# taylor

Note Title 11/28/2005

Proof of Taylor's Theorem with remainder: Suppose $f(x)$ is $N+1$ times differentiable near $x=a$. Then

$$f(x) = \sum_{n=1}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_N(x)$$

where $$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$$

for some $c$ between $a$ and $x$.

I will write out the proof in case $N=1$. Let

$$T_1(x,a) = f(a) + \frac{f'(a)}{1!}(x-a)$$

be the first order Taylor

polynomial. Notice that

$$f(x) = T_1(x,a) + M(x-a)^2$$

where $M = \dfrac{f(x) - T_1(x,a)}{(x-a)^2}$.

I must show that $M = \dfrac{f''(c)}{2!}$

for some $c$ between $x$ and $a$.

Now here is a very sneaky, but great, idea. Define

$$\Phi(t) = f(x) - T_1(x,t) - M(x-t)^2.$$

(Notice that we have replaced the constant $a$ with a variable $t$.)

I claim that $\Phi(x) = 0$ and $\Phi(a) = 0$, and so $\Phi$ satisfies the conditions of Rolle's Theorem.

Indeed,

$$T_1(x,x) = f(x) + \frac{f'(x)}{1!}(x-x) = f(x)$$

and so

$$\Phi(x) = f(x) - f(x) + M(x-x)^2 = 0,$$

and $M$ was chosen exactly so that $\Phi(a) = 0$. So Rolle's Theorem gives a $c$ between $a$ and $x$ where $\Phi'(c) = 0$. But

$$\Phi'(t) = -\frac{d}{dt}\left[f(t) + \frac{f'(t)}{1!}(x-t)\right]$$

$$-\frac{d}{dt}\left[M(x-t)^2\right]$$

$$= -f'(t) - f''(t)(x-t) + f'(t) + 2M(x-t)$$

$$= -f''(t)(x-t) + 2M(x-t).$$

Now plug in $x = c$ to see that

$$\Phi'(c) = -f''(c)(x-c) + 2M(x-c) = 0,$$

i.e., $M = \frac{f''(c)}{2}$. Done!

The same proof works for big N. Try it.