MA 181 Exam 1 Solutions

1. a) Let $u = x^3 + 1$ $\quad du = 3x^2\,dx$

So $x^2\,dx = \frac{1}{3}du$

When $x = 0$, $u = 0^3 + 1 = 1$

When $x = 2$, $u = 2^3 + 1 = 9$.

$$\text{So } \int_{x=0}^{2} \frac{1}{(x^3+1)^{1/2}} \left(x^2\,dx\right) = \int_{u=1}^{9} u^{-1/2}\left(\tfrac{1}{3}du\right)$$

$$= \frac{1}{3}\left[\frac{1}{\left(-\frac{1}{2}+1\right)} u^{-\frac{1}{2}+1}\right]_1^9 = \frac{2}{3}\left(\sqrt{9} - \sqrt{1}\right)$$

$$= \frac{2}{3}(3-1) = \underline{\underline{\frac{4}{3}}}$$

1. b) $u = x^3 + 1$, $x^2\,dx = \frac{1}{3}du$

as above, $x^3 = u - 1$.

$$\int x^5\sqrt{x^3+1}\,dx = \int x^3\sqrt{x^3+1}\left(x^2\,dx\right)$$

$$= \int (u-1)\,u^{1/2}\left(\tfrac{1}{3}du\right)$$

$$= \frac{1}{3}\int u^{3/2} - u^{1/2}\, du = \frac{1}{3}\left[\frac{1}{\left(\frac{5}{2}\right)} u^{5/2} - \frac{1}{\left(\frac{3}{2}\right)} u^{3/2}\right]$$

$$= \frac{1}{3}\left[\frac{2}{5}(x^3+1)^{5/2} - \frac{2}{3}(x^3+1)^{3/2}\right]$$

2. $y = F(x) = \int_0^u e^{t^2}\, dt$ where $u = \sqrt{x}$.

$$F'(x) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^{u^2}\left(\frac{1}{2}x^{-1/2}\right)$$

$$= \frac{1}{2} e^{(\sqrt{x})^2} \frac{1}{\sqrt{x}} = \frac{e^x}{2\sqrt{x}}.$$

Notice that $\int_0^u e^{t^2}\, dt > \int_0^u 1\, dt = u$.

So $F(x) = \int_0^{\sqrt{x}} e^{t^2}\, dt > \sqrt{x}$.

Hence $F(x) \to \infty$ as $x \to \infty$.

$$\lim_{x\to\infty} \frac{x}{F(x)} \underset{\left(\frac{\infty}{\infty}\right)}{\overset{L'H}{=}} \lim_{x\to\infty} \frac{1}{F'(x)} = \lim_{x\to\infty} \frac{2\sqrt{x}}{e^x}$$

$$\underset{\left(\frac{\infty}{\infty}\right)}{\overset{L'H}{=}} \lim_{x\to\infty} \frac{\left(\frac{1}{\sqrt{x}}\right)}{e^x} = \lim_{x\to\infty} \frac{1}{\sqrt{x}\, e^x} = 0$$

3. a) MVT for derivatives:

$$f(b) - f(a) = \underline{\underline{f'(c)\,(b-a)}}$$

b) MVT for integrals:

$$\int_a^b f(x)\,dx = \underline{\underline{f(c)\,(b-a)}}$$

4. The volume obtained by revolving the area A about the y-axis is

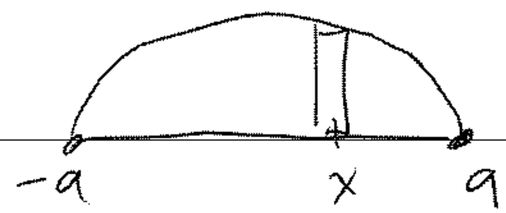
$$V = \int_a^b (2\pi x)\,(f(x) - g(x))\,dx,$$

using the cylindrical shell method.

And $\bar{y} = \frac{1}{A}\int_a^b x\,[f(x) - g(x)]\,dx$.

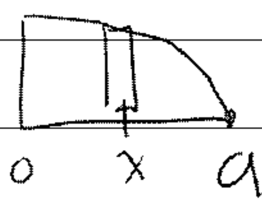
So $V = 2\pi\left(\underbrace{\int_a^b x\,(f(x) - g(x))\,dx}_{A\bar{y}}\right)$

and the Theorem of Pappus, $V = 2\pi\bar{y}A$ is obtained.

5. a) 

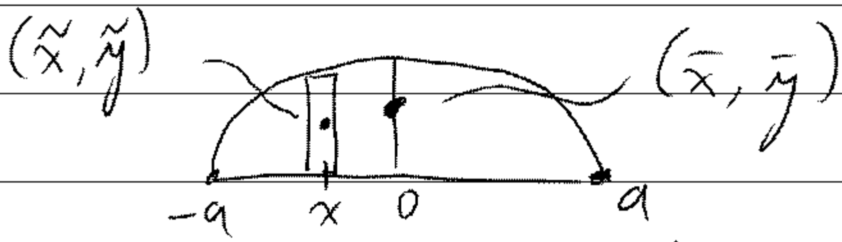
$y = \frac{b}{a}\sqrt{a^2 - x^2}$ (washers)

$$V = \int_{-a}^{a} \pi \left[\frac{b}{a}\sqrt{a^2 - x^2}\right]^2 dx$$

b) 

(cylindrical shells)

$$V = \int_0^a 2\pi x \left[\frac{b}{a}\sqrt{a^2 - x^2}\right] dx$$

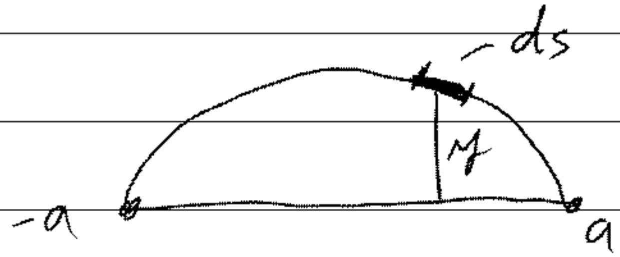
c) 

$$\bar{y} = \frac{\int \tilde{y}\, dm}{\int dm} = \frac{\int_{-a}^{a} \left(\frac{1}{2}\frac{b}{a}\sqrt{a^2 - x^2}\right)\sqrt{a^2 - x^2}\, dx}{\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx}$$

d) $$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(-a\sin t)^2 + (b\cos t)^2}\, dt$$

e)



$0 \le t \le \pi$ gives the top half.

$$A = \int 2\pi y \, ds$$

$$= \int_0^{\pi} 2\pi (b\sin t) \sqrt{a^2\sin^2 t + b^2\cos^2 t}\, dt$$

or $$A = \int_{-a}^{a} 2\pi \left(\frac{b}{a}\sqrt{a^2 - x^2}\right) \sqrt{1 + \left[\frac{-bx}{a\sqrt{a^2-x^2}}\right]^2}\, dx$$