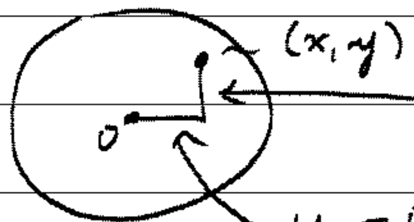
10/9/2009

Exam 1 sol$^n$s.

① $f(x+iy) = u(x,y) + i\underbrace{v(x,y)}_{\equiv 0}$

C-R equ's
$$\left.\begin{aligned} u_x &= v_y \equiv 0 \\ u_y &= -v_x \equiv 0 \end{aligned}\right\} \nabla u \equiv 0.$$

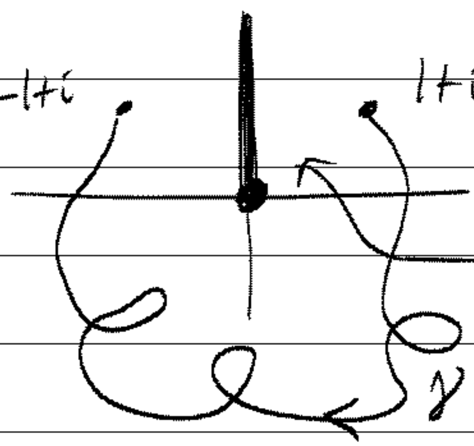



$u_y = 0 \Rightarrow u$ const. along vertical

$u_x = 0 \Rightarrow u$ const. along horizontal

So $u$ is constant.

②




misses Imaginary Axis and 0.

Define $\log z = \mathrm{Ln}|z| + i\theta$, $\theta \in \arg z$, $-\frac{3\pi}{2} < \theta < \frac{\pi}{2}$

$$\int_\gamma \frac{1}{z}\,dz = \int_\gamma \frac{d}{dz}[\log z]\,dz = [\log z]_{1+i}^{-1+i}$$

$$= \left(\mathrm{Ln}|-1+i| + i\left(-\frac{5\pi}{4}\right)\right) - \left(\mathrm{Ln}|1+i| + i\frac{\pi}{4}\right)$$

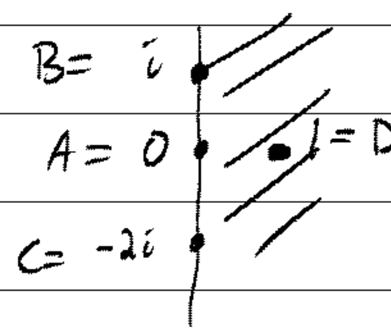
$$= -\frac{3\pi}{2} i$$

(3) $(1-z)(1+z+\cdots+z^N) = 1+z+z^2+\cdots+z^N$
$\quad - (z+z^2+\cdots+z^N+z^{N+1})$
$\quad = 1-z^{N+1}$

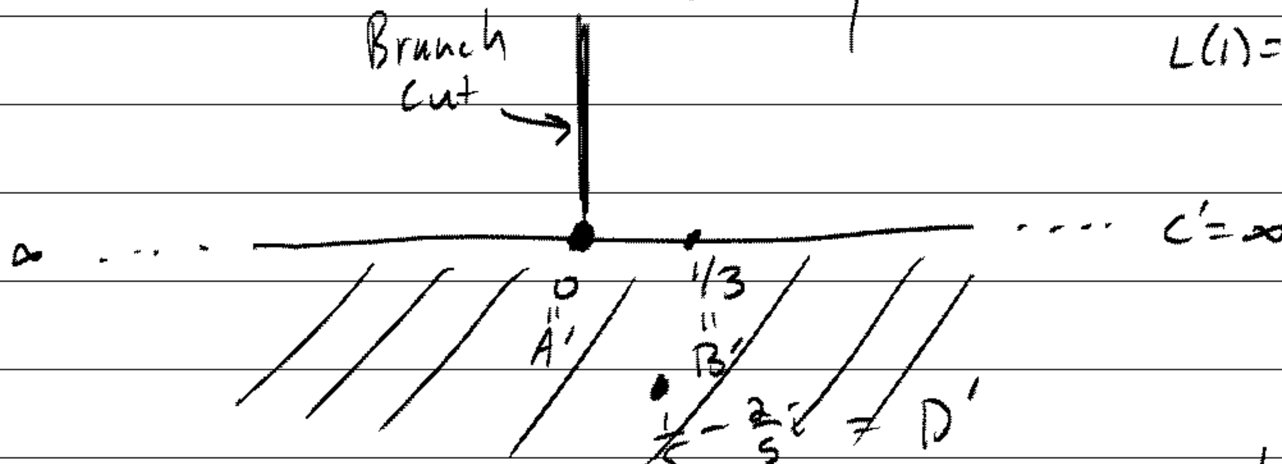
$$(1+z+\cdots+z^N) = \frac{1-z^{N+1}}{1-z}, \quad z \neq 1.$$

$$1+\cos\theta+\cos 2\theta+\cdots+\cos N\theta =$$
$$\mathrm{Re}\left[1+e^{i\theta}+e^{i2\theta}+\cdots+e^{iN\theta}\right] =$$
$$\mathrm{Re}\left[1+e^{i\theta}+(e^{i\theta})^2+\cdots+(e^{i\theta})^N\right] = \mathrm{Re}\left[\frac{1-e^{i\theta(N+1)}}{1-e^{i\theta}}\right]$$

(4) $L(z) = \dfrac{z}{z+2i}$



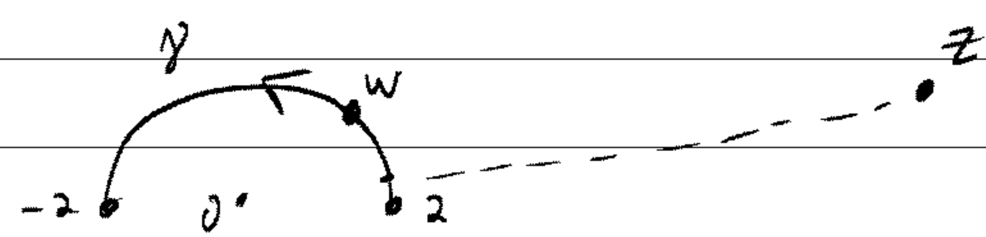

$L(0)=0$
$L(-2i)=\infty$
$L(i)=1/3$
$L(1)=\frac{1}{1+2i}=\frac{1-2i}{5}=\frac{1}{5}-\frac{2}{5}i$




$\log z$ same as in prob. 2. $\quad \sqrt{L(z)} = e^{\frac{1}{2}\log L(z)}$.

Note: $\log z = \mathrm{Log}\, z$ on Lower Half Plane.

(5)



$$\Bigg|\underbrace{\int_\gamma \frac{1}{w-z}\,dw}_{F(z)}\Bigg| \le \left(\max_{w\in\gamma} \frac{1}{|w-z|}\right) \underbrace{\text{Length}(\gamma)}_{2\pi}$$

$$|w-z| \ge \Big|\underbrace{|w|}_{2}-|z|\Big| = |z|-2 \quad \text{if } \begin{cases}|w|=2\\ |z|>2,\end{cases}$$

So $\displaystyle |F(z)| \le \frac{1}{|z|-2}\cdot 2\pi$ if $|z|>2$.

Hence $\displaystyle \lim_{z\to\infty} F(z) = 0.$

(6) $z = x+iy$. $\quad z^2 = (x^2-y^2)+i2xy$

$$e^{z^2} = e^{(x^2+y^2)}\, e^{i2xy}$$

$$|e^{z^2}| = e^{x^2-y^2}\,\underbrace{|e^{i2xy}|}_{=1} = e^{x^2-y^2}$$

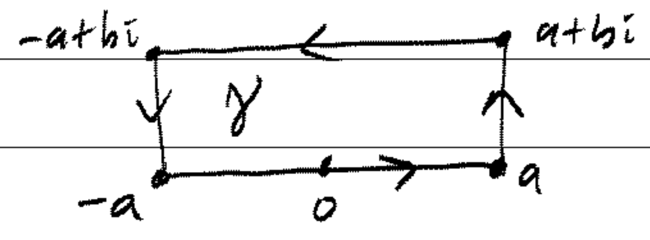
$x^2-y^2 < x^2+y^2$ if $y\neq 0$.

$e^{x^2-y^2} < e^{x^2+y^2}$ if $y \neq 0$.

Hence $|e^{z^2}| < e^{|z^2|}$ if $y\neq 0$.

Equality happens exactly when $y=0$.

<u>Exciting Problem at end of VI.</u>



How to compute $\int_{-\infty}^{\infty} e^{-t^2} \cos bt \, dt$.

Trick: Compute $\int_\gamma e^{-z^2}\,dz = 0$.

Need to know that $e^{-z^2}$ has an analytic anti-derivative.
It does, by power series argument:

$$e^{-z^2} = 1 + (-z^2) + \frac{(-z^2)^2}{2!} + \frac{(-z^2)^3}{3!}$$

Want $F(z)$ with $F'(z) = e^{-z^2}$.

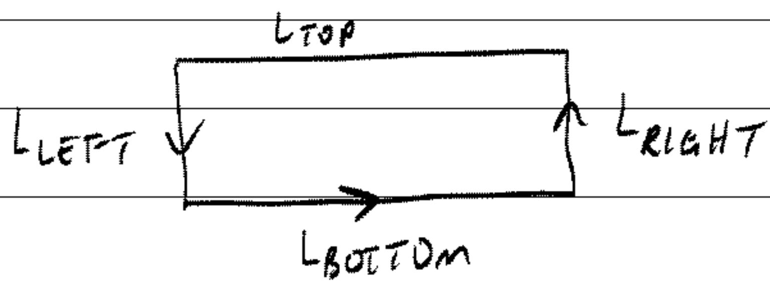
$$\text{Try} \quad F(z) = z - \frac{z^3}{3} + \frac{z^5}{5\cdot 2!} - \frac{z^7}{7\cdot 3!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)\,n!}$$

$\longleftarrow$ Chap V, converges to analytic $F$ and $F'(z) = e^{-z^2}$.
(Use Ratio Test.)

$$\text{so} \quad \int_\gamma e^{-z^2}\,dz = \int_\gamma F'(z)\,dz = \left[F(z)\right]_{\text{START}}^{\text{END}} = 0.$$

Plan:




BOTTOM: $z(t) = t$, $-a \le t \le a$.

$$\int_{L_{BOTTOM}} e^{-z^2}\,dz = \int_{-a}^{a} e^{-t^2}\cdot 1 \cdot dt \longrightarrow \sqrt{\pi} \quad \text{as } a \to \infty.$$

$$\left(\int_{-\infty}^{\infty} e^{-t^2}\,dt\right)^2 = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy\right)$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{-e^{-r^2}}{2}\right]_0^{\infty} d\theta = \int_0^{2\pi}\left[0-\left(-\tfrac{1}{2}\right)\right]d\theta$$

$$= \pi$$

Read VI, 12 and problems for later.