# Practice Problems, Final Exam

1. $\dfrac{x^2-1}{x^2-x} = \dfrac{(x-1)(x+1)}{x(x-1)} = \dfrac{x+1}{x} \xrightarrow[x\to 1]{} \dfrac{1+1}{1} = 2$

2. $\left[(x^2+1)\operatorname{Tan} x\right]' = (2x)\operatorname{Tan} x + (x^2+1)\operatorname{Sec}^2 x$

3. $\lim\limits_{x\to -1^-}(x^2+a) = 1+a \overset{\text{want}}{=} -1-8 = \lim\limits_{x\to -1^+} x^3-8$

$a=-10$

4. Sandwich Theorem: $-|x| \le x\operatorname{Cos}\frac{1}{x} \le |x|$.

5. $\dfrac{f(x)-f(1)}{x-1} = \dfrac{\frac{1}{x+3}-\frac{1}{4}}{x-1} = \dfrac{4-(x+3)}{4(x+3)(x-1)} = \dfrac{-(x-1)}{4(x+3)(x-1)}$

$= \dfrac{-1}{4(x+3)} \xrightarrow{x\to 1} \dfrac{-1}{4(1+3)} = \dfrac{-1}{16}$ or use $f'(1)$.

6. $f(x) = x^3-x-5$. Use Intermediate Value Theorem.

$f(-2)<0,\ f(-1)<0,\ f(0)<0,\ \underbrace{f(1)<0,\ f(2)>0}_{\text{zero between 1 and 2.}}$

7. $\Big[\underbrace{\dfrac{1-x}{1+x}}_{f(x)}\Big]' = \dfrac{(-1)(1+x)-(1)(1-x)}{(1+x)^2} = \dfrac{-2}{(1+x)^2}$

$f'(1) = \dfrac{-2}{(1+1)^2} = -\dfrac{1}{2}$.

8. $y = \ln(1-x^2) + (\sin x)^2$

$y' = \frac{1}{(1-x^2)}(-2x) + 2(\sin x)[\cos x].$

9. From above, $f'(x) = \frac{-2}{(1+x)^2}$. So $f''(x) = \frac{(-2)(-2)}{(1+x)^3}$

10. $xy^2 - x^2 + y + 5 = 0$

$1\cdot y^2 + x(2yy') - 2x + y' = 0$

$y'(2xy+1) = 2x - y^2$

$y' = \frac{2x-y^2}{2xy+1}$ At $(-2,1)$, we get $y' = \frac{2(-2)-1^2}{2\cdot1\cdot(-2)+1}$

11. $f(x) = 3x^2 + 6x - 10$ $f'(x) = 6x + 6.$

Critical point: $x=-1$ $f(-1) = 3-6-10 = -13 \Leftarrow$ min

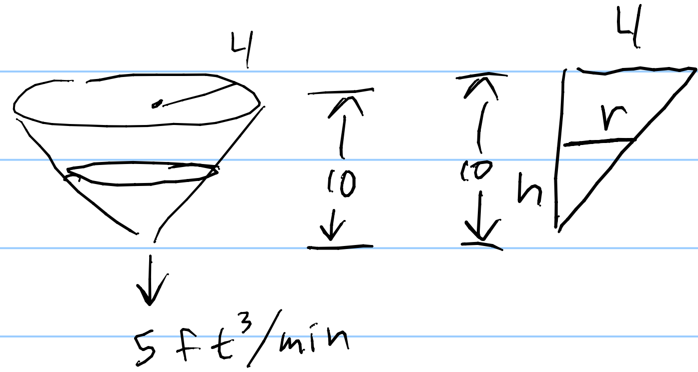
End points: $x=-2$ $f(-2) = 12-12-10 = -10$

$x = 2$ $f(2) = 12+12-10 = 14 \Leftarrow$ max

12. $f(x) \approx f(a) + f'(a)(x-a).$ $x = 3.02$, $a = 3.$

$f(x) \approx 5 + (-2)(3.02-3) = 5 - .04 = 4.96$

13.




$\frac{h}{r} = \frac{10}{4}$

$r = \frac{2}{5}h$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h = \frac{4}{75}\pi h^3$$

$$-5 = \frac{dV}{dt} = \frac{4}{25}\pi h^2 \frac{dh}{dt}.$$

When $h = 5$, $-5 = \frac{4}{25}\pi 5^2 \frac{dh}{dt}$, so $\frac{dh}{dt} = \frac{-5}{4\pi}$.

So $h$ is <u>decreasing</u> at a rate of $\frac{5}{4\pi}$ ft/min when $h=5$.

14. $A = 2xy = 2x\sqrt{a^2-x^2}$. $A=0$ at endpoints $x=0, a$.

$A' = 2\sqrt{a^2-x^2} + 2x\frac{1}{2}(a^2-x^2)^{-1/2}(-2x) = 0$ when

$\sqrt{a^2-x^2} = \frac{x^2}{\sqrt{a^2-x^2}}$, $a^2-x^2 = x^2$, $2x^2 = a^2$, $x = \frac{a}{\sqrt{2}}$ at max.

Max $A = 2\frac{a}{\sqrt{2}}\sqrt{a^2-\left(\frac{a}{\sqrt{2}}\right)^2} = \sqrt{2}\,a\sqrt{\frac{a^2}{2}} = a^2$.

15. $\frac{f(7)-f(2)}{7-2} = f'(c)$ for some $c$ between 2 and 7.

$= \frac{10-4}{5} = \frac{6}{5}$

16. $R = R_0 e^{-kt}$. $R_0 = 18$. $2 = 18e^{-k(2)}$. $\frac{1}{9} = e^{-2k}$

$\ln\frac{1}{9} = -2k$, $k = \frac{1}{2}\ln 9$, $k = \ln 3$. Now start with $R_0 = 12$:

$R = 12e^{-kt} = 4$ when $\frac{1}{3} = e^{-kt}$ or $\ln\frac{1}{3} = -kt$.

So $t = \frac{\ln 3}{k} = \frac{\ln 3}{\ln 3} = \underline{1}$.

17. $g(x) = 4x^3 - 3x^4$

$g'(x) = 12x^2 - 12x^3 = 12x^2(1-x)$

$g''(x) = 24x - 36x^2 = 12x(2-3x)$

$+++++++++\;|\;+++++\;|\;------$ Sign $g'$

$\qquad\qquad 0 \qquad\quad 1$

$----------\;|\;++++\;|\;--------$ Sign $g''$

$\qquad\qquad 0 \quad \frac{2}{3}$

$g$ is decreasing for $x > 1$. $g$ does <u>not</u> have a relative extreme value at $(0,0)$. The graph of $g$ is concave <u>down</u> for $x < 0$.

18. $f(x) = \dfrac{2}{\sqrt{1+x^2}} = 2(1+x^2)^{-1/2}$. $f'(x) = -1(1+x^2)^{-3/2}(2x)$

$f'(x) = \dfrac{-2x}{(1+x^2)^{3/2}}$ $\qquad +++++\;|\;----$ Sign $f'$

$\qquad\qquad\qquad\qquad 0$

$f$ is increasing for $x < 0$.

19. $f'(x) = (x-1)^2(x+2)(x-5)$

$+++++\;|\;------\;|\;----------\;|\;++++$ Sign $f'$

$\qquad -2 \qquad\qquad 1 \qquad\qquad 5$

$\qquad \uparrow \qquad\qquad\qquad\qquad \uparrow$

local max $\qquad\qquad$ local min

20. $y = \displaystyle\int_1^u \sqrt{t^2+1}\,dt$ where $u = 2x$.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \sqrt{u^2+1}\,(2) = \sqrt{(2x)^2+1}\,(2)$$

$$\frac{dy}{dx} = 2\sqrt{4x^2+1} = 2\sqrt{4\cdot 2+1} = 6 \text{ when } x=\sqrt{2}.$$

21. $I = \int_3^4 x\sqrt{25-x^2}\,dx$, $u = 25-x^2$ $du = -2x\,dx$

When $x=3$ : $u = 25-3^2 = 16$ $x\,dx = -\frac{1}{2}du$

When $x=4$ : $u = 25-4^2 = 9$

$$I = \int_{u=16}^{9} \sqrt{u}\left(-\frac{1}{2}du\right) = \frac{1}{2}\int_9^{16} u^{1/2}\,du$$

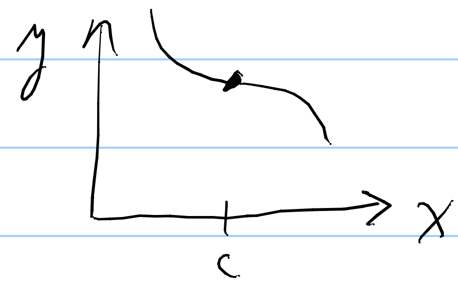
$$= \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_9^{16} = \frac{1}{3}\left(16^{3/2} - 9^{3/2}\right) = \frac{2}{3}\left(4^3 - 3^3\right)$$

$$= \frac{1}{3}(64-27) = \frac{37}{3}.$$

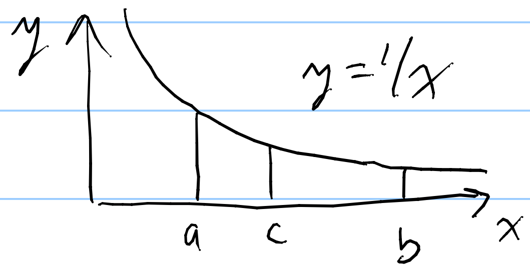
22. $$\frac{x^2+2x}{3x^2+4}\,\frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} = \frac{1+\frac{2}{x}}{3+\frac{4}{x^2}} \longrightarrow \frac{1+0}{3+0} = \frac{1}{3} \text{ as } x\to\infty.$$

23. $$\lim_{x\to 0} \frac{2x - \sin^{-1}x}{2x + \tan^{-1}x} \overset{L'H}{=} \lim_{x\to 0} \frac{2 - \frac{1}{\sqrt{1-x^2}}}{2 + \frac{1}{1+x^2}} = \frac{2-1}{2+1} = \frac{1}{3}$$

24. $f''(x) > 0$ for $x < c$ : $f$ concave up left of $c$.
$f'(c) = 0$ : horizontal tangent at $x = c$.
$f'(c) < 0$ for $x > c$ : $f$ decreasing right of $c$.




25.




$$\int_a^c \frac{1}{x}\,dx = \int_c^b \frac{1}{x}\,dx$$

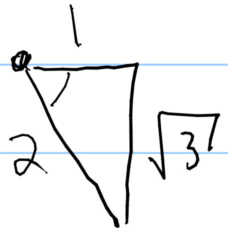
$$\ln c - \ln a = \ln b - \ln c$$

$$\ln c = \frac{1}{2}(\ln b + \ln a) = \frac{1}{2}\ln ab = \ln\sqrt{ab}.$$

so $c = \sqrt{ab}$.

26. $$A = \int_{-\sqrt{3}}^{1} \frac{1}{1+x^2}\,dx = \left[\operatorname{Tan}^{-1} x\right]_{-\sqrt{3}}^{1} = \frac{\pi}{4} - \left(-\frac{\pi}{3}\right)$$

$$= \frac{3\pi + 4\pi}{12} = \frac{7\pi}{12}.$$




27. $$\left[e^{2x} \ln\sqrt{1+x}\right]' = \left(2e^{2x}\right)\ln\sqrt{1+x} +$$

$$e^{2x}\frac{1}{\sqrt{1+x}}\left(\frac{1}{2}(1+x)^{-1/2}\cdot 1\right)$$

$$= 2e^{2x}\left(\frac{1}{2}\ln(1+x)\right) + \frac{e^{2x}}{2(\sqrt{1+x})^2}$$

$$= e^{2x}\ln(1+x) + \frac{e^{2x}}{2(1+x)}$$

28. $$\left(x^{\sin x}\right)' = \left(e^{\sin x \ln x}\right)' =$$

$$e^{\sin x \ln x}\left(\cos x \ln x + \sin x\left(\frac{1}{x}\right)\right)$$

$$= x^{\sin x}\left(\frac{\sin x}{x} + (\cos x)\ln x\right)$$

29. $y = \mathrm{Tan}^{-1} u$ where $u = e^{3x}$.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{1+u^2}\left(3e^{3x}\right) = \frac{3e^{3x}}{1+(e^{3x})^2} = \frac{3e^{3x}}{1+e^{6x}}$$

30. $\sqrt{4-x^2} = 2\sqrt{1-\left(\frac{x}{2}\right)^2}$, $u = \frac{x}{2}$, $du = \frac{1}{2}dx$

When $x = 0$ : $u = \frac{0}{2} = 0$

When $x = \sqrt{3}$ : $u = \sqrt{3}/2$

$$I = \int_0^{\sqrt{3}} \frac{1}{2\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx = \int_{u=0}^{\sqrt{3}/2} \frac{1}{\sqrt{1-u^2}}\,du =$$

$$= \left[\sin^{-1} u\right]_0^{\sqrt{3}/2} = \frac{\pi}{3} - 0 = \frac{\pi}{3}$$

31. $u = 1 + 2x \qquad du = 2\,dx$

$x = \frac{1}{2}(u-1)$

When $x = 0$ : $u = 1$.

When $x = 4$ : $u = 9$.

$$I = \int_{x=0}^{4} \frac{x}{\sqrt{1+2x}}\,dx = \int_{u=1}^{9} \frac{\frac{1}{2}(u-1)}{\sqrt{u}} \left(\frac{1}{2}du\right)$$

$$= \frac{1}{4}\int_1^9 u^{1/2} - u^{-1/2}\,du = \frac{1}{4}\left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^9$$

$$= \frac{1}{4}\left[\left(\frac{2}{3}9^{3/2} - 2\,9^{1/2}\right) - \left(\frac{2}{3} - 2\right)\right]$$

$$= \frac{1}{4}\left[\frac{2}{3}\cdot 3^3 - 2\cdot 3 - \frac{2}{3} + 2\right]$$

$$= \frac{1}{4}\left[18 - 6 - \frac{2}{3} + 2\right] = \frac{1}{4}\cdot\frac{40}{3} = \frac{10}{3}.$$

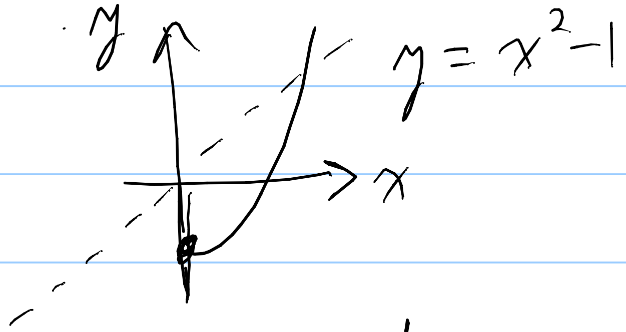
32. $u = 1 + e^x$, $du = e^x\,dx$, When $x = 0$ : $u = 2$

when $x = 1$ : $u = 1+e$

$$I = \int_{x=0}^{1} \frac{e^x}{1+e^x}\,dx = \int_{u=2}^{1+e} \frac{du}{u} = \ln u \Big|_2^{1+e} =$$

$$I = \ln(1+e) - \ln 2 = \ln \frac{1+e}{2}.$$

33.




Inverse graph is the reflection about the line $y = x$.