Solutions to MA 527 Exam 1 (1A Green, EPE)

1. a) $\begin{bmatrix} 1 & 1 & 0 & 2 & 3 \\ 1 & 1 & 0 & 5 & 6 \\ 1 & 1 & 0 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 6 & 6 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \end{matrix}$ $\leftarrow$ Done ! or...

$\sim \begin{bmatrix} 1 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

1. b) Dim Column Space = Dim Row Space

= Rank (A) = $\underline{\underline{2}}$

1.c)

$$\begin{array}{ccccc} & & & & \\ \left[\begin{array}{cccc|c} 1 & 1 & -2 & -3 & 0 \\ 0 & 0 & 1 & -4 & 0 \end{array}\right] & \leftarrow \text{row echelon!} \\ \end{array}$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ x_1 & x_2 & x_3 & x_4 \\ \text{bound} & \text{free} & \text{bound} & \text{free} \end{array}$$

Let $x_2 = t_1$
$x_4 = t_2$

E2: $x_3 - 4(\underbrace{t_2}_{x_4}) = 0$ $\quad \boxed{x_3 = 4t_2}$

E1: $x_1 + (\underbrace{t_1}_{x_2}) - 2(\underbrace{4t_2}_{x_3}) - 3(\underbrace{t_2}_{x_4}) = 0$

$x_1 = -t_1 + 11t_2$

List: $\begin{cases} x_1 = -t_1 + 11t_2 \\ x_2 = t_1 \\ x_3 = 4t_2 \\ x_4 = t_2 \end{cases}$

$$\vec{x} = t_1 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 11 \\ 0 \\ 4 \\ 1 \end{pmatrix}$$

Basis vectors for null space $\mathbb{B} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 11 \\ 0 \\ 4 \\ 1 \end{pmatrix} \right\}$

Another way: Let free vars $\begin{cases} x_2 = 1 \\ x_4 = 0 \end{cases}$. Get $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Let free vars $\begin{cases} x_2 = 0 \\ x_4 = 1 \end{cases}$. Get $\begin{pmatrix} 11 \\ 0 \\ 4 \\ 1 \end{pmatrix}$

2. $\det(\mathbb{A} - \lambda \mathbb{I}) = \det \begin{bmatrix} 1-\lambda & 2 & 3 & 4 \\ 0 & 2-\lambda & 1 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix}$

$=$ product of diagonal entries (upper Δ)

$= (1-\lambda)(2-\lambda)^3 = 0$

e-vals: $\lambda = 1, 2$

<u>For $\lambda=1$:</u> $(\mathbb{A} - 1 \cdot \mathbb{I})\vec{a} = \vec{0}$

$$\left[\begin{array}{cccc|c} 0 & 2 & 3 & 4 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_1 free

x_2, x_3, x_4 bound

Let $x_1 = t$.

$x_2 = 0$ E1

$x_3 = 0$ E2

$x_4 = 0$ E3

$\vec{x} = t\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Basis for e-space for $\lambda=1$ is $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}$

<u>For $\lambda=2$:</u> $(\mathbb{A} - 2 \cdot \mathbb{I})\vec{a} = \vec{0}$

$$\left[\begin{array}{cccc|c} -1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{array}{c} \left[\begin{array}{cccc|c} 1 & -2 & -3 & -4 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ x_1 \text{ bound} & x_2 \text{ free} & x_3 \text{ bound} & x_4 \text{ free} \end{array} \end{array}$$

Let $x_2 = t_1$

$x_4 = t_2$

E2: $x_3 = 0$

E1: $x_1 - 2\overset{x_2}{(t_1)} - 3\overset{x_3}{(0)} - 4\overset{x_4}{(t_2)} = 0$

$x_1 = 2t_1 + 4t_2$

List
$$\begin{cases} x_1 = 2t_1 + 4t_2 \\ x_2 = t_1 \\ x_3 = 0 \\ x_4 = t_2 \end{cases}$$

$$\vec{x} = t_1 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Basis for e-space for $\lambda=2$: $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

3. $\det(A-\lambda I) = \det \begin{bmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix}$ ← expand along row 2

$$= (+1)(-\lambda)\det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = -\lambda(\lambda^2-1)$$

$$= -\lambda(\lambda-1)(\lambda+1) = 0$$

e-vals : $\lambda = 0, \pm 1$

For $\lambda=1$: $\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

x_1, x_3 bnd ↑ x_3 free

Let $x_3 = t$. E2: $x_2 = 0$

E1: $x_1 - (\underbrace{t}_{x_3}) = 0$ $x_1 = t$

List: $\begin{cases} x_1 = t \\ x_2 = 0 \\ x_3 = t \end{cases}$ $\vec{x} = t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $\vec{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is e-vect.

Length $= \sqrt{1^2+0^2+1^2} = \sqrt{2}$

Normalize e-vect: $\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$

For $\lambda = -1$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Get $\vec{a} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Normalized $\vec{a}$: $\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$

For $\lambda = 0$:

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_1, x_3 bnd.

x_2 free

Let $x_2 = t$.

E2: $x_3 = 0$

E1: $x_1 = 0$

List $\begin{cases} x_1 = 0 \\ x_2 = t \\ x_3 = 0 \end{cases}$ $\quad \vec{x} = t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

e-vect $\vec{a} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Has length 1!

No need to normalize.

$$P = \begin{bmatrix} \text{normalized} \\ \text{e-vects in cols} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} \text{lined up} \\ \text{e-vals down} \\ \text{diag.} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

4. $\lambda = -1 \pm 2i$

$\operatorname{Re}\lambda < 0$ Spiral in., Asmp. Stable

or attractive.

One Complex sol'n $= \begin{pmatrix} 5 \\ -1+2i \end{pmatrix} e^{(-1+2i)t} =$

$$= \left[\begin{pmatrix} 5 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right] e^{-t} e^{2it}$$

$$= \left[\begin{pmatrix} 5 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right] \left(e^{-t} \cos 2t + i e^{-t} \sin 2t \right)$$

$$= \left[\underbrace{\begin{pmatrix} 5 \\ -1 \end{pmatrix} e^{-t} \cos 2t \underset{\uparrow \, i^2 = -1}{-} \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^{-t} \sin 2t}_{\text{one real sol}^n \ \vec{x}_1} \right] +$$

$$+\ i\left[\underbrace{\begin{pmatrix}5\\-1\end{pmatrix}e^{-t}\sin 2t+\begin{pmatrix}0\\2\end{pmatrix}e^{-t}\cos 2t}_{\text{second real sol}^{n}\ \vec{x}_2}\right]$$

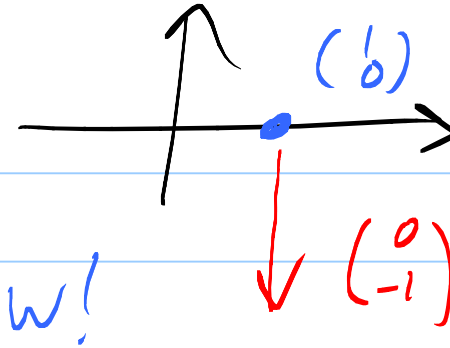
$$\text{Gen}^{l}\ \text{Sol}^{n} = c_1\vec{x}_1 + c_2\vec{x}_2$$

$$= c_1e^{-t}\left(\begin{pmatrix}5\\-1\end{pmatrix}\cos 2t-\begin{pmatrix}0\\2\end{pmatrix}\sin 2t\right)+c_2e^{-t}\left(\begin{pmatrix}5\\-1\end{pmatrix}\sin 2t+\begin{pmatrix}0\\2\end{pmatrix}\cos 2t\right)$$

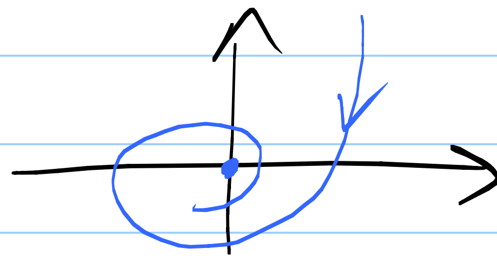
$$= c_1e^{-t}\begin{pmatrix}5\cos 2t\\ -\cos 2t-2\sin 2t\end{pmatrix}+c_2e^{-t}\begin{pmatrix}5\sin 2t\\ -\sin 2t+2\cos 2t\end{pmatrix}$$

CW or CCW? Test field at $\begin{pmatrix}1\\0\end{pmatrix}$:

$\vec{x}\,'$ at $\begin{pmatrix}1\\0\end{pmatrix} = A\vec{x} = \begin{pmatrix}0\\-1\end{pmatrix}$



CW!



5. $\begin{cases} x_1 = y \\ x_2 = y' \end{cases}$

ODE: $y'' + y - \frac{1}{2}y^2 = 0$

System: $x_1' = y' = x_2$

$x_2' = y'' = -y + \frac{1}{2}y^2$ ← from ODE

$$= -x_1 + \tfrac{1}{2}x_1^2$$

Answer: $\begin{cases} \dfrac{dx_1}{dt} = x_2 \\[2ex] \dfrac{dx_2}{dt} = -x_1 + \frac{1}{2}x_1^2 \end{cases}$

6. Linearized system: $\vec{x}\,' = \mathbb{A}\vec{x}$

where $\mathbb{A} = J\big|_{(-2,2)} = \begin{bmatrix} -1+x_2 & 1+x_1 \\ -1 & -1 \end{bmatrix}\Bigg|_{(-2,2)}$

$$= \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

e-vals $\det \begin{bmatrix} 1-\lambda & -1 \\ -1 & -1-\lambda \end{bmatrix} = \lambda^2 - 1 - 1 = \lambda^2 - 2 = 0$

$\lambda^2 = 2$

$\boxed{\lambda = \pm\sqrt{2}}$

Saddle Point

Unstable