MATH 425, Exam 1

Each problem is 20 points

(20) **1.** Find all complex numbers z such that $z^4 = -16$. Express them in polar form $re^{i\theta}$ and cartesian form $a + ib$.

(20) **2.** **a)** Using the notation $f(x+iy) = u(x,y)+iv(x,y)$, write down the Cauchy-Riemann equations and state exactly what is needed to deduce that f is an analytic function on a domain Ω.

b) Show that $f(x+iy) = e^y e^{ix}$ is *not* analytic on $\mathbb{C}$.

(20) **3.** **a)** Define a branch of a complex log function and use it to compute $\int_\gamma \frac{1}{z}\,dz$ where γ is any curve that starts at $3i$ and ends at 2, avoiding the set

$$\{z = re^{i\theta} : r \geq 0,\ \frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}\}.$$

b) Compute $\int_\gamma \frac{1}{z^2}\,dz$. Explain your reasoning.

(20) **4.** Let C_R denote the half circle parametrized by $z(t) = Re^{it}$, $0 \leq t \leq \pi$. Use careful estimates to show that

$$\int_{C_R} \frac{z-2}{z^7+5}\,dz$$

tends to zero as $R \to \infty$.

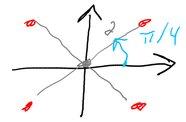
(20) **5.** Compute

$$I = \int_{C_2} \frac{e^{2z}}{(z-1)^2(z-5)}\,dz$$

where C_2 is the counterclockwise circle of radius **two** about the origin. Explain.

Exam 1 solutions

1. $\underbrace{(re^{i\theta})^4}_{r^4e^{i4\theta}} = -16 = 16e^{i\pi}$



Need $r^4 = 16$, $4\theta = \pi + 2\pi n$, $n \in \mathbb{Z}$.

So $r = 2$, $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ to get four distinct roots.

$z = 2e^{i\pi/4}, 2e^{i3\pi/4}, 2e^{i5\pi/4}, 2e^{i7\pi/4} = \pm\sqrt{2} \pm i\sqrt{2}$.

2. a) u, v need to be C^1-smooth on Ω and $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

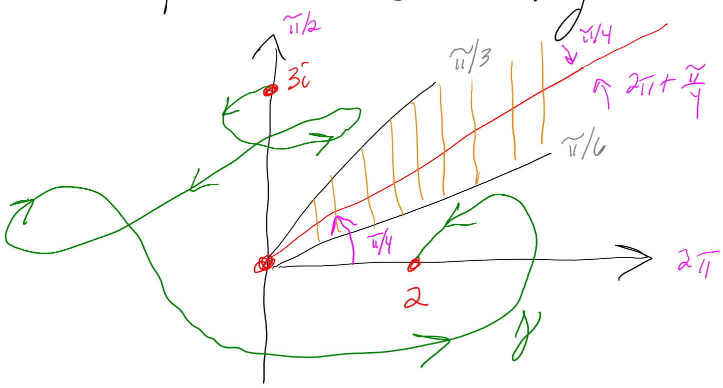
b) $e^y e^{ix} = \underbrace{e^y \cos x}_{u} + i\underbrace{e^y \sin x}_{v}$

$\begin{cases} u_x = -e^y \sin x \neq e^y \sin x = v_y. & \text{C-R eqn \#1 fails } (x \neq 2\pi n.) \\ u_y = e^y \cos x \neq -e^y \cos x = -v_x & \text{C-R eqn \#2 fails } (x \neq \frac{\pi}{2} + 2\pi n) \end{cases}$

So f not analytic on $\mathbb{C}$.

3. $\log_{\pi/4} z = \ln|z| + i\theta$ where $\theta \in \arg z$ with $\frac{\pi}{4} < \theta < 2\pi + \frac{\pi}{4}$

is analytic in an open set containing the curve γ:



$$\int_\gamma \frac{1}{z}\,dz = \int_\gamma \frac{d}{dz}\left[\log_{\pi/4} z\right] dz = \log_{\pi/4} 2 - \log_{\pi/3} 3i$$

$$= \left(\ln 2 + i(2\pi)\right) - \left(\ln 3 + i\frac{\pi}{2}\right) = \ln\frac{2}{3} + i\frac{3\pi}{2}$$

b) $\int_\gamma \frac{1}{z^2}\,dz = \int_\gamma \frac{d}{dz}\left[-\frac{1}{z}\right] dz = -\frac{1}{2} - \left(-\frac{1}{3i}\right) = -\frac{1}{2} - \frac{1}{3}i$

4. $$\left|\int_{C_R} \frac{z-2}{z^7+5}\,dz\right| \le \left(\max_{C_R} \left|\frac{z-2}{z^7+5}\right|\right) \text{Length}(C_R)$$

If $|z| = R > 5^{1/7}$, then $$\left|\frac{z-2}{z^7+5}\right| \le \frac{|z|+2}{\left||z^7|-5\right|} \le \underbrace{\frac{R+2}{|R^7-5|}}_{= \frac{R+2}{R^7-5} \text{ when } |z|=R > 5^{1/7}}.$$

$|z^7+5| \ge \left||z|^7 - 5\right|$ denom est.

So $\boxed{|I| \le \frac{R+2}{R^7-5}(\pi R)}$ and this $\to 0$ as $R \to \infty$.

5. Let $f(z) = e^{2z}/(z-5)$. f is analytic inside and on C_2.

Higher order Cauchy integral formula $f'(1) = \frac{1!}{2\pi i}\int_{C_2} \frac{f(z)}{(z-1)^{1+1}}\,dz$

yields $$I = 2\pi i\, f'(1) = 2\pi i \cdot \left[\frac{2e^{2z}(z-5) - e^{2z}(1)}{(z-5)^2}\right]\Bigg|_{z=1}$$

$$= 2\pi i \,\frac{2e^2(-4) - e^2}{4^2} = \underline{\underline{-\frac{9\pi e^2}{8} i}}$$