MA 527 EXAM 2 A Key



1. (15) Find the Laplace transform $F(s)$ of $f(t) = e^{3t}u(t-2)$.

$$= e^{3[(t-2)+2]}u(t-2)$$

$$= e^{6}\underbrace{e^{3(t-2)}}_{g(t-2)}u(t-2)$$

$$g(t) = e^{3t}, \quad G(s) = \frac{1}{s-3}$$

$F(s) =$ $e^{6}e^{-2s} \cdot \dfrac{1}{s-3}$

2. (15) Find the inverse Laplace transform $f(t)$ of $F(s) = \dfrac{se^{-s}}{(s+2)^2+9}$.

$$= \frac{[(s+2)-2]}{(s+2)^2+3^2}e^{-s}$$

$$= \underbrace{\left(\frac{(s+2)}{(s+2)^2+3^2} - \frac{2}{3}\frac{3}{(s+2)^2+3^2}\right)}_{G(s)}e^{-s}$$

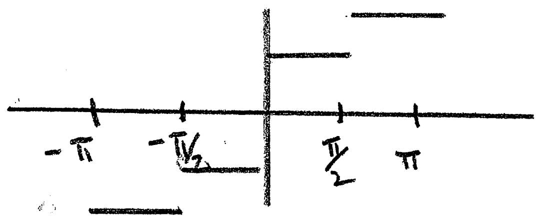
$$g(t) = e^{-2t}\cos 3t - \frac{2}{3}e^{-2t}\sin 3t$$

$u(t-1)g(t-1) =$

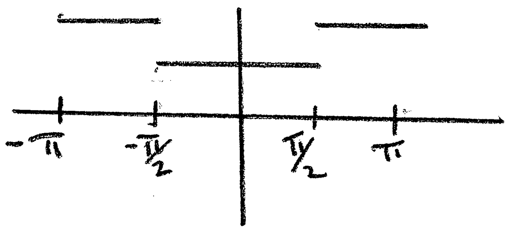
$f(t) =$ $u(t-1)\left[e^{-2(t-1)}\left(\cos 3(t-1) - \frac{2}{3}\sin 3(t-1)\right)\right]$

3. For the function $f(x)$ defined for $0 < x < \pi$ by $f(x) = 1$ if $0 < x < \pi/2$, and $f(x) = 2$ if $\pi/2 < x < \pi$,

a) (5) sketch the graph of the odd extension of $f$ for $-\pi < x < \pi$,




b) (5) sketch the graph of the even extension of $f$ for $-\pi < x < \pi$,




c) (20) compute the first two coefficients $b_1$ and $b_2$ of the sine series for $f$ on $(0, \pi)$.

$$b_1 = \frac{2}{\pi}\int_0^{\pi/2} \sin x\,dx + \frac{4}{\pi}\int_{\pi/2}^{\pi} \sin x\,dx = \frac{2}{\pi}\left(-\cos x\Big|_0^{\pi/2} - 2\cos x\Big|_{\pi/2}^{\pi}\right)$$

$$= \frac{2}{\pi}(0+1+2-0) = 6/\pi$$

$$b_2 = \frac{2}{\pi}\int_0^{\pi/2} \sin 2x\,dx + \frac{4}{\pi}\int_{\pi/2}^{\pi} \sin 2x\,dx = \frac{2}{\pi}\left(-\frac{\cos 2x}{2}\Big|_0^{\pi/2} - \frac{2\cos 2x}{2}\Big|_{\pi/2}^{\pi}\right)$$

$$= \frac{2}{\pi}\left(\frac{1+1}{2} - \frac{2+2}{2}\right) = -2/\pi$$

| | |
|---|---|
| $b_1 = 6/\pi$ | $b_2 = -2/\pi$ |

4. (10) Given that the Fourier series for $f(x) = x$ for $-\pi < x < \pi$ is

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx, \text{ compute the sum } \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}}{n}\right)^2 = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\| $$

$$\frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$\boxed{\pi^2/6}$$

$$\|$$

$$\frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\frac{\pi^3}{3} = \frac{2\pi^2}{3}, \quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2\pi^2}{4\cdot 3} = \frac{\pi^2}{6}$$

5. (5) Which of the following functions are eigenfunctions associated to the Sturm-Liouville problem

$$y''(x) + \lambda y(x) = 0 \qquad \text{with} \quad y'(0) = 0 \text{ and } y(\pi) = 0?$$

Here, $n$ denotes a positive integer.

A. $\sin nx$
B. $\cos nx$
C. $\sin(2n + \frac{1}{2})x$
D. $\cos(2n + \frac{1}{2})x$
E. 1

$$y'(0) = -(2n+\tfrac{1}{2})\sin 0 = 0$$

$$y(\pi) = \cos(2n\pi + \tfrac{\pi}{2}) = \cos\tfrac{\pi}{2} = 0$$

6. (15) Let

$$f(x) = \begin{cases} 1 & 0 < x < 2\pi \\ 0 & \text{otherwise.} \end{cases}$$

Compute the Fourier cosine transform $F(w)$ of $f$.

Fourier cosine transform $F(w) =$

$$\boxed{\sqrt{\frac{2}{\pi}}\ \frac{\sin 2\pi w}{w}}$$

$$\sqrt{\frac{2}{\pi}} \int_0^{2\pi} \cos wx \, dx = \sqrt{\frac{2}{\pi}}\ \frac{1}{w}\left(\sin 2\pi w - \sin 0\right)$$

7. (10) Let $F(w)$ denote your answer from problem ~~5~~ 6. Evaluate the two constants

$$A = \int_0^\infty F(w) \cos \pi w \, dw \qquad \text{and} \qquad B = \int_0^\infty F(w) \cos 2\pi w \, dw.$$

$$A = \sqrt{\frac{\pi}{2}}\, f(\pi) = \sqrt{\frac{\pi}{2}} \cdot 1$$

$$B = \sqrt{\frac{\pi}{2}}\ \frac{1}{2}(0+1) = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

$$\boxed{A = \sqrt{\frac{\pi}{2}} \qquad B = \frac{1}{2}\sqrt{\frac{\pi}{2}}}$$