MA 181 Exam 2 solutions

1. a) $\int \sin^2(2x)\,dx = \int \frac{1}{2}(1-\cos 4x)\,dx$

$= \frac{1}{2}\left(x - \frac{1}{4}\sin 4x\right) + C$

b) $I = \int \frac{2x-3}{x^2+2x+5}\,dx = \int \frac{2x-3}{(x+1)^2+4}\,dx$

Let $u = x+1$. Then $x = u-1$ and $dx = du$.

So $I = \int \frac{2(u-1)-3}{u^2+4}\,du =$

$\int \frac{2u}{u^2+4}\,du - 5\int \frac{1}{u^2+2^2}\,du =$

$= \ln|u^2+4| - 5\cdot\frac{1}{2}\tan^{-1}\frac{u}{2} + C$

$= \ln|x^2+2x+5| - \frac{5}{2}\tan^{-1}\frac{x+1}{2} + C$

c) $I = \int \underbrace{x}_{u}\ \underbrace{e^{-2x}\,dx}_{dv}$

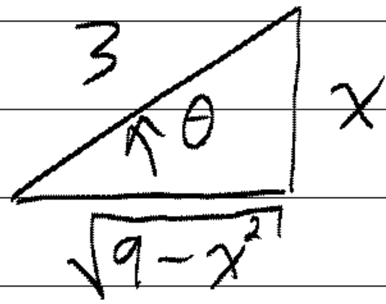
$u = x \quad du = dx$

$dv = e^{-2x}\,dx \quad v = -\frac{1}{2}e^{-2x}$

$$I = uv - \int v\,du = x\left(-\tfrac{1}{2}e^{-2x}\right) - \int\left(-\tfrac{1}{2}e^{-2x}\right)dx$$

$$= -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + C$$

2. $x = 3\sin\theta \qquad dx = 3\cos\theta\, d\theta$



$$\sqrt{9-x^2} = 3\cos\theta$$

$$I = \int \frac{x^3}{\sqrt{9-x^2}}\,dx = \int \frac{(3\sin\theta)^3}{3\cos\theta}\,(3\cos\theta\, d\theta)$$

$$= 27\int \sin^3\theta\, d\theta = 27\int \sin^2\theta\,(\sin\theta\, d\theta)$$

Let $u = \cos\theta$. $du = -\sin\theta\, d\theta$. $\sin^2\theta = 1-\cos^2\theta$.

$$I = 27\int (1-u^2)(-du) =$$

$$= 27\left(-u + \tfrac{1}{3}u^3\right) + C$$

$$= -27\cos\theta + 9\cos^3\theta + C$$

$$= -27\left(\frac{\sqrt{9-x^2}}{3}\right) + 9\left(\frac{\sqrt{9-x^2}}{3}\right)^3 + C$$

3. $A_n = \left(1-\frac{3}{n}\right)^n \qquad (1^\infty)$

$$\ln A_n = n\ln\left(1-\frac{3}{n}\right) = \frac{\ln\left(1-\frac{3}{n}\right)}{\left(\frac{1}{n}\right)} \quad \left(\frac{0}{0}\right)$$

$$\lim_{n\to\infty} \ln A_n \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{\left(1-\frac{3}{n}\right)}\left(\frac{3}{n^2}\right)}{\left(-\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{-3}{\left(1-\frac{3}{n}\right)}$$

$$= -3.$$

Hence $A_n = e^{\ln A_n} \longrightarrow e^{-3}$ as $n\to\infty$.

4. Let $a_n = \frac{\ln n}{n^{8/7}}$ and $b_n = \frac{1}{n^p}$.

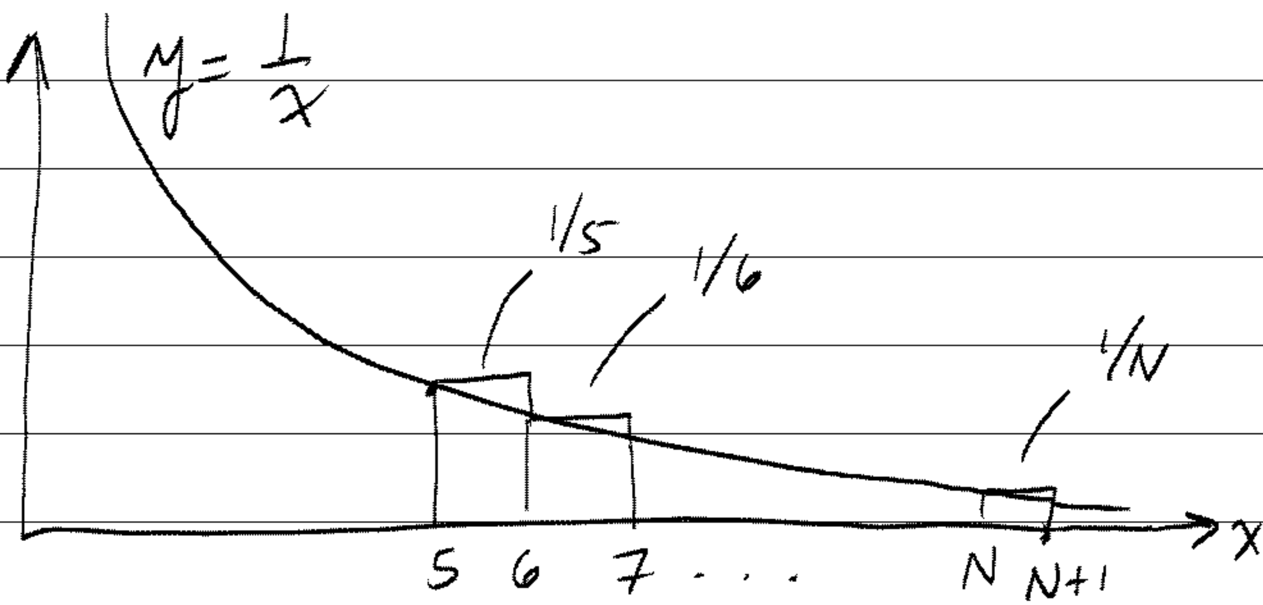
$$\frac{a_n}{b_n} = \frac{\ln n}{n^{8/7-p}} \longrightarrow 0 \text{ as } n\to\infty \text{ if } \frac{8}{7}-p>0,$$

i.e., if $p < \frac{8}{7}$.

But the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ only converges if $p>1$.

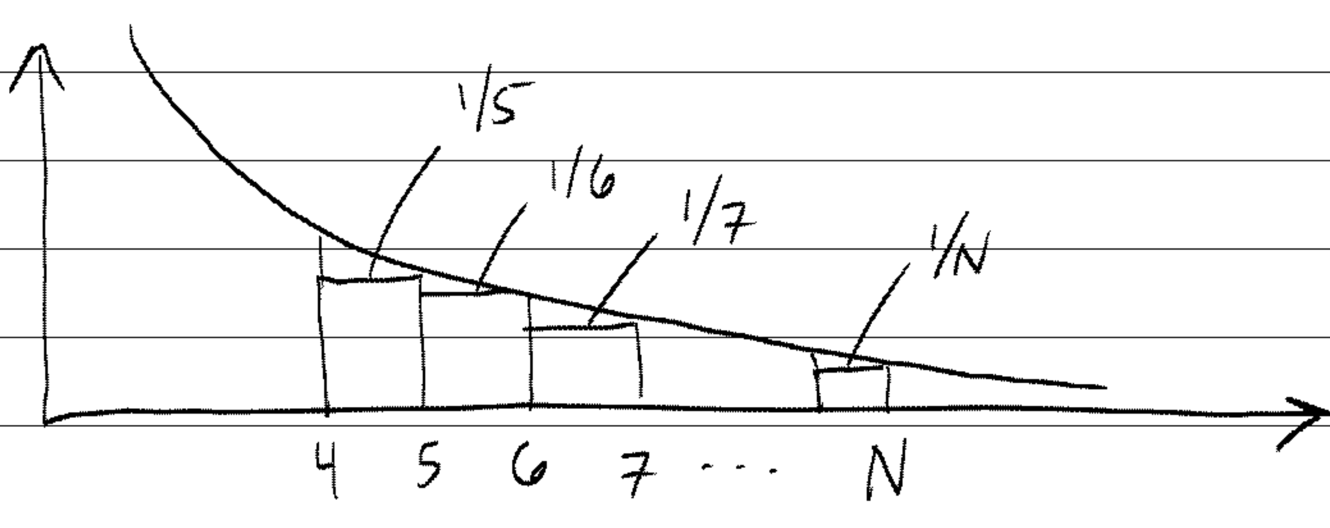
Hence, the Limit Comparison Theorem yields that $\sum_{n=1}^{\infty} \frac{\ln n}{n^{8/7}}$ converges exactly when $1<p<\frac{8}{7}$.

5. a)



$$\sum_{n=5}^{N} \frac{1}{n} > \int_5^{N+1} \frac{1}{x}\,dx$$

b)



$$\sum_{n=5}^{N} \frac{1}{n} < \int_{4}^{N} \frac{1}{x}\, dx$$

The inequality in part (a) shows that

$$\sum_{n=5}^{N} \frac{1}{n} > \ln(N+1) - \ln 5 \longrightarrow \infty \text{ as } N \to \infty.$$

6. A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent when $\sum_{n=1}^{\infty} |a_n|$ converges (to a finite value). It is conditionally convergent if it converges, but not absolutely. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is an example of a series that is conditionally convergent.