Final Exam Solutions

Note Title 12/13/2005

1. $\frac{d}{dx}\int_{-x}^{x} \frac{(t+1)^2}{1+t^5}\,dt =$

$$\frac{(x+1)^2}{1+x^5} - \frac{(-x+1)^2}{1+(-x)^5}\cdot(-1)$$

2. a) $u = 1+x^2$, so $x^2 = u-1$

$du = 2x\,dx$

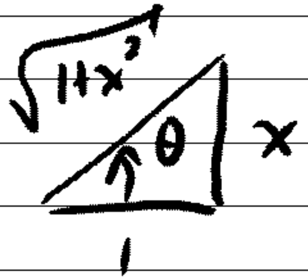
$$\int \frac{x^3}{(1+x^2)^{1/3}}\,dx = \frac{1}{2}\int \frac{x^2\,(2x\,dx)}{(1+x^2)^{1/3}}$$

$$= \frac{1}{2}\int \frac{(u-1)\,du}{u^{1/3}} = \frac{1}{2}\int u^{2/3} - u^{-1/3}\,du$$

$$= \frac{1}{2}\left[\frac{1}{\left(\frac{5}{3}\right)}u^{5/3} - \frac{1}{\left(\frac{2}{3}\right)}u^{2/3}\right] + C$$

$$= \frac{1}{2}\left[\frac{3}{5}(1+x^2)^{5/3} - \frac{3}{2}(1+x^2)^{2/3}\right] + C$$

b)



$$x = \tan\theta \qquad dx = \sec^2\theta\, d\theta$$

$$\sqrt{1+x^2} = \sec\theta$$

$$(1+x^2)^2 = \left(\sqrt{1+x^2}\right)^4 = \sec^4\theta$$

$$\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1}{\sec^4\theta}(\sec^2\theta)\,d\theta$$

$$= \int \frac{1}{\sec^2\theta}\,d\theta = \int \cos^2\theta\, d\theta =$$

$$= \int \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C$$

$$= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$$

$$= \frac{1}{2}\tan^{-1}x + \frac{1}{2}\,\frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{\sqrt{1+x^2}} + C$$

$$= \frac{1}{2}\tan^{-1}x + \frac{x}{2(1+x^2)} + C$$

c) $u = \ln x \quad dv = x^2\,dx$

$du = \frac{1}{x}\,dx \quad v = \frac{1}{3}x^3$

$$\int x^2 \ln x\,dx = \int u\,dv = uv - \int v\,du$$

$$= \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \left(\frac{1}{x}\,dx\right)$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$$

d) $$\int \frac{x-2}{x^2-4x+8}\,dx = \int \frac{(x-2)}{(x-2)^2+4}\,dx$$

$u = (x-2) \quad du = dx$

$$I = \int \frac{u}{u^2+4}\,du \qquad v = u^2+4 \quad dv = 2u\,du$$

$$= \int \frac{\frac{1}{2}dv}{v} = \frac{1}{2}\ln|v| + C$$

$$= \frac{1}{2} \ln(u^2+4) + C$$

$$= \frac{1}{2} \ln\left((x-2)^2+4\right) + C$$

$$= \frac{1}{2} \ln\left[x^2-4x+8\right] + C$$

e) $u = x \qquad dv = e^{-2x}\,dx$

$du = dx \qquad v = -\frac{1}{2}e^{-2x}$

$$\int x e^{-2x}\,dx = \int u\,dv = uv - \int v\,du$$

$$= x\left(-\frac{1}{2}e^{-2x}\right) - \int\left(-\frac{1}{2}e^{-2x}\right)dx$$

$$= -\frac{1}{2}x e^{-2x} - \frac{1}{4}e^{-2x} + C$$

3. Ratio Test: $\lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)}{x^n/n} \right| =$

$$= \lim_{n\to\infty} \frac{n}{n+1}|x| = |x| \begin{matrix} <1 & \text{Conv.} \\ >1 & \text{Div.} \end{matrix}$$

IF $x=1$: $\displaystyle\sum_{1}^{\infty}\frac{x^n}{n}=\sum_{1}^{\infty}\frac{1}{n}$ DIVERGES.

IF $x=-1$: $\displaystyle\sum_{1}^{\infty}\frac{x^n}{n}=-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\cdots$

Converges by Alt. Ser. Test.

So the series converges if $-1\leq x<1$

and diverges if $x<-1$ or $x\geq 1$.

b) $\displaystyle\frac{1}{1-x}=1+x+x^2+x^3+\cdots$ if $|x|<1$.

$$\int_0^x \frac{1}{1-t}\,dt=\sum_{n=0}^{\infty}\int_0^x t^n\,dt$$

$$-\ln|1-x|=\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}=x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots$$

4. $$\int\frac{dy}{y(1+y)}=\int dx=x+C$$

$$\int\frac{1}{y}-\frac{1}{1+y}\,dy=x+C$$

$$\ln|y| - \ln|1+y| = x + C$$

$$\ln\left|\frac{y}{1+y}\right| = x + C$$

$$\left|\frac{y}{1+y}\right| = e^{x+C} = e^{C}e^{x}$$

$$\frac{y}{1+y} = \underbrace{\pm e^{C}}_{K} e^{x}.$$

Plug in $y=2$ when $x=0$:

$$\frac{2}{1+2} = Ke^{0} = K \qquad \boxed{K = \frac{2}{3}}$$

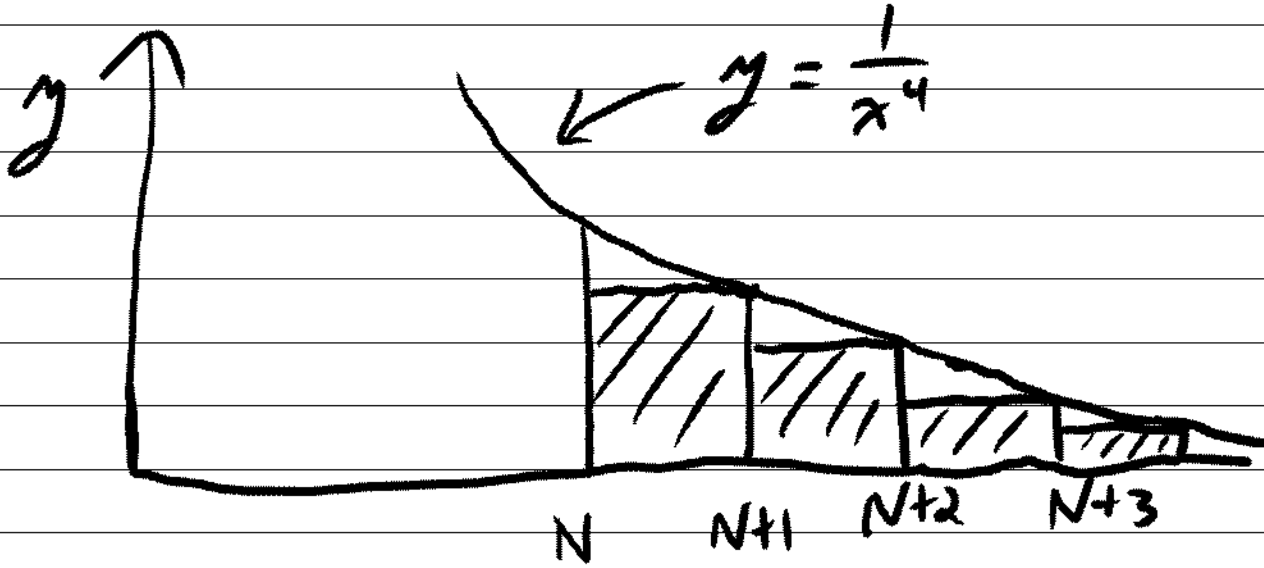
Solve $\frac{y}{1+y} = Ke^{x}$ for y:

$$y = (1+y)Ke^{x} = Ke^{x} + yKe^{x}$$

$$y(1-Ke^{x}) = Ke^{x}$$

$$y = \frac{ke^x}{1-ke^x} = \frac{2/3\, e^x}{1-\frac{2}{3}e^x}$$

$$y = \frac{2e^x}{3-2e^x}.$$

5. 

$$S - \sum_{n=1}^{N} \frac{1}{n^4} = \sum_{n=N+1}^{\infty} \frac{1}{n^4} < \int_N^\infty \frac{1}{x^4}\,dx$$

$$= \lim_{B\to\infty} \int_N^B x^{-4}\,dx = \lim_{B\to\infty} \left[-\frac{1}{3}x^{-3}\right]_N^B$$

$$= \frac{1}{3N^3} < 10^{-3} \quad \text{if} \quad 10^3 < 3N^3 \text{ or}$$

$$N > \frac{10}{\sqrt[3]{3}} .$$

6. Mean Value Theorem shows there is a point α between 0 and 1 where $f'(\alpha) = \frac{f(1) - f(0)}{1-0} = 5$, so $|f'|$ cannot stay less than 5. Rolle's Theorem shows there is a point β between 0 and 2 where $f'(\beta) = 0$. Now the Mean Value Theorem applied to $f'(x)$ yields a point c between α and β where $f''(c) = \frac{f'(\alpha) - f'(\beta)}{\alpha - \beta}$

$$= \frac{5-0}{\alpha-\beta} .$$ So $|f''(c)| = \frac{5}{|\alpha - \beta|} > \frac{5}{2}$.

So $|f''|$ cannot stay less than $5/2$.