1. Define

$$A = \begin{bmatrix} 1 & 2 & 0 & 8 & 9 \\ 1 & 2 & 0 & 5 & 6 \\ 1 & 2 & 0 & 2 & 3 \end{bmatrix}.$$

(5) (a) Use row operations to find an echelon form of A.

$$\begin{pmatrix} 1 & 2 & 0 & 8 & 9 \\ 0 & 0 & 0 & -3 & -3 \\ 0 & 6 & 0 & -6 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 8 & 9 \\ 0 & 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 1 & 2 & 0 & 8 & 9 \\ 0 & 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}$$

(5) (b) What is the dimension of the column space of A?

dim= 2

(10) (c) Find a basis for the space of solutions (the null space) of the system

$$\begin{bmatrix} 1 & 1 & 2 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

x_2, x_4 free

$x_4 = 1 \quad x_2 = 0$

$\Rightarrow x_3 = -4 \quad x_1 = 8+3 = 11$

$x_4 = 0 \quad x_2 = 1$

$\Rightarrow x_3 = 0 \quad x_1 = -1$

basis vectors

$$\begin{pmatrix} 11 \\ 0 \\ -4 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

2. (20) The matrix $A = \begin{bmatrix} 2 & 2 & 3 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ has 2 eigenvalues λ_1 and λ_2. Find λ_1 and λ_2 and basis vectors for the corresponding eigenspaces.

$\lambda_1 = 1$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

x_2, x_4 free

$x_2 = 0 \quad x_4 = 1$

$\Rightarrow x_3 = 0 \quad x_1 = -4$

$x_2 = 1 \quad x_4 = 0$

$\Rightarrow x_3 = 0, \; x_1 = -2$

$\lambda_2 = 2$

$$\begin{pmatrix} 0 & 2 & 3 & 4 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

x_1 free

$\Rightarrow x_4 = 0, \; x_3 = 0, \; x_2 = 0$

$\lambda_1 = 1$

basis for eigenspace = $\begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda_2 = 2$

basis for eigenspace = $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

3. (20) Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Find an orthogonal matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

$$p(\lambda) = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - 0) + 1(0 + \lambda)$$

$$= -\lambda^3 + \lambda = \lambda(1 - \lambda^2)$$

$\lambda = 0$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda = 1$

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda = -1$

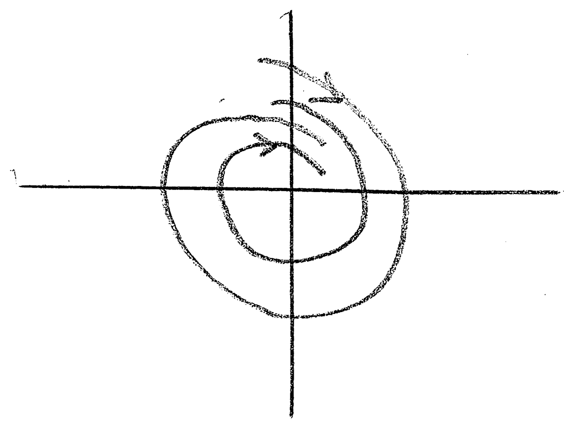
$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{P = \begin{pmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}$$

$$\boxed{D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}$$

4. (20) The matrix $A = \begin{bmatrix} 0 & 5 \\ -1 & -2 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1+2i$ with eigenvector $\begin{pmatrix} -1-2i \\ 1 \end{pmatrix}$ and $\lambda_2 = -1-2i$ with eigenvector $\begin{pmatrix} -1+2i \\ 1 \end{pmatrix}$. Determine the type and stability of the origin for the system $dy/dt = Ay$. Find a <u>real</u> general solution, and sketch some trajectories in the phase plane (indicate directions of trajectories).



At $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\frac{dy}{dt} = \begin{pmatrix} 0 & 5 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

type: Spiral

stability: stable and attractive

real general solution

$$c_1 e^{-t}\left(\begin{pmatrix} -1 \\ 1 \end{pmatrix}\cos 2t + \begin{pmatrix} 2 \\ 0 \end{pmatrix}\sin 2t\right) + c_2 e^{-t}\left(\begin{pmatrix} -1 \\ 1 \end{pmatrix}\sin 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix}\cos 2t\right)$$

5. (10) For the equation $y'' - y + y^2/2 = 0$, convert to a corresponding system
$$dx_1/dt = f_1(x_1, x_2) \qquad dx_2/dt = f_2(x_1, x_2).$$
(Do not attempt to solve the system.)

$x_1 = y$

$x_2 = y'$

system

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = x_1 - x_1^2/2$$

6. (10) Given that $(-2, 2)$ is a critical point of the non-linear system
$$dx_1/dt = -x_1 + x_2 + x_1 x_2 \qquad dx_2/dt = -x_1 - x_2,$$
find the type and stability of the system at $(-2, 2)$.

$$\frac{d\tilde{x}_1}{dt} = -(x_1 - 2) + (\tilde{x}_2 + 2) + (\tilde{x}_1 - 2)(\tilde{x}_2 + 2)$$

$$\frac{d\tilde{x}_2}{dt} = -(\tilde{x}_1 - 2) - (\tilde{x}_2 + 2)$$

linearized system

$$\frac{d\tilde{x}_1}{dt} = \tilde{x}_1 - \tilde{x}_2$$

$$\frac{d\tilde{x}_2}{dt} = -\tilde{x}_1 - \tilde{x}_2$$

$$p(\lambda) = \begin{vmatrix} 1-\lambda & -1 \\ -1 & -1-\lambda \end{vmatrix}$$

$$= (1-\lambda)(1+\lambda) - 1$$

$$= -2 + \lambda^2$$

$$\lambda = \pm\sqrt{2}$$

type and stability

saddle point

unstable