

## MATH 425, Practice Problems

1. Prove that if a twice continuously differentiable harmonic function vanishes on an open subset of a domain, then it must be zero on the whole domain.
2. Suppose that  $f$  and  $g$  are analytic on a disc about  $a$  and that each has a zero of multiplicity  $N$  at  $a$ . Prove that  $\lim_{z \rightarrow a} \frac{f(z)}{g(z)} = \frac{f^{(N)}(a)}{g^{(N)}(a)}$ .
3. Let  $f(z) = \frac{1}{1+z^2}$ . Find the first three terms in the power series  $\sum_{n=0}^{\infty} a_n(z-2)^n$  for  $f$  centered at  $z=2$ . What is the radius of convergence of the series? Explain.
4. What is the radius of convergence of the power series for  $\frac{\sin z}{z}$  centered at  $z=\pi$ ?
5. From Sarason, V.10.2, V.14.1, V.16.2, VI.12.2, VII.8.1, VII.11.3, VII.13.1-2, VII.14.1, VII.16.1-2
6. Suppose  $a_1, a_2, \dots, a_n$  are points on the unit circle. For a point  $z$  on the unit circle, let  $p(z)$  denote the product of the distances from  $z$  to each of the  $n$  points on the circle. Use the Maximum Modulus Principle to prove that there is a point  $z$  on the circle such that the product  $p(z)$  is exactly equal to one.
7. State Liouville's Theorem.
8. Find a formula involving only elementary functions (like polynomials, log, and square roots) for the inverse hyperbolic sine function. Hint: Solve  $z = \sinh w = \frac{1}{2}(e^w - e^{-w})$  for  $w$ .
9. Find all functions  $f$  analytic in a disc  $D_R(0)$  that satisfy  $f(0) = e^i$  and  $|f(z)| \leq 1$  for all  $z$  in the disc.
10. Assume  $b_0 = 1$ . Constants  $b_2, b_4, b_6, \dots$  are generated via the recursion formula

$$b_{n+2} = \frac{4(n+3)(n-5)}{7(n+1)(n+2)} b_n.$$

What is the radius of convergence of the power series

$$\sum_{n=0}^{\infty} b_{2n} z^{2n} = b_0 + b_2 z^2 + b_4 z^4 + \dots?$$

11. If the complex numbers  $a_n$  tend to zero as  $n$  tends to infinity, show that the radius of convergence of the power series  $\sum_{n=0}^{\infty} a_n z^n$  is at least one.
12. Let  $C_R$  denote the half circle parameterized by  $z(t) = Re^{it}$  for  $0 \leq t \leq \pi$ . Prove that

$$\int_{C_R} \frac{e^{iz}}{z^4 + 1} dz$$

tends to zero as  $R$  tends to infinity.