MA 527 Exam 2 Fall 2002 Name ____________________

1a. (15) Find the (Laplace) convolution $u(t-\pi) * \sin(t)$.

$$\int_0^t u(\tau-\pi)\sin(t-\tau)\,d\tau = \begin{cases} 0 & t<\pi \\ \int_\pi^t \sin(t-\tau)\,d\tau & \end{cases}$$

$$= u(t-\pi)\left(1-\cos(t-\pi)\right) = u(t-\pi)(1+\cos t)$$

u(t-π) $*$ $\sin(t)$= $u(t-\pi)\,(1+\cos t)$

1b. (5) Let $F(s) = \dfrac{e^{-\pi s}}{s(s^2+1)}$. Using Problem 1a above, find the inverse Laplace transform $f(t)$ of $F(s)$.

f(t)= $u(t-\pi)\,(1+\cos t)$

2. (15) Find the Laplace transform $F(s)$ of $f(t) = tu(t-1)$.

$$F(s) = e^{-s}\mathcal{L}(t+1) = e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right)$$

$$\boxed{F(s)= \quad e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right)}$$

3. (15) Find the inverse Laplace transform $f(t)$ of $F(s) = \dfrac{se^{-s}}{(s+3)^2+4}$.

$$\mathcal{L}^{-1}\left(\frac{s+3}{(s+3)^2+4} - \frac{3}{2}\frac{2}{(s+3)^2+4}\right) = e^{-3t}\left(\cos 2t - \frac{3}{2}\sin 2t\right.$$

$$\boxed{f(t) = e^{-3(t-1)}\left(\cos 2(t-1) - \frac{3}{2}\sin 2(t-1)\right)u(t-1)}$$

4. (15) Compute the Fourier sine series for $f(x) = x \quad 0 \le x \le 1$.

$$b_n = 2\int_0^1 x \sin n\pi x \, dx$$

$$= -\frac{2}{n\pi}\left(x \cos n\pi x \Big|_0^1 - \int_0^1 \cos n\pi x \, dx \right.$$

$$= \frac{-2}{n\pi}\left(\cos n\pi - \frac{\sin n\pi x}{n\pi} \Big|_0^1 \right)$$

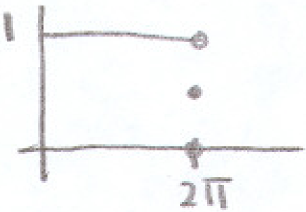
$$= -\frac{2}{n\pi} \cos n\pi = \frac{2}{n\pi}(-1)^{n+1}$$

$f(x) =$

$$\boxed{\sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1} \sin n\pi x}$$

5a. (15) Let

$$f(x) = \begin{cases} 1 & 0 < x < 2\pi \\ 0 & \text{otherwise.} \end{cases}$$



Compute the Fourier cosine transform of f.

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos wx \, dx = \sqrt{\frac{2}{\pi}} \int_0^{2\pi} \cos wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin wx}{w} \Big|_0^{2\pi} = \sqrt{\frac{2}{\pi}} \frac{\sin 2\pi w}{w}$$

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin 2\pi w}{w} \cos wx \, dx$$

$$= \begin{cases} 0 & |x| > 2\pi \\ 1/2 & x = \pm 1/2 \\ 1 & |x| < 2\pi \end{cases}$$

Fourier cosine transform

$$\boxed{\sqrt{\frac{2}{\pi}} \frac{\sin 2\pi w}{w}}$$

5b. (5) Using Problem 5a above, evaluate

$$\frac{2}{\pi} \int_0^{\infty} \frac{\sin 2\pi w \cos 2\pi w}{w} dw$$

$$\boxed{1/2}$$

6. (15) Let

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Compute the complex Fourier transform of f.

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x} e^{-iwx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{-1-iw} e^{(-1-iw)x} \Big|_0^\infty$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{1+iw}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1-iw}{1+w^2}$$

$$\boxed{\frac{1}{\sqrt{2\pi}} \frac{1-iw}{1+w^2}}$$