 1, \qquad \textbf{Cauchy Process}$$

For any $a > 0$, the two processes

$$\{\eta_{(at)}\,;\ t \geq 0\} \quad \text{and} \quad \{a^{1/\alpha}\eta_t\,;\ t \geq 0\},$$

have the same finite dimensional distributions **(self-similarity).**

In the same way, the transition probabilities scale similarly to those for BM:

$$p_t^\alpha(x) = t^{-d/\alpha} p_1^\alpha(t^{-1/\alpha} x)$$

9. Subordinators

A subordinator is a one-dimensional Lévy process $\{T_t\}$ such that

(i) $T_t \geq 0$ a.s. for each $t > 0$

(ii) $T_{t_1} \leq T_{t_2}$ a.s. whenever $t_1 \leq t_2$

Theorem (Bertoin, p.73: Laplace transforms)

$$E(e^{-\lambda T_t}) = e^{-t\psi(\lambda)},\ \lambda > 0,$$

$$\psi(\lambda) = b\lambda + \int_0^\infty \left(1 - e^{-\lambda s}\right) \nu(ds)$$

$b \geq 0$ and the Lévy measure satisfies $\nu(-\infty, 0) = 0$ and $\int_0^\infty \min(s,1)\nu(ds) < \infty$. ψ is called the Laplace exponent of the subordinator.

Example ($\alpha/2$–Stable subordinator): $\psi(\lambda) = \lambda^{\alpha/2}$, $0 < \alpha < 2$ gives the with $b = 0$ and

$$\nu(ds) = \frac{\alpha/2}{\Gamma(1-\alpha/2)} s^{-1-\alpha/2}\, ds$$

Example 2 (Relativistic stable subordinator): $0 < \alpha < 2$ and $m > 0$, $\Psi(\lambda) = (\lambda + m^{2/\alpha})^{\alpha/2} - m$.

$$\nu(ds) = \frac{\alpha/2}{\Gamma(1-\alpha/2)} e^{-m^{2/\alpha} s} s^{-1-\alpha/2} ds$$

Example 3 (Gamma subordinator): $\Psi(\lambda) = \log(1+\lambda)$.

$$\nu(ds) = \frac{e^{-s}}{s} ds$$

Many others: "Geometric stable subordinators, iterated geometric stable subordinators, Bessel subordinators,. . . "

Theorem (**Applebaum, p. 53**)

If X is an arbitrary Lévy process and T is a subordinator independent of X, then $Z_t = X_{T_t}$ is a Lévy process. For any Borel $A \subset \mathbb{R}^d$,

$$p_{Z_t}(A) = \int_0^\infty p_{X_s}(A) p_{T_t}(ds)$$

When $X_t = B_t$ Brownian motion, Z_t is called subordinate Brownian motion. When T_t is the $\alpha/2$ stable subordinator and X is BM, Z is the α rotationally invariant stable process of **Example 8**.

For the Lévy process $\{X(t); t \geq 0\}$, define

$$T_t f(x) = E[f(X(t))|X_0 = x] = E_0[f(X(t) + x)], \qquad f \in \mathcal{S}(\mathbb{R}^d).$$

This is a Feller semigroup (takes $C_0(\mathbb{R}^d)$ into itself). Setting

$$p_t(A) = P_0\{X_t \in A\} \ \text{ (the distribution of } X_t\text{)}$$

we see that (by Fourier inversion formula)

$$T_t f(x) = \int_{\mathbb{R}^d} f(x+y) p_t(dy) = p_t * f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{t\rho(\xi)} \widehat{f}(\xi) d\xi$$

with generator

$$\begin{aligned} Af(x) &= \left.\frac{\partial T_t f(x)}{\partial t}\right|_{t=0} = \lim_{t\to 0} \frac{1}{t}\Big(E_x[f(X(t)] - f(x)\Big) \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \rho(\xi) \hat{f}(\xi) d\xi = \text{a pseudo diff operator, in general} \end{aligned}$$

From the Lévy–Khintchine formula (and properties of the Foruier transform),

$$Af(x) = \sum_{i=1} b_i \partial_i f(x) \quad + \quad \frac{1}{2} \sum_{i,j} a_{i,j} \partial_i \partial_j f(x)$$
$$\quad + \quad \int \Big[f(x+y) - f(x) - y \cdot \nabla f(x) \chi_{\{|y|<1\}} \Big] \nu(dy)$$

Examples:

- Standard Brownian motion:

$$Af(x) = \frac{1}{2} \Delta f(x)$$

- Poisson Process of intensity λ:

$$Af(x) = \lambda \Big[f(x+1) - f(x) \Big]$$

- Rotationally Invariant Stable Processes of order $0 < \alpha < 2$, **Fractional Diffusions:**

$$\begin{aligned} Af(x) &= -(-\Delta)^{\alpha/2} f(x) \\ &= A_{\alpha,d} \int \frac{f(y) - f(x)}{|x-y|^{d+\alpha}} dy \end{aligned}$$

Many questions on the "fine" potential theoretic properties of solutions for $(-\Delta)^{\alpha/2}$ have been studied by many authors in recent years. **Examples:**

- Regularity of heat kernels, general solutions of "heat equation", Sobolev, log-Sobolev inequalities, "intrinsic ultracontractivity," ...
- "Boundary" regularity of solutions, including boundary Harnack principle, "gauge theorems," Fatou theorems, Martin boundary, ...

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(1) "Potential theory of suborndinate BM," (2007), Song-Vondracek
(2) "Potential Theory for Lévy Processes," (2002) Bogdan–Stozs–Sztonyk.
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(4) Also, have a look at the references given in the above papers and search for other recent papers of the authors already mentioned.

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I am interested in the “fine” spectral theoretic properties of these processes

- Estimates on eigenvalues, including the ground state $\lambda_{1,\alpha}$ and the spectral gap $\lambda_{2,\alpha} - \lambda_{1,\alpha}$, Number of “nodal” domains (Courant–Hilbert Nodal domain Theorem), geometric properties of eigenfunctions, including a “Brascamp–Lieb” log–concavity type theorem for $\varphi_{1,\alpha}$, ...

From now on $X_t = X_t^\alpha$ is rotationally invariant stable with symbol

$$\rho(\xi) = -|\xi|^\alpha, \qquad 0 < \alpha \leq 2.$$

Let D be a bounded connected subset of $\mathbb{R}^d$. The first exit time of X_t^α from D is

$$\tau_D = \inf\{t > 0 : X_t^\alpha \notin D\}$$

Heat Semigroup in D is the self-adjoint operator

$$T_t^D f(x) = E_x\Big[f(X_t^\alpha); \tau_D > t\Big], \qquad f \in L^2(D)$$

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$$\begin{aligned} T_t^D f(x) &= E_x\Big[f(X_t^\alpha); \tau_D > t\Big], \qquad f \in L^2(D) \\ &= \int_D p_t^{D,\alpha}(x,y) f(y) dy, \end{aligned}$$

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$$p_t^{D,\alpha}(x,y) = p_t^\alpha(x-y) - E^x(\tau_D < t; p_{t-\tau_D}^\alpha(X_{\tau_D}^\alpha, y)).$$

(K.L. Chung and Z. Zhao, "From Brownian motion to Schrödinger equations" Springer)

$p_t^{D,\alpha}(x,y)$ is called the **Heat Kernel for the stable process in D.**

$$\begin{aligned} p_t^{D,\alpha}(x,y) \le p_t^{\alpha}(x-y) \le p_1^{\alpha}(0)t^{-d/\alpha} &= \left(\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-|\xi|^\alpha}d\xi\right)t^{-d/\alpha} \\ &= t^{-d/\alpha}\frac{\omega_d}{(2\pi)^d\alpha}\int_0^\infty e^{-s}s^{(\frac{n}{\alpha}-1)}ds \\ &= t^{-d/\alpha}\frac{\omega_d\Gamma(d/\alpha)}{(2\pi)^d\alpha} \end{aligned}$$

The general theory of heat semigroups (**R. Bass "Probabilistic Techniques in Analysis," B. Davies, "Heat Semigroups"**) gives an orthonormal basis of eigenfunctions

$$\{\varphi_{m,\alpha}\}_{m=1}^{\infty} \quad on \quad L^2(D)$$

with eigenvalues $\{\lambda_{m,\alpha}\}$ satisfying

$$0 < \lambda_{1,\alpha} < \lambda_{2,\alpha} \le \lambda_{3,\alpha} \le \cdots \to \infty$$

That is,

$$T_t^D\varphi_{m,\alpha}(x) = e^{-\lambda_{m,\alpha}t}\varphi_{m,\alpha}(x), \quad x \in D.$$

$$
\begin{aligned}
p_t^{D,\alpha}(x,y) &= \sum_{m=1}^{\infty} e^{-\lambda_{m,\alpha} t} \varphi_{m,\alpha}(x)\varphi_{m,\alpha}(y) \\
&= e^{-\lambda_{1,\alpha} t} \varphi_{1,\alpha}(x)\varphi_{1,\alpha}(y) + \sum_{m=2}^{\infty} e^{-\lambda_{m,\alpha} t} \varphi_{m,\alpha}(x)\varphi_{m,\alpha}(y)
\end{aligned}
$$

$$
\begin{aligned}
p_t^{D,\alpha}(x,y) &= \sum_{m=1}^{\infty} e^{-\lambda_{m,\alpha} t}\varphi_{m,\alpha}(x)\varphi_{m,\alpha}(y) \\
&= e^{-\lambda_{1,\alpha} t}\varphi_{1,\alpha}(x)\varphi_{1,\alpha}(y) + \sum_{m=2}^{\infty} e^{-\lambda_{m,\alpha} t}\varphi_{m,\alpha}(x)\varphi_{m,\alpha}(y)
\end{aligned}
$$

Theorem (From "Intrinsic Ultracontractivity")

$$
e^{-(\lambda_{2,\alpha}-\lambda_{1,\alpha})t} \le \sup_{x,y\in D}\left|\frac{e^{\lambda_{1,\alpha} t} p_t^{D,\alpha}(x,y)}{\varphi_{1,\alpha}(x)\varphi_{1,\alpha}(y)} - 1\right| \le C(D,\alpha)e^{-(\lambda_{2,\alpha}-\lambda_{1,\alpha})t},\ \ t \ge 1.
$$

For $\alpha = 2$ this is valid for "many" domains but not all. For $0 < \alpha < 2$, valid for any bounded domain.

See Smits, "Spectral Gaps and Rates to Equilibrium" Mich. Math. J. p. Vol 43 (1997), page 154.

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Apply the semigroup to the function $f(x) = 1,\ x \in D$

$$T_t^D f(x) = E_x[1_D(X_t^\alpha); \tau_D > t] = \int_D p_t^{D,\alpha}(x,y)dy$$

So that

$$\begin{aligned} P_x\{\tau_D > t\} &= \sum_{m=1}^{\infty} e^{-t\lambda_{m,\alpha}} \varphi_{m,\alpha}(x) \int_D \varphi_{m,\alpha}(y)dy \\ &= e^{-t\lambda_{1,\alpha}} \varphi_{1,\alpha}(x) \int_D \varphi_{1,\alpha}(y)dy + \sum_{m=2}^{\infty} e^{-t\lambda_{m,\alpha}} \varphi_{m,\alpha}(x) \int_D \varphi_{m,\alpha}(y)dy \end{aligned}$$

Theorem (Implied by the Intrinsic Ultracontractivity result)

$$\lim_{t\to\infty} e^{t\lambda_{1,\alpha}} P_x\{\tau_D > t\} = \varphi_{1,\alpha}(x) \int_D \varphi_{1,\alpha}(y)dy \tag{3}$$

and

$$\lim_{t\to\infty} \frac{1}{t} \log P_x\{\tau_D > t\} = -\lambda_{1,\alpha}, \tag{4}$$

uniformly for $x \in D$.

If I want to study the eigenfunction $\varphi_{1,\alpha}$ and $\lambda_{1,\alpha}$ and how these are affected by the geometry of the domain D, I should (better, must, ...) study the distribution of the exit time τ_D of the process. That is, study

$$P_x\{\tau_D > t\}$$

as a function of D, $x \in D$, $t > 0$.

But:

$$\begin{aligned} P_x\{\tau_D > t\} &= P_z\{X_s^\alpha \in D;\ \forall s, 0 < s \le t\} \\ &= \lim_{m\to\infty} P_z\{X_{jt/m}^\alpha \in D, j = 1, 2, \dots, m\} \\ &= \lim_{m\to\infty} \int_D \cdots \int_D p_{t/m}^\alpha(x - x_1) \cdots p_{t/m}^\alpha(x_m - x_{m-1}) dx_1 \dots dx_m \end{aligned}$$

$$p_t^\alpha(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} e^{-t|\xi|^\alpha} d\xi$$

In the same way (integrating against a delta function at y)

$$p_t^{D,\alpha}(x,y) = \lim_{m\to\infty} \int_D \cdots \int_D p^{\alpha}_{t/m}(x - x_1)\cdots p^{\alpha}_{t/m}(y - x_{m-1})dx_1 \dots dx_{m-1},$$

Alternatively, for Brownian motion, if $0 = t_0 < t_1 < \dots < t_m < t$, then the conditional finite-dimensional distribution

$$P_{z_0}\{B_{t_1} \in dx_1, \dots, B_{t_m} \in dx_m \mid B_t = y\},$$

is given by

$$\frac{p^2_{t-t_m}(z_n, y)}{p^2_t(z_0, y)} \prod_{i=1}^{m} p^2_{t_i - t_{i-1}}(z_i, z_{i-1}),$$

Reference:

- I. Karatzas and E. Shreve, Brownian Motion and Stochastic Calculus, Springer-Verlag, New York, 1991–page 395.
- Similar formula holds for "stable bridges", Bertoin, page 226.

Do we really need the stables?

With $X_t^\alpha = B_{T_t}$ with

$$E(e^{-\lambda T_t}) = e^{-t\lambda^{\alpha/2}},\ \lambda > 0,$$

we have

$$p_t^\alpha(x) = \int_0^\infty p_s^{(2)}(x)\, g_{\alpha/2}(t,s)ds = \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{|x|^2/4s} g_{\alpha/2}(t,s)\, ds$$

where

$$g_{\alpha/2}(t,s) = \text{density of } \ T_t$$

Note: $\alpha = 1/2$,

$$T_t = \inf\{s > 0 : B_s = \frac{t}{\sqrt{2}}\}$$

and

$$g_{1/2}(t,s) = \left(\frac{t}{2\sqrt{\pi}}\right) s^{-3/2} e^{-t^2/4s}, \quad \text{Bochner subordination}$$

$$
\begin{aligned}
&P_x\{X^\alpha_{t_1} \in D, \dots, X^\alpha_{t_m} \in D\} \\
&= \int_D \cdots \int_D \prod_{i=1}^m p^\alpha_{t_i - t_{i-1}}(x_i - x_{i-1})\, dx_1 \dots dx_n \\
&= \int_0^\infty \dots \int_0^\infty \left(\int_D \cdots \int_D \prod_{i=1}^m p^2_{s_i}(x_i - x_{i-1})\, dx_1 \dots dx_n \right) \\
&\times \prod_{i=1}^n g_{\alpha/2}(t_i - t_{i-1}, s_i)\, ds_1 \dots ds_m \\
&= \int_0^\infty \dots \int_0^\infty P_x\{B_{2s_1} \in D, B_{2(s_1+s_2)} \in D, \dots, B_{2(s_1+s_2+\dots+s_n)} \in D\} \\
&\times \prod_{i=1}^m g_{\alpha/2}(t_i - t_{i-1}, s_i)\, ds_1 \dots ds_m.
\end{aligned}
$$

Must study the function

$$
\Phi_m(x, D) = \int_D \cdots \int_D \prod_{i=1}^m p^2_{t_i}(x_i - x_{i-1})\, dx_1 \dots dx_m, \qquad x_0 = x
$$

No order on t_i.

For $A \subset \mathbb{R}^d$, A^* =ball centered at the origin and same volume as A. $\chi_A^* = \chi_{A^*}$
$f : \mathbb{R}^d \to \mathbb{R}$,

$$f^*(x) = \int_0^\infty \chi^*_{\{|f|>t\}}(x)dt$$

(Compare this with)

$$|f(x)| = \int_0^\infty \chi_{\{|f|>t\}}(x)dt$$

Properties:

$$f^*(x) = f^*(y), \quad |x| = |y|, \quad f^*(x) \geq f^*(y), \quad |x| \leq |y|$$

$$\{x : f^*(x) > t\} = \{x : |f(x)| > t\}^* \text{ (same level sets)}$$

$$\Rightarrow m\{x : f^*(x) > \lambda\} = m\{x : |f(x)| > \lambda\}$$

Theorem (Luttinger 1973)

Let $f_1, \dots, f_m$ be nonnegative functions in R^d and let $f_1^*, \dots, f_m^*$ be their symmetric decreasing rearrangement. Then for any $x_0 \in D$ we have

$$\int_{D^m} \prod_{j=1}^{m} f_j(x_j - x_{j-1}) dx_1 \cdots dx_m \leq \int_{\{D^*\}^m} f_1^*(x_1) \prod_{j=2}^{m} f_j^*(x_j - x_{j-1}) dx_1 \cdots dx_m.$$

D^* =ball center at zero and and same volume as D

Theorem (Brascamp–Lieb–Lutinger (1975), (1977))

$Q_j : \mathbb{R}^d \to [0, \infty)$ and $1 \leq j \leq r$. a_{jk}, $1 \leq j \leq r, 1 \leq k \leq m$ real numbers.

$$\int_{(\mathbb{R}^d)^m} \prod_{j=1}^{r} Q_j\Big(\sum_{k=1}^{m} a_{jk} z_k\Big) dz_1 \cdots dz_m \leq \int_{(\mathbb{R}^d)^m} \prod_{j=1}^{r} Q_j^*\Big(\sum_{k=1}^{m} a_{jk} z_k\Big) dz_1 \cdots dz_m$$

- Roots lie in inequalities of Hardy–Littlewood–Pólya–Riesz

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F_1(x_1) H(x_2 - x_1) F_2(x_2) dx_1 dx_2 \leq *$$

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Theorem (R. B. Latala, Méndez, 2001 ($d=2$), Méndez 2003, $d \geq 3$)

$D \subset \mathbb{R}^d$ convex of finite inradius r_D and S infinite strip of inradius r_D Let $f_1, \ldots, f_m$ be nonnegative radially symmetric decreasing on $\mathbb{R}^d$. For any $z_0 \in \mathbb{R}^d$,

$$\int_D \cdots \int_D \prod_{j=1}^m f_j(z_j - z_{j-1})\, dz_1 \cdots dz_m \leq$$

$$\int_S \ldots \int_S f_1(z_1) \prod_{j=2}^m f_j(z_j - z_{j-1})\, dz_1 \cdots dz_m.$$

Theorem

$f_j : \mathbb{R}^d \to [0,1]$, $h_j : \mathbb{R}^d \to [0,\infty)$, $1 \leq j \leq m$, radial symmetric decreasing. Then

$$\int \cdots \int [1 - \prod_{j=1}^m (1 - f_j(z_j))] \prod_{j=1}^m h_j(z_j - z_{j-1}) dz_0 \ldots dz_m$$

$$\geq \int \cdots \int [1 - \prod_{j=1}^m (1 - f_j^*(z_j))] \prod_{j=1}^m h_j(z_j - z_{j-1}) dz_0 \ldots dz_m$$

Corollary (**Isoperimetric Inequality for stable processes and eigenvalues**)

$$\Phi_m(x, D) \le \Phi_m(0, D^*)$$

$$P_x\{\tau_D^\alpha > t\} \le P_0\{\tau_{D^*}^\alpha > t\}$$

$$\sup_{x \in D} E_x(\tau_D^\alpha) \le E_0(\tau_{D^*}^\alpha)$$

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$$\lambda_{1,\alpha}(D^*) \le \lambda_{1,\alpha}(D) \quad \textit{The Faber-Krahn Theorem}$$

$$Cap_\alpha(A) \ge Cap_\alpha(A^*),$$

(α-capacity version of a theorem of Polya–Szego. Proved by Watanabe 1984, conjectured by Mattila 1990, Proved by Betsakos 2003, P. Méndez 2006)

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Corollary (**Isoperimetric Inequality for the "partition function"**)

$$\begin{aligned} Z_t^\alpha(D) &= \sum_{m=1}^{\infty} e^{-t\lambda_{m,\alpha}(D)} = \int_D p_t^{\alpha,D}(x,x)dx \\ &\leq \int_{D^*} p_t^{\alpha,D^*}(x,x)dx \leq \sum_{m=1}^{\infty} e^{-t\lambda_{m,\alpha}(D^*)} = Z_t^\alpha(D^*) \end{aligned}$$

Amongst all regions of equal volume the ball minimizes surface area. It follows from "trace inequality" and

Theorem (M. Kac, "Can one hear the shape of a drum?")

With $\alpha = 2$, $|\partial D|$=surface area of boundary of D,

$$Z_t^2(D) \sim C_d t^{-d/2} vol(D) - C_d' t^{-(d-1)/2} |\partial D| + o(t^{-(d-1)/2}, \quad t \to 0$$

The first term is trivial from

$$\begin{aligned} P_t^{2,D}(x,y) &= \frac{1}{(4\pi t)^{d/2}} e^{\frac{-|x-y|^2}{4t}} P_x\{\tau_D > t | B_t = y\} \\ &= \text{free motion times Brownian bridge in D} \end{aligned}$$

A detour into Weyl's asymptotics

$$\lim_{t\to 0} t^{\gamma}\int_0^{\infty} e^{-t\lambda} d\mu(\lambda) = A \Rightarrow \lim_{a\to\infty} a^{-\gamma}\mu[0,a) = \frac{A}{\Gamma(\gamma+1)}$$

Theorem (Weyl's Formula, $\alpha = 2$. $N_D(\lambda) = \#\{j \geq 1 : \lambda_j \leq \lambda\}$)

$$N_D(\lambda) \sim C_d vol(D)\lambda^{d/2}, \quad \lambda \to \infty$$

More difficult (and no probabilistic treatment exists):

$$N_D(\lambda) \sim C_d vol(D)\lambda^{d/2} - C_d'|\partial D|\lambda^{(d-1)/2} + o(\lambda^{(d-1)/2})$$

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Theorem (R.B. and T. Kulczycki (2007). $0 < \alpha \leq 2$)

$$\int_D p_t^{\alpha,D}(x,x)dx \sim C_{d,\alpha}t^{-d/\alpha}vol(D) - C_d't^{-(d-1)/\alpha}|\partial D| + o(t^{-(d-1)/\alpha}$$

as $t \to 0$. *This gives Weyl's version for all* $0 < \alpha \leq 2$.

The $$ Question: Is there an α–version of the more general Weyl?

Question

Amongst all convex domains $D \subset \mathbb{R}^d$ of inradius 1, which one has the largest exit time for Brownian motion? Also, lowest eigenvalue? **Answer:** *The infinite strip:*

$$S = \mathbb{R}^{d-1} \times (-1, 1)$$

Theorem (For D convex with inradius 1.)

$$\Phi_m(x, D) \leq \Phi_m(0, S), \quad x \in D$$

R.B. Méndez-Latala (2001), $d = 2$ and (2003), $d \geq 3$. (Convexity is essential here!)

Corollary (For D convex with inradius 1 and $0 < \alpha \leq 2$.)

$$P_x\{\tau_D > t\} \quad \leq \quad P_0\{\tau_S > t\} = P_0\{\tau_{(-1,1)} > t\} \tag{5}$$

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Corollary (For D convex with inradius 1 and $0 < \alpha \leq 2$.)

$$P_x\{\tau_D > t\} \leq P_0\{\tau_S > t\} = P_0\{\tau_{(-1,1)} > t\} \quad (5)$$

$$\lambda_{1,\alpha}(-1, 1) \leq \lambda_{1,\alpha}(D) \quad (6)$$

Definition: $F : \mathbb{R}^d \to \mathbb{R}$ is said to be **log-concave** if

$$\log F(\beta x + (1-\beta)y) \geq \beta \log F(x) + (1-\beta) \log F(y), \quad x, y \in \mathbb{R}^d$$

or

$$F(\beta x + (1-\beta)y) \geq F(x)^\beta F(y)^{1-\beta}$$

Examples:

$$F(x) = \frac{1}{(4\pi)^{d/2}} e^{-|x|^2/4}$$

and

$$F(x) = \chi_D(x),$$

$D \subset \mathbb{R}^d$ is convex, are log–concave.

Theorem (Prékopa (1971))

Convolutions of log-concave functions are log-concave.

Corollary ($D \subset \mathbb{R}^d$ convex)

For Brownian motion, the function $\Phi_m(x, D)$ is log-concave.

Corollary (Brascamp-Lieb (1979))

For any bounded convex domain $D \subset \mathbb{R}^d$ and for Brownian motion, the function $P_x\{\tau_D > t\}$ is log-concave and therefore so is the "ground state" eigenfunction $\varphi_{1,2}(x)$. In fact, this holds for the "ground state" eigenfunction for the Scrödinger operator $-\Delta + V$ where $V : D \to [0, \infty)$ is convex.

Note: Unfortunately we cannot conclude the same for $0 < \alpha < 2$. Why?

Question ($D \subset \mathbb{R}^d$ convex, $0 < \alpha < 2$)

Are the functions $P_x\{\tau_D > t\}$ and/or $\varphi_{1,\alpha}(x)$ log-convex?

Known only for $\alpha = 1$, $D = (-1, 1)$. In fact, in this case the functions are concave, just like for $\alpha = 2$.

Several other partial results are known for special doubly symmetric domains in the plane.

Definition

$D \subset \mathbb{R}^d$ be a convex symmetric relative to each coordinate axes. J any line segment in D parallel to the x_1-axis intersecting ∂D only at the two points. $F : D \to \mathbb{R}$, is *mid–concave* on J if it is concave on mid half of J. F mid–concave along the x_1–axis if it is mid–concave on every such segment contained in D parallel to the x_1–axis. Same for mid–concavity along the x_2-axis, $\cdots$. F mid–concave on D if it is mid–concave along each coordinate axes.

Theorem (R.B.-Méndez-Kulczycki, 2006)

$Q \subset R^d$ a rectangle. $\Phi_m(x, Q) = P_x\{X^\alpha_{t_1} \in Q, \dots, X^\alpha_{t_m} \in Q\}$ is mid–concave in Q for any $0 < \alpha \leq 2$. In addition, if $x = (x_1, \dots, x_n) \in Q$, then

$$\frac{\partial}{\partial x_i} F(x) \geq 0, \;\; \textit{if } x_i < 0, \;\; \textit{and } \frac{\partial}{\partial x_i} F(x) \leq 0, \;\; \textit{if } x_i > 0.$$

But (recall, $x_0 = x$)

$$\Phi_m(x) = \int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{i=1}^{m} p^{(2)}_{t_i}(x_{i-1} - x_i)\, dx_1 \dots dx_m, \;\; \text{not concave on } (-1, 1) \text{ for all } t_i.$$

Corollary

Let $Q = (-a_1, a_1) \times (-a_2, a_2) \times \cdots \times (-a_d, a_d)$, $0 < a_i < \infty$ *for all* $i = 1, 2, \ldots, d$, *be a rectangle in* $\mathbb{R}^d$. $\varphi_{1,\alpha}$ $0 < \alpha < 2$ *is mid–concave on* Q. *In addition, if* $x = (x_1, \ldots, x_n) \in Q$, *then*

$$\frac{\partial}{\partial x_i}\varphi_1^\alpha(x) \geq 0, \text{ if } x_i < 0, \text{ and } \frac{\partial}{\partial x_i}\varphi_1^\alpha(x) \leq 0, \text{ if } x_i > 0.$$

Theorem

For $n \geq 1$, *set*

$$D(n) = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \, x_1 \in (-n, n), \, x_2 \in \left(-1 + \frac{|x_1|}{n}, 1 - \frac{|x_1|}{n} \right) \right\}.$$

For n *large enough,* $\varphi_{1,2}$ *is not mid–concave on* $D(n)$.

- Probabilistic reasons: Many people (C. Borell, S. Bobkov, C. Houdré, M. Ledoux) have studied "isoperimetric inequalties" "Cheeger's inequalities", etc., for log–concave measures. Here is a "beautiful result:" For any probability measure μ on $\mathbb{R}^d$ define its "spectral gap" by

$$Specg(\mu) = \inf_{f \in Lip(1)} \left\{ \frac{\int_{\mathbb{R}^d} |\nabla f|^2 d\mu}{\int_{\mathbb{R}^d} |f|^2 d\mu} : \int_{\mathbb{R}^d} f \, d\mu = 0 \right\}$$

Theorem (Bobkov 1996)

Suppose $d\mu = f(x)dx$, f log-concave and $diam(\mu) = diam(supp(f)) < \infty$. Then

$$Specg(\mu) \geq \frac{c}{diam^2(\mu)}$$

Theorem (R. Smits 1997. Under same assumptions as Bobkov)

$$Specg(\mu) \geq \frac{\pi^2}{diam^2(\mu)}$$

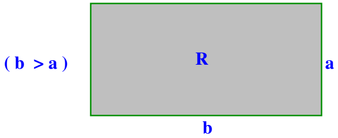
$H = -\Delta + V$ with Dirichlet conditions in the bounded convex domain $D \subset \mathbb{R}^d$ of finite diameter d_D, $V \geq 0$ is bounded and convex in D. We have

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \ldots.$$

Conjecture: (Problem #44 in Yau's 1990 "open problems in geometry")

$$gap(D,V) = \lambda_2 - \lambda_1 > \frac{3\pi^2}{d_D^2}$$

with the lower bound approached when $V = 0$ and the domain becomes a thin rectangular box.
False for nonconvex domains even with $V = 0$.



$$\lambda_2(R) - \lambda_1(R) = \left(\frac{4\pi^2}{b^2} + \frac{\pi^2}{a^2}\right) - \left(\frac{\pi^2}{b^2} + \frac{\pi^2}{a^2}\right) = \frac{3\pi^2}{b^2},$$

Theorem (I.M. Singer, B. Wang, S.T. Yau and S.S.T. Yau (1985))

$$gap(D, V) \geq \frac{\pi^2}{4d_D^2}$$

Several improvements by many, many, others. Full conjecture only known for $D = (-1, 1)$ *(R. Lavin (1994)*

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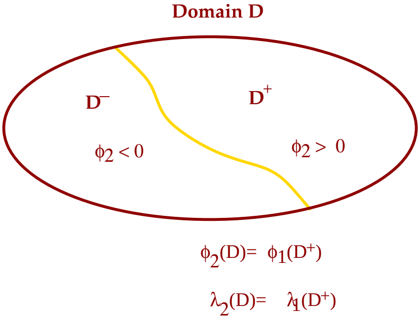
Several improvements by many, many, others. Full conjecture only known for $D = (-1, 1)$ *(R. Lavin (1994)*

If $d\mu = \varphi_1^2(x)\,dx$, then $Specg(\mu) = \lambda_2 - \lambda_1$. Since φ_1^2 is log–concave, Smits gives:

Corollary (R. Smits, 1997–best general bound up-to-date)

$$gap(D, V) \geq \frac{\pi^2}{d_D^2}$$

Take $V = 0$ for now. (**A. Melas, 1998 UCLA Ph.D Thesis. The "nodal domains" for φ_2 look like in the picture**)



More General Conjecture: D convex of diameter d_D. Let

$$I = \left(\frac{-d_D}{2}, \frac{d_D}{2}\right), \quad I^+ = \left(0, \frac{d_D}{2}\right)$$

$$\frac{\int_{D^+} P_t^{D^+}(z,z)dz}{\int_D P_t^D(z,z)dz} \leq \frac{\int_{I^+} P_t^{I^+}(z,z)dz}{\int_I P_t^I(z,z)dz}$$

Conjecture Equivalent to (In terms of Partition Function):

$$\frac{Z_t^2(D^+)}{Z_t^2(D)} = \frac{\sum_{j=1}^{\infty} e^{-t\lambda_j(D^+)}}{\sum_{j=1}^{\infty} e^{-t\lambda_j(D)}} \leq \frac{\sum_{j=1}^{\infty} e^{-t\lambda_j(I^+)}}{\sum_{j=1}^{\infty} e^{-t\lambda_j(I)}} = \frac{Z_t^2(I^+)}{Z_t^2(I)}$$

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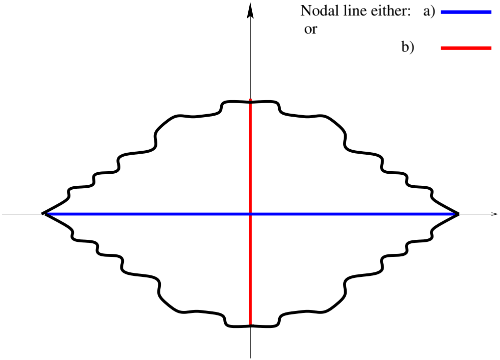
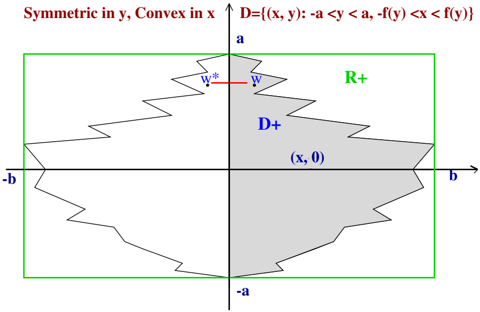
L. Payne (here at Cornell) (1973)





Theorem (R.B. Médez-Hernández, 2002)

Suppose D, and D^+ and b are as in the picture.

$$\frac{P_{(x,0)}\{\tau_{D^+} > t\}}{P_{(x,0)}\{\tau_D > t\}} \leq \frac{P_x\{\tau_{(0,b)} > t\}}{P_x\{\tau_{(-b,b)} > t\}}$$

Same as:

$$\frac{\Phi_m(D^+,(x,0))}{\Phi_m(D,(x,0))} \leq \frac{\Phi_m((0,b),x)}{\Phi_m((-b,b),x)} \quad \textit{(F.D.D., again!)}$$

Two domains in

3-dimensions:

Right Half

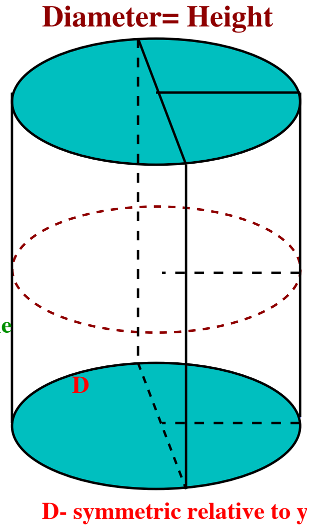
and

Bottom Half

Which Half has the

smallest Dirichlet

eigenvalue?



The maximum (and the minimum) of the "first" non-constant Neumann eigenfunction for bounded convex domains are attained on the boundary and only on the boundary of the domain.

Many partial results: R.B.-K.Burdzy (1999), D.Jerison-N.Darishavilli (2000), M. Pascu (2001), R. Bass–K. Burdzy (2000), R.B.-M. Pang (2003), R.B. M.Pang-Pascu (2004), R.Atar K.Burdzy (2005)

Counterexample: K. Burdzy-W. Werner (2000), K. Burdzy (2005)

Believed to be true for any simply connected domain, conjectured to be true for any convex domain.

Unknown even for an arbitrary triangle in the plane!

Conjecture: The maximum and minimum for the first nonconstant eigenfunction for the semigroup of Brownian motion conditioned to remain forever in a convex domain are attained on the boundary and only on the boundary of the domain.

That is, the function φ_2/φ_1 attains its maximum and minimum on the boundary and only on the boundary of D.

Theorem (R.B. Médez-Hernández, 2006)

The conditional "Hot Spots" conjecture is true for symmetric domains in the plane as those shown above. The maximum (and minimum) of the function

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Proof: Via finite dimensional distributions!

Theorem

Let D be a bounded domain in $\mathbb{R}^2$ which is symmetric and convex with respect to both axes.

(i) *If $z_1 = (x, y_1) \in D^+$, $z_2 = (x, y_2) \in D^+$ and $y_1 < y_2$, then*

$$\frac{P_{z_1}\{\tau_{D^+} > t\}}{P_{z_1}\{\tau_D > t\}} < \frac{P_{z_2}\{\tau_{D^+} > t\}}{P_{z_2}\{\tau_D > t\}},$$

for any $t > 0$. In particular, the function

$$\Psi(z, t) = \frac{P_z\{\tau_{D^+} > t\}}{P_z\{\tau_D > t\}},$$

for each $t > 0$ arbitrarily fixed, cannot have a maximum at an interior point of D^+.

(ii) *If $z_1 = (x_1, y) \in D^+$ and $z_2 = (x_2, y) \in D^+$ with $|x_2| \leq |x_1|$, then*

$$\frac{P_{z_1}\{\tau_{D^+} > t\}}{P_{z_1}\{\tau_D > t\}} \leq \frac{P_{z_2}\{\tau_{D^+} > t\}}{P_{z_2}\{\tau_D > t\}},$$

for any $t > 0$.

Corollary

$D \subset \mathbb{R}^2$ as in Theorem φ_2 be such that its nodal line is the intersection of the x-axis with the domain. Without LOG, $\varphi_2 > 0$ in D^+ and $\varphi_2 < 0$ in D^-. Set $\Psi = \varphi_2/\varphi_1$.

(i) *If $z_1 = (x, y_1) \in D^+$ and $z_2 = (x, y_2) \in D^+$ with $y_1 < y_2$, then*

$$\Psi(z_1) < \Psi(z_2).$$

(ii) *If $z_1 = (x, y_1) \in D^-$ and $z_2 = (x, y_2) \in D^-$ with $y_2 < y_1$, then*

$$\Psi(z_1) < \Psi(z_2).$$

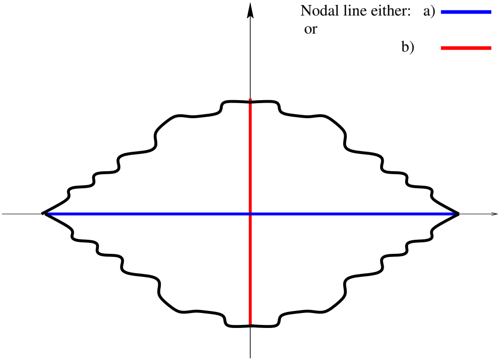
In particular, Ψ cannot attain a maximum nor a minimum in the interior of D.

(iii) *If $z_1 = (x_1, y) \in D^+$ and $z_2 = (x_2, y) \in D^+$ with $|x_2| < |x_1|$, then*

$$\Psi(z_1) \leq \Psi(z_2). \tag{7}$$

Corollary (Exact analogue of D. Jerison and N. Nadirashvili (2000) for classical "hot–spots")

Suppose $D \subset \mathbb{R}^2$ is a bounded domain with piecewise smooth boundary which is symmetric and convex with respect to both coordinate axes and that φ_2 is as in Theorem 1.2. Then strict inequality holds in (7) unless D is a rectangle. The maximum and minimum of Ψ on $\overline{D}$ are achieved at the points where the y-axis meets ∂D and, except for the rectangle, at no other points.



“To a hammer everything is a nail”

Can finite dimensional distributions make my morning coffee?”

Mow my lawn?

don’t know that,. . .

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yet!