MA 181 Exam 2 sol'ns.

1, a) $\int \cos^3 x \sin^2 x \, dx$

$= \int \underbrace{\cos^2 x}_{(1-\sin^2 x)} \underline{\sin^2 x} \underbrace{(\cos x \, dx)}_{d(\underbrace{\sin x}_{u})}$

$= \int (1-u^2) u^2 \, du$

$= \int u^2 - u^4 \, du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C$

$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$

b) $I = \int x^3 \underbrace{\ln x}_{u} \, dx$

$dv = x^3 dx$

$V = \frac{1}{4}x^4$

$u = \ln x$

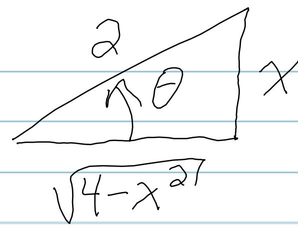
$du = \frac{1}{x} dx$

$I = \int u \, dv = uv - \int v \, du$

$= (\ln x)\left(\frac{x^4}{4}\right) - \int \frac{1}{4}x^4 \cdot \left(\frac{1}{x} dx\right) = \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3 dx$

$= \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$

c) $I = \int \frac{x^2}{\sqrt{4-x^2}}\, dx$



$x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$\sqrt{4-x^2} = 2\cos\theta$

$$I = \int \frac{(2\sin\theta)^2}{(2\cos\theta)}\,(2\cos\theta\, d\theta)$$

$$= 4\int \sin^2\theta\, d\theta = 4\int \frac{1}{2}(1-\cos 2\theta)\, d\theta$$

$$= 2\theta - \sin 2\theta + C$$

$$= 2\sin^{-1}\frac{x}{2} - \sin\left(2\sin^{-1}\frac{x}{2}\right) + C$$

Better way: $\sin 2\theta = 2\sin\theta\cos\theta$

$$= 2\,\frac{x}{2}\,\frac{\sqrt{4-x^2}}{2}$$

$$I = 2\sin^{-1}\frac{x}{2} - \frac{1}{2}x\sqrt{4-x^2} + C$$

2. a) $$\frac{\text{degree } 6 \checkmark}{x^2(x-1)^3(x^2+2x+2)} = \frac{A}{x^2} + \frac{B}{x} +$$

$$\frac{C}{(x-1)^3} + \frac{D}{(x-2)^2} + \frac{E}{(x-1)} + \frac{Fx+G}{x^2+2x+2}$$

b) $$\frac{4x^2+2x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$4x^2+2x+1 = A(x^2+1) + (Bx+C)x$$

$$4x^2+2x+1 = \underbrace{(A+B)}_{4}x^2 + \underbrace{C}_{2}x + \underbrace{A}_{1}$$

$A=1$

$C=2$

$A+B=4 \Leftarrow B = 4-A = 4-1 = 3$

$$\frac{1}{x} + \frac{3x+2}{x^2+1}$$

3. a) $$\sum_{n=1}^{\infty} \frac{n}{n^2+n+1}$$

$$\underbrace{\frac{n}{n^2+n+1}}_{a_n} \sim \frac{n}{n^2} = \underbrace{\frac{1}{n}}_{b_n} \quad \text{for big } n.$$

Limit comparison: $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} =$

$$\lim_{n\to\infty} \frac{\left(\frac{n}{n^2+n+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n^2}{n^2+n+1} \cdot \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} =$$

$$\lim_{n\to\infty} \frac{1}{1+\frac{1}{n}+\frac{1}{n^2}} = \frac{1}{1+0+0} = 1$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ Diverges. Hence, so does the given series by Limit Comparison.

b) $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$$= \lim_{n\to\infty} \frac{\left[\frac{(n+1)^3}{3^{n+1}}\right]}{\left[\frac{n^3}{3^n}\right]} = \lim_{n\to\infty} \frac{1}{3}\left[\frac{n+1}{n}\right]^3$$

$$= \lim_{n\to\infty} \frac{1}{3}\left(\frac{1+\frac{1}{n}}{1}\right)^3 = \frac{1}{3}\cdot 1 = \frac{1}{3} < 1$$

Ratio Test implies series converges.

Or Root Test $\sqrt[n]{a_n} = \left(\frac{n^3}{3^n}\right)^{1/n} = \frac{\left(n^{1/n}\right)^3}{3}$

$$\to \frac{1^3}{3} = \frac{1}{3} < 1. \checkmark$$

c) $\sum_{n=1}^{\infty} (-1)^n \dfrac{n^2}{n(n+10)}$

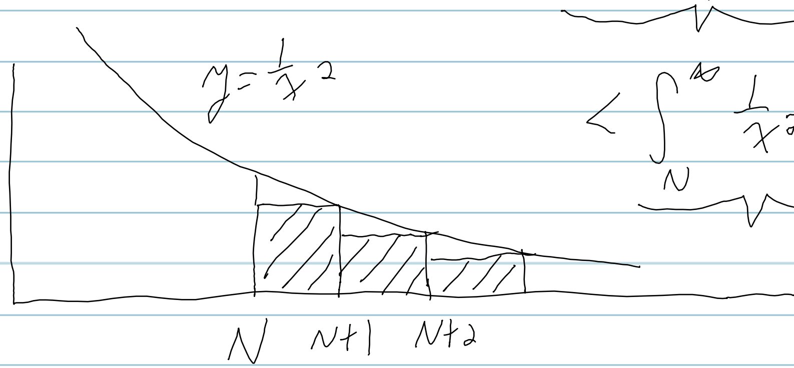
$$a_n = \frac{n^2}{n^2+10n} \cdot \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \frac{1}{1+\frac{10}{n}} \to 1$$

$(-1)^n a_n$ DO NOT tend to zero!

n-th term test yields that series diverges.

4. a) $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ $\qquad L = \sum_{n=1}^{\infty} \dfrac{1}{n^2}$

$$\left| L - \sum_{n=1}^{N} \frac{1}{n^2} \right| = \underbrace{\sum_{n=N+1}^{\infty} \frac{1}{n^2}}_{} \overset{\text{want}}{<} 10^{-2}$$



$$< \underbrace{\int_N^{\infty} \frac{1}{x^2}\,dx}_{} < 10^{-2}$$

$$\lim_{B\to\infty} \left[\frac{-1}{x} \right]_N^B = \lim_{B\to\infty} \left(\frac{-1}{B} - \left(\frac{-1}{N} \right) \right) =$$

$\dfrac{1}{N} \overset{\text{want}}{<} 10^{-2}$ $\qquad$ Need $\dfrac{1}{N} < \dfrac{1}{100}$, $\qquad N > 100$.

$N = 101$.

b) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n} = L$

$$\left| \left(1 - \frac{1}{2} + \frac{1}{3} + \cdots + \underbrace{(-1)^{N+1} \cdot \frac{1}{N}}_{(-1)^{N+1} a_N} \right) - L \right| < \underbrace{a_{N+1}}_{\frac{1}{N+1}}$$

want $< 10^{-2}$. $\frac{1}{N+1} < \frac{1}{100}$, $N+1 > 100$

$N = 100$. $\underline{N > 99}$.

c) $\sum_{n=1}^{\infty} \frac{1}{n!}$ = Taylor Series $(e^x - 1)$ at $x = 1$.

$$e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

$$|f(x) - P_N(x)| = |R_N(x)|$$

$$\left| (e-1) - \sum_{n=1}^{N} \frac{1}{n!} \right| = \left| \frac{f^{(N+1)}(c)}{(N+1)!} (\underset{x=1}{1} - \underset{a=0}{0})^{N+1} \right|$$

But $f(x) = e^x - 1$ $\quad$ c between 0 and 1.

$f^{(N+1)}(x) = e^x$ $\quad$ So $e^c < e^1 < 3$.

So $f^{(N+1)}(c) = e^c$ $\quad$ ERROR $< \frac{3}{(N+1)!} \overset{\text{want}}{<} \frac{1}{100}$

$$\frac{3}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} = \frac{1}{240} < \frac{1}{100}$$

$\uparrow$ $N+1$

$$\boxed{N=5}$$

5. $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$

Ratio Test $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{\left(\frac{x^{n+1}}{(n+1)3^{n+1}}\right)}{\left(\frac{x^n}{n3^n}\right)}\right|$

$= \lim_{n\to\infty} \frac{|x|}{3}\cdot\frac{n}{n+1} = \frac{|x|}{3}$ < 1 conv. / > 1 div.

$|x| < 3 \leftarrow$ absolute convergence.

$|x| > 3 \leftarrow$ divergence.

$x = \pm 3$ delicate.

$x=3$; $\sum_{n=1}^{\infty} \frac{3^n}{n\cdot 3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$x=-3$; $\sum_{n=1}^{\infty} \frac{(-3)^n}{n\cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ Conditionally Convergent!