

1. Define

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 & 3 \\ 1 & 1 & 0 & 5 & 6 \\ 1 & 1 & 0 & 8 & 9 \end{bmatrix}$$

(5) (a) Use row operations to find an echelon form of A.

$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 6 & 6 \end{bmatrix} \rightsquigarrow$$

5 pts

Reduced echelon
OK too

$$\begin{bmatrix} 1 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(5) (b) What is the dimension of the column space of A?

dim = 2

5 pts

(10) (c) Find a basis for the space of solutions (the null space) of the system

$$\begin{bmatrix} 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -4 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

\uparrow x_2 \uparrow x_4
 free

$x_2 = 1, x_4 = 0$

$x_2 = 0, x_4 = 1$

$x_3 = 4 \cdot 1 = 4$
 $x_1 = -1 \cdot 0 + 2 \cdot 4 + 3 \cdot 1 = 11$

$x_3 = -4 \cdot 0 = 0$
 $x_1 = -1 + 2 \cdot 0 + 3 \cdot 0$

$$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 11 \\ 0 \\ 4 \\ 1 \end{pmatrix}$$

basis vectors

$$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 11 \\ 0 \\ 4 \\ 1 \end{pmatrix}$$

5 pts each

2. (20) The matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ has 2 eigenvalues λ_1 and λ_2 . Find λ_1 and λ_2 and basis vectors for the corresponding eigenspaces.

$$\det(A - \lambda I) = (1 - \lambda)(2 - \lambda)^3 \quad \lambda = 1, 2, 2, 2$$

For $\lambda = 1$: $\begin{bmatrix} 0 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ x_1 free
 $x_2 = x_3 = x_4 = 0$ $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

For $\lambda = 2$: $\begin{bmatrix} -1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ x_2, x_4 free
 $x_3 = 0$

$$x_2 = 1, x_4 = 0$$

$$x_3 = 0$$

$$x_1 = 2 \cdot 1 + 4 \cdot 0 = 2$$

$$\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$x_2 = 0, x_4 = 1$$

$$x_3 = 0$$

$$x_1 = 2 \cdot 0 + 4 \cdot 1 = 4$$

$$\begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

5 pts each

e-vals
5 pts

$$\lambda_1 = 1$$

basis for eigenspace =

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 2$$

basis for eigenspace =

$$\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

3. (20) Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Find an orthogonal matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix} = -\lambda \det \begin{bmatrix} \lambda & 1 \\ 1 & -\lambda \end{bmatrix} = -\lambda(\lambda^2 - 1)$$

$\lambda = 0, \pm 1$ For $\lambda = 0$: $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$
 $x_1 = 0, x_3 = 0, x_2$ free
 Get $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ unit length ✓

For $\lambda = 1$: $\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$
 $x_2 = 0$
 x_3 free
 $x_3 = 1 \Rightarrow x_1 = 1$

Get $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Normalized: $\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$

For $\lambda = -1$: $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$
 $x_2 = 0$
 x_3 free
 $x_3 = 1 \Rightarrow x_1 = -1$
 $x_2 = 0$
 5 pts each col

Get $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

Normalized: $\begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$

$$P = \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

3 pts for normalized cols

2 pts

4. (20) The matrix $A = \begin{bmatrix} 0 & 5 \\ -1 & -2 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1 + 2i$ with eigenvector $\begin{pmatrix} 5 \\ -1 + 2i \end{pmatrix}$ and $\lambda_2 = -1 - 2i$ with eigenvector $\begin{pmatrix} 5 \\ -1 - 2i \end{pmatrix}$. Determine the type and stability of the origin for the system $dy/dt = Ay$. Find a real general solution, and sketch some trajectories in the phase plane (indicate directions of trajectories).

$\text{Re } \lambda = -1 < 0$ Spiral in

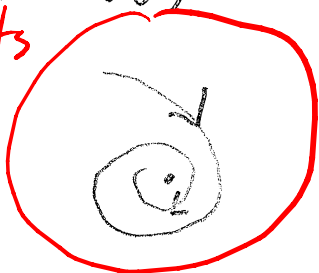
Complex solⁿ $\left[\begin{pmatrix} 5 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right] (e^{-t} \cos 2t + i e^{-t} \sin 2t)$

$$= \underbrace{\left[\begin{pmatrix} 5 \\ -1 \end{pmatrix} e^{-t} \cos 2t - \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^{-t} \sin 2t \right]}_{\vec{x}_1} + i \underbrace{\left[\begin{pmatrix} 5 \\ -1 \end{pmatrix} e^{-t} \sin 2t + \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^{-t} \cos 2t \right]}_{\vec{x}_2}$$

\vec{x}' at $\begin{pmatrix} 1 \\ 0 \end{pmatrix} =$ First col of $A = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ CW

3 pts
CW
in



type

Spiral in
5

stability

Asymptotically Stable
5

real general solution

$$c_1 e^{-t} \begin{pmatrix} 5 \cos 2t \\ -\cos 2t - 2 \sin 2t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 5 \sin 2t \\ -\sin 2t + 2 \cos 2t \end{pmatrix}$$

5 pts

5. (10) For the equation $y'' + y - y^2/2 = 0$, convert to a corresponding system
 $dx_1/dt = f_1(x_1, x_2)$ $dx_2/dt = f_2(x_1, x_2)$.
 (Do not attempt to solve the system.)

$$\begin{aligned} x_1 &= y \\ x_2 &= y' \\ x_1' &= y' = x_2 \\ x_2' &= y'' = -y + y^2/2 \\ &= -x_1 + \frac{1}{2}x_2^2 \end{aligned}$$

system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 + \frac{1}{2}x_2^2 \end{aligned}$$

10 pts

6. (10) Given that $(-2, 2)$ is a critical point of the non-linear system
 $dx_1/dt = -x_1 + x_2 + x_1x_2$ $dx_2/dt = -x_1 - x_2$,
 find the type and stability of the system at $(-2, 2)$.

Linearized system: $\vec{x}' = A\vec{x}$ where

$$A = J_{(-2, 2)} = \begin{bmatrix} -1+x_2 & 1+x_1 \\ -1 & -1 \end{bmatrix} \Big|_{(-2, 2)} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

5 pts

e-vals: $(1-\lambda)(-1-\lambda) - 1 = \lambda^2 - 2 = 0$ $\lambda = \pm\sqrt{2}$

type and stability

Saddle point
 Unstable

5 pts