Exam 2 solutions (2 A Green Pro Ed version)

1. $f(t) = e^{3t}\,u(t-2) = e^{3[(t-2)+2]}\,u(t-2)$

$$= u(t-2)\,\underbrace{e^{6}e^{3(t-2)}}_{g(t-2)}$$

So $g(t) = e^{6}e^{3t}$

$$\mathcal{L}[f(t)] = \mathcal{L}[u(t-2)\,g(t-2)] = e^{-2s}G(s)$$

$$= e^{-2s}\cdot e^{6}\cdot\frac{1}{s-3}$$

or $\mathcal{L}[e^{3t}h(t)] = H(s-3) = \dfrac{e^{-2(s-3)}}{(s-3)}$ if $h(t) = u(t-2)$

since $\mathcal{L}[u(t-2)] = \dfrac{e^{-2s}}{s}$.

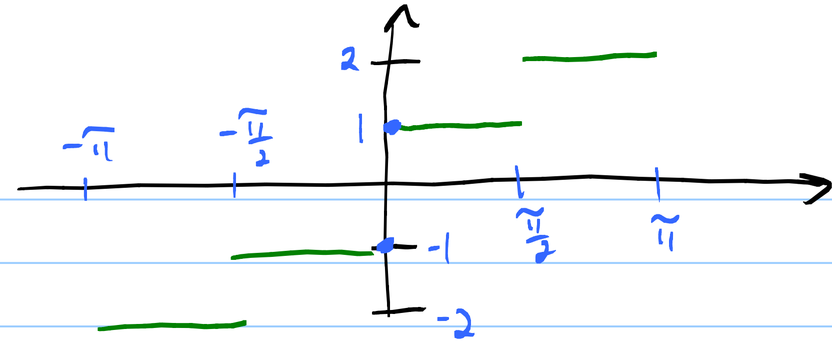
2. $F(s) = e^{-s}\,\dfrac{[(s+2)-2]}{(s+2)^2+3^2}$

$$= e^{-s}\left[\underbrace{\frac{(s+2)}{(s+2)^2+3^2}}_{\mathcal{L}[e^{-2t}\cos 3t]} - \frac{2}{3}\cdot\underbrace{\frac{3}{(s+2)^2+3^2}}_{\mathcal{L}[e^{-2t}\sin 3t]}\right]$$

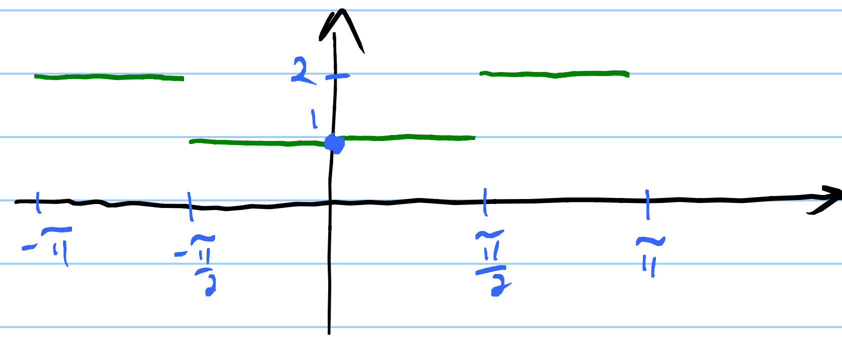
So $f(t) = u(t-1)\left[e^{-2(t-1)}\cos 3(t-1) - \frac{2}{3}e^{-2(t-1)}\sin 3(t-1)\right]$

3\.

a)



b)



c)

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} 1\cdot\sin nx\,dx + \int_{\pi/2}^{\pi} 2\cdot\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left(\left[-\frac{1}{n}\cos nx\right]_0^{\pi/2} + 2\left[-\frac{1}{n}\cos nx\right]_{\pi/2}^{\pi}\right)$$

$$b_1 = \frac{2}{\pi}\left(\left(\left(-\underbrace{\cos\frac{\pi}{2}}_{0}\right) - \left(-\underbrace{\cos 0}_{1}\right)\right) + 2\left(\left(-\underbrace{\cos\pi}_{-1}\right) - \left(-\underbrace{\cos\frac{\pi}{2}}_{0}\right)\right)\right)$$

$$= \frac{2}{\pi}(1+2) = \underline{\underline{6/\pi}}$$

$$b_2 = \frac{2}{\pi}\left[\left(\left(-\frac{1}{2}\underbrace{\cos\pi}_{-1}\right) - \left(-\frac{1}{2}\underbrace{\cos 0}_{1}\right)\right) + 2\left(\left(-\frac{1}{2}\underbrace{\cos 2\pi}_{1}\right) - \left(-\frac{1}{2}\underbrace{\cos\pi}_{-1}\right)\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{2} + \frac{1}{2} + 2\left(-\frac{1}{2} - \frac{1}{2}\right)\right] = \underline{\underline{-\frac{2}{\pi}}}$$

4. Use Parseval's Identity:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = 2a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{x}{2}\right)^2 dx = 2\cdot 0^2 + \sum_{n=1}^{\infty}\left(0^2 + \left[\frac{(-1)^{n+1}}{n}\right]^2\right)$$

$f(x) = \frac{x}{2}$ for EPE

$b_n = \frac{(-1)^{n+1}}{n}$ for EPE

$$\underbrace{\frac{1}{\pi}\left[\frac{1}{4\cdot 3}x^3\right]_{-\pi}^{\pi}} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{1}{\pi}\,\frac{1}{2\cdot 3}\,\pi^3 = \underline{\underline{\frac{\pi^2}{6}}}$$

5. All the given functions satisfy ODE for some λ, but only $\underline{\underline{D.}}$ $\cos(2n+\frac{1}{2})x$ satisfies the boundary conditions $y'(0)=0$ and $y(\pi)=0$.

$= -(2n+\frac{1}{2})\sin 0 = 0$ $\qquad = \cos(2n\pi + \frac{\pi}{2}) = \cos\frac{\pi}{2} = 0$

6. $$\mathcal{F}_c[f] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos wx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{2\pi} 1\cdot\cos wx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{1}{w}\cdot\sin wx\right]_{x=0}^{2\pi}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{w} \cdot \left(\sin w 2\pi - \underbrace{\sin w \cdot 0}_{\sin 0 = 0} \right) \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin 2\pi w}{w} = F(w)$$

7.

$$\tilde{\mathcal{F}}_c \left[\underbrace{\tilde{\mathcal{F}}_c [f]}_{F(w)} \right] = \begin{cases} (f \text{ where } f \text{ continuous.}) \\ (\text{midpoint of jumps of } f \text{ at jumps.}) \end{cases}$$

$$A = \sqrt{\frac{\pi}{2}} \, \tilde{\mathcal{F}}_c [F(w)](\pi) = \sqrt{\frac{\pi}{2}} \, \underbrace{f(\pi)}_{1} = \sqrt{\frac{\pi}{2}}$$

$$B = \sqrt{\frac{\pi}{2}} \, \tilde{\mathcal{F}}_c [F(w)](2\pi) = \sqrt{\frac{\pi}{2}} \left(\underbrace{\text{Mid point of jump}}_{= 1/2} \right)$$

$$= \frac{1}{2} \sqrt{\frac{\pi}{2}}$$