Solutions to Exam 2

1. $$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Take $n=3$, $f(z) = e^{2z}$, $a=0$ to see

$$2\cdot 2\cdot 2\, e^{2\cdot 0} = \frac{3!}{2\pi i}\int_C \frac{e^{2z}}{z^4}\,dz.$$

$$\text{So } I = \frac{8\cdot 2\pi i}{6} = \frac{8\pi i}{3}.$$

2. The Jukovsky map $J(z) = \frac{1}{2}(z + \frac{1}{z})$ maps $\{z : |z| > 1\}$ one-to-one onto $\mathbb{C} - [-1,1]$. $J(1/z)$ maps $D_1(0) - \{0\}$ 1-1 onto $\mathbb{C} - [-1,1]$ since $1/z$ maps the outside of the unit circle 1-1 onto the inside minus 0. But $J(\frac{1}{z}) = J(z)$. Now J^{-1} is an analytic map that maps $\mathbb{C} - [-1,1]$ one-to-one onto $D_1(0) - \{0\}$. If f is entire and

misses $[-1,1]$, then $J^{-1}(f(z))$ is a $\underline{\text{bounded}}$ entire function. Liouville's $\Longrightarrow$ $J^{-1}(f(z)) \equiv c$, a constant. Hence $f(z) \equiv J(c)$, a constant.

3. Use the Ratio Test:

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{\frac{(n+1)^{(n+1)}}{(n+1)!} z^{(n+1)^2}}{\frac{n^n}{n!} z^{n^2}}\right|$$

$$= \frac{n!}{(n+1)!} \cdot \frac{(n+1)(n+1)^n}{n^n} |z|^{2n+1}$$

$$= \left(1+\frac{1}{n}\right)^n |z|^{2n+1}$$

$\left(1+\frac{1}{n}\right)^n \longrightarrow e$ as $n \to \infty$, so this sequence is bounded.

Hence $\left|\frac{u_{n+1}}{u_n}\right| \longrightarrow 0$ if $|z|<1$ and the series converges.

$$\left|\frac{u_{n+1}}{u_n}\right| \to \infty \quad \text{if} \quad |z| > 1, \text{ and}$$

so the series diverges (because the terms do not go to zero). Therefore, $R = 1$.

4. $f(z) = \dfrac{1+z}{1-z}$ $\quad f'(z) = \dfrac{1\cdot(1-z) - (-1)(1+z)}{(1-z)^2} = \dfrac{2}{(1-z)^2}$

$f'''(z) = \dfrac{4}{(1-z)^3}$. Near $z = i$,

$$f(z) = f(i) + f'(i)(z-i) + \frac{f''(i)}{2!}(z-i)^2 + \cdots$$

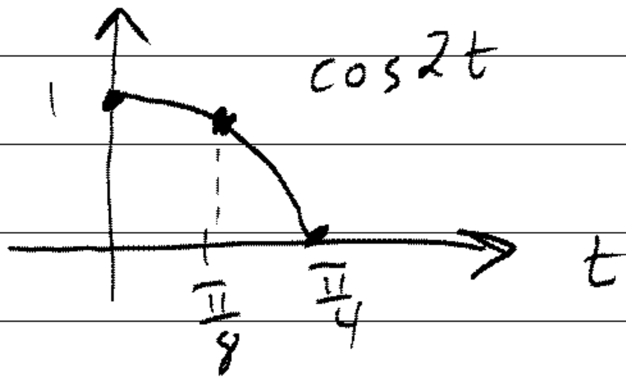
$$= \left(\frac{1+i}{1-i}\right) + \frac{2}{(1-i)^2}(z-i) + \frac{2}{(1-i)^3}(z-i)^3 + \cdots$$

The radius of convergence is $\sqrt{2}$, which is the distance from i to the nearest singular point of $f(z)$ at $z = 1$.

5. $$\left|\int_{C_R} e^{-z^2}\, dz\right| \le \left(\max_{C_R} \left|e^{-z^2}\right|\right)\left(\frac{\pi}{8} R\right)$$

$$\left|e^{-(Re^{it})^2}\right| = \left|e^{-R^2 e^{i2t}}\right| =$$

$$\left| e^{-R^2 \cos 2t} \cdot e^{-iR^2 \sin 2t} \right| = e^{-R^2 \cos 2t}$$



$\cos 2t \geq \cos 2 \cdot \frac{\pi}{8} = \frac{1}{\sqrt{2}}$

if $0 \leq t \leq \frac{\pi}{8}$.

So $-R^2 \cos 2t \leq -R^2/\sqrt{2}$

on the interval.

$$\max_{C_R} \left| e^{-z^2} \right| = e^{-R^2/\sqrt{2}}.$$

So $|I| \leq e^{-R^2/\sqrt{2}} \left(\frac{\pi}{8} \cdot R \right)$

and this tends to zero as $R \to \infty$.