Lecture Notes in Analysis[1]

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In memory of my father José and
brothers Sergio and Javier.
From them I learned the true meaning
of the words faith, courage, and determination.

Preface

These notes are based on a one semester course, Mathematics 545, which I have taught at Purdue University for several years. The background of most of the students who enroll in the course consists, for the most part, of nothing more than the basics of measure theory, for example, Chapters 3, 4 and 11 of Royden *Real Analysis*, or Chapters 1 and 3 of Rudin *Real and Complex Analysis*. My goals for the course have been to present some of the other basic and essential tools of real analysis, such as differentiation of monotone functions, absolute continuity, signed measures, the Radon–Nikodym theorem, Fubini's theorem, the basic theory of the Fourier transform, and in addition to introduce material which is rarely part of such an introductory course. These other topics include the Hardy–Littlewood maximal function, approximations to the identity, interpolation theorems, singular integrals, fractional integration, and square functions. I make an effort to present some of the basic applications of these topics to, for example, the basic boundary value problems for the Laplacian and the heat equation in the upper half space of $\mathbb{R}^n$ and the Hörmander multiplier theorem.

A serious difficulty I encountered the first time I taught this course was that despite the large number of excellent introductory analysis books on the market, there is no book available that starts at such an elementary level and develops some of the more advanced topics mentioned above. The recommended books for the course are usually Royden's *Real Analysis*, Rudin's *Real and Complex Analysis*, Torchinsky's *Real Variables* and *Fourier Analysis*, Stein's *Singular Integrals and Differentiability Properties of Functions*, and Stein and Weiss' *Introduction to Fourier Analysis on Euclidean Spaces*. The material in these notes consists of pieces from each one of these references with many additions and subtractions, keeping in mind my overall goals of the course. I have made an effort to go through the basic material as quickly as possible without compromising rigor and completeness in any serious way.

There are a number of exercises in these notes but probably not as many as one would like to see in a text of this level. These are, for the most part, interwoven with the material as it is being presented. Many of the exercises are used in subsequent results and others are used for various generalizations and extensions. For this reason, the exercises are an essential part of these notes.

The notes consist of ten chapters. The more elementary/introductory material is contained in Chapters 1–4. These Chapters include differentiation of monotone functions, absolute continuity, Fubini's theorem, the Radon-Nykodym theorem, applications to the duality of L^p–spaces and properties of convolutions. Chapter 6 is devoted to the Fourier transform. One item that is discussed here, which is not often taught in introductory analysis courses, is the inversion formula for regular Borel measures on the real line. Such a formula is very useful in applications to limit theorems in probability theory. Even though probability is not part of this course, I never miss the opportunity to informally point out various connections when they arise naturally. The more advanced chapters are Chapters 5, and 7–10, and I devote much of the fifteen weeks of the semester to these topics. These chapters include some of the standard tools of harmonic analysis: the Hardy-Littlewood maximal operator, the Calderón–Zygmund decomposition, the basic theory of singular integrals and applications to the Riesz transforms and the Beurling–Ahlfors operator (both in the plane and in $\mathbb{R}^n$), fractional integration and its connections to the inequalities of Sobolev and Nash, and a brief introduction to the classical Lusin and Littlewood–Paley square functions and their applications to the Hörmander multiplier theorem.

I wish to express my thanks to the many students who have taken this course over the years. Many of these students made valuable contributions to the material presented here. In particular, the notes were carefully read by Pedro Méndez who made innumerable suggestions for improvements. Pedro certainly deserves very special thanks. It is a pleasure to thank my colleague, friend and next door-office neighbor, Christoph Neugebauer, for his very useful comments on earlier versions of the notes. Chris teaches some best courses in real analysis at Purdue and these notes would have no reason to exist if I had been able to convince him to write his excellent lecture notes for Math 544 and 545. I would like to thank Tom Carroll for his careful reading and detailed comments on several chapters. I thank Betty Gick for all her work in typing parts of the first draft of these notes. I am deeply greateful to Francisco Blanco-Silva who transformed the AMS–TeX file into a LaTeX file and in the process made many valuable mathematical comments and corrections. I thank the National Science Foundation for supporting my research during the years when these notes were written.

By far my biggest thanks are reserved for my family, Rosa, Nidia and Carisa. Their love has always been my source of strength and inspiration. Without them not only would these kind of projects be much more difficult but I would be completely lost in this world.

Finally, it is always a great pleasure for me to see the mathematical growth that many of the students achieve as the semester progresses and as they begin to master and appreciate some of the basic material on singular integrals and Littlewood–Paley theory presented in these notes. I have been fortunate to have had seven Ph.D students, Robert Smits, Arthur J. Lindeman, Pedro Méndez, Dahae You, Prabhu Janakiraman, Nane Erkan, and Ambica Rajagopal all of whom began their studies in analysis from these notes and just a couple of years later wrote beautiful research papers using many of the tools and ideas presented here. What more can a teacher ask for?

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