Error Estimate for Riemann Sums

Suppose that $f(x)$ is continuously differentiable on $[a,b]$. Let $M = \text{Max}\{|f'(x)| : a \le x \le b\}$. For a given positive integer N, let R_N denote the "right endpoint" Rieman Sum for $I = \int_a^b f(x)\,dx$, i.e.,

$$R_N = \sum_{n=1}^{N} f\left(a + n\frac{(b-a)}{N}\right)\underbrace{\left(\frac{b-a}{N}\right)}_{\Delta x}.$$

I shall prove that $|I - R_N| \le \dfrac{M(b-a)^2}{N}$.

Let $x_n = a + \dfrac{n(b-a)}{N}$ denote the n-th right endpoint of our subdivision of $[a,b]$.

Let $x_0 = a$. Note that

$$I = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{N-1}}^{x_N} f(x)\,dx.$$

Now the Mean Value Theorem for integrals yields that there is a point c_n between x_{n-1} and x_n such that

$$\int_{x_{n-1}}^{x_n} f(x)\,dx = f(c_n)\Delta x$$

where $\Delta x = x_n - x_{n-1} = \frac{b-a}{N}$.

Hence, $$I = \sum_{n=1}^{N} \int_{x_{n-1}}^{x_n} f(x)\,dx = \sum_{n=1}^{N} f(c_n)\Delta x.$$

(How interesting! This Riemann Sum gives the exact integral!)

Now we may write

$$R_N - I = \sum_{n=1}^{N} f(x_n)\Delta x - \sum_{n=1}^{N} f(c_n)\Delta x$$

$$= \sum_{n=1}^{N} [f(x_n) - f(c_n)]\Delta x.$$

Next, we may use the Mean Value Theorem for derivatives to assert that there is a point d_n between c_n and x_n such that

$$f(x_n) - f(c_n) = f'(d_n)(x_n - c_n),$$

Now $R_N - I = \sum_{n=1}^{N} \left(f'(d_n)(x_n - c_n)\right) \Delta x$

and we may use the estimates

$|f'(d_n)| \leq M$ and $|x_n - c_n| \leq \Delta x$ to

obtain the basic estimate

$$|R_N - I| \leq \sum_{n=1}^{N} |f'(d_n)|\,|x_n - c_n|\,\Delta x$$

$$\leq \underbrace{\sum_{n=1}^{N} M(\Delta x)^2}_{N M (\Delta x)^2 = NM\left(\frac{b-a}{N}\right)^2}$$

and this completes the proof