 Name Key

1. Define

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 & 3 \\ 1 & 1 & 0 & 5 & 6 \\ 1 & 1 & 0 & 8 & 9 \end{bmatrix}.$$

(5) (a) Use row operations to find an echelon form of A.

$$\sim \begin{bmatrix} 1 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 6 & 6 \end{bmatrix} \sim$$

$$\boxed{\begin{bmatrix} 1 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

(5) (b) What is the dimension of the column space of A? dim= 2

(10) (c) Find a basis for the space of solutions (the null space) of the system

$$\begin{bmatrix} 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -4 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

x_2 x_4 free

$\boxed{x_2 = 1, x_4 = 0}$

$x_3 = -4 \cdot 0 = 0$

$x_1 = -1 + 2 \cdot 0 + 3 \cdot 0$

$$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$\boxed{x_2 = 0, x_4 = 1}$

$x_3 = 4 \cdot 1 = 4$

$x_1 = -1 \cdot 0 + 2 \cdot 4 + 3 \cdot 1 = 11$

$$\begin{pmatrix} 11 \\ 0 \\ 4 \\ 1 \end{pmatrix}$$

basis vectors $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 11 \\ 0 \\ 4 \\ 1 \end{pmatrix}$

2. (20) The matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ has 2 eigenvalues λ_1 and λ_2. Find λ_1 and λ_2 and basis vectors for the corresponding eigenspaces.

$\det(A - \lambda I) = (1-\lambda)(2-\lambda)^3 \quad \lambda = 1, 2, 2, 2$

For $\lambda = 1$: $\begin{bmatrix} 0 & 2 & 3 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ x_1 free, $x_2 = x_3 = x_4 = 0$ $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

For $\lambda = 2$: $\begin{bmatrix} -1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ x_2, x_4 free, $x_3 = 0$

$\boxed{x_2 = 1,\ x_4 = 0}$

$x_3 = 0$

$x_1 = 2 \cdot 1 + 4 \cdot 0 = 2$

$\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\boxed{x_2 = 0,\ x_4 = 1}$

$x_3 = 0$

$x_1 = 2 \cdot 0 + 4 \cdot 1 = 4$

$\begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\lambda_1 = 1$

basis for eigenspace = $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\lambda_2 = 2$

basis for eigenspace = $\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

3. (20) Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Find an orthogonal matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix} = -\lambda \det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = -\lambda(\lambda^2 - 1)$$

$\lambda = 0, \pm 1$

For $\lambda = 0$: $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = 0$, $x_3 = 0$, x_2 free

Get $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. unit length ✓

For $\lambda = 1$: $\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x_2 = 0$, x_3 free $\boxed{x_3 = 1}$ $x_1 = 1$

Get $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Normalized: $\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$

For $\lambda = -1$: $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x_2 = 0$, x_3 free $\boxed{x_3 = 1}$ $x_1 = -1$, $x_2 = 0$

Get $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

Normalized: $\begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$

$$P = \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

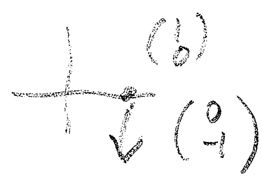
4. (20) The matrix $A = \begin{bmatrix} 0 & 5 \\ -1 & -2 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1+2i$ with eigenvector $\begin{pmatrix} 5 \\ -1+2i \end{pmatrix}$ and $\lambda_2 = -1-2i$ with eigenvector $\begin{pmatrix} 5 \\ -1-2i \end{pmatrix}$. Determine the type and stability of the origin for the system $dy/dt = Ay$. Find a <u>real</u> general solution, and sketch some trajectories in the phase plane (indicate directions of trajectories).

$\operatorname{Re}\lambda = -1 < 0$ Spiral in

Complex sol'n $\left[\begin{pmatrix} 5 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 2 \end{pmatrix}\right]\left(e^{-t}\cos 2t + i e^{-t}\sin 2t\right)$

$$= \Big[\underbrace{\begin{pmatrix} 5 \\ -1 \end{pmatrix} e^{-t}\cos 2t - \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^{-t}\sin 2t}_{\vec{x}_1}\Big] + i\Big[\underbrace{\begin{pmatrix} 5 \\ -1 \end{pmatrix} e^{-t}\sin 2t + \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^{-t}\cos 2t}_{\vec{x}_2}\Big]$$

$\vec{x}\,'$ at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ = First col of $A = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$



CW

type: Spiral in

stability: Asymptotically Stable

real general solution

$$c_1 e^{-t}\begin{pmatrix} 5\cos 2t \\ -\cos 2t - 2\sin 2t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 5\sin 2t \\ -\sin 2t + 2\cos 2t \end{pmatrix}$$

5. (10) For the equation $y'' + y - y^2/2 = 0$, convert to a corresponding system
$$dx_1/dt = f_1(x_1, x_2) \qquad dx_2/dt = f_2(x_1, x_2).$$
(Do not attempt to solve the system.)

$x_1 = y$ $\quad$ $x_1' = y' = x_2$

$x_2 = y'$ $\quad$ $x_2' = y'' =$

$-y + y^2/2$

$= -x_1 + \frac{1}{2}x_2^2$

system

$x_1' = x_2$

$x_2' = -x_1 + \frac{1}{2}x_2^2$

6. (10) Given that $(-2, 2)$ is a critical point of the non-linear system
$$dx_1/dt = -x_1 + x_2 + x_1x_2 \qquad dx_2/dt = -x_1 - x_2,$$
find the type and stability of the system at $(-2, 2)$.

Linearized system: $\vec{x}' = A\vec{x}$ where

$$A = J_{(-2,2)} = \left.\begin{bmatrix} -1 + x_2 & 1 + x_1 \\ -1 & -1 \end{bmatrix}\right|_{(-2,2)} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

e-vals: $(1-\lambda)(-1-\lambda) - 1 = \lambda^2 - 2 = 0$ $\quad \boxed{\lambda = \pm\sqrt{2}}$

type and stability

Saddle point

Unstable