MA 527 Exam 2 B Key

1. (15) Find the Laplace transform $F(s)$ of $f(t) = e^{2t}u(t-3)$.

$$= e^{2[(t-3)+3]}\, u(t-3)$$

$$= e^{6}\, \underbrace{e^{2(t-3)}}_{g(t-3)}\, u(t-3)$$

$$g(t) = e^{2t}, \quad G(s) = \frac{1}{s-2}$$

$$\boxed{F(s) = e^{6}\, e^{-3s} \cdot \frac{1}{s-2}}$$

2. (15) Find the inverse Laplace transform $f(t)$ of $F(s) = \dfrac{se^{-s}}{(s+3)^2+4}$.

$$= \frac{[(s+3)-3]}{(s+3)^2+2^2}\, e^{-s}$$

$$= \underbrace{\left(\frac{(s+3)}{(s+3)^2+2^2} - \frac{3}{2}\, \frac{2}{(s+3)^2+2^2} \right)}_{G(s)} e^{-s}$$

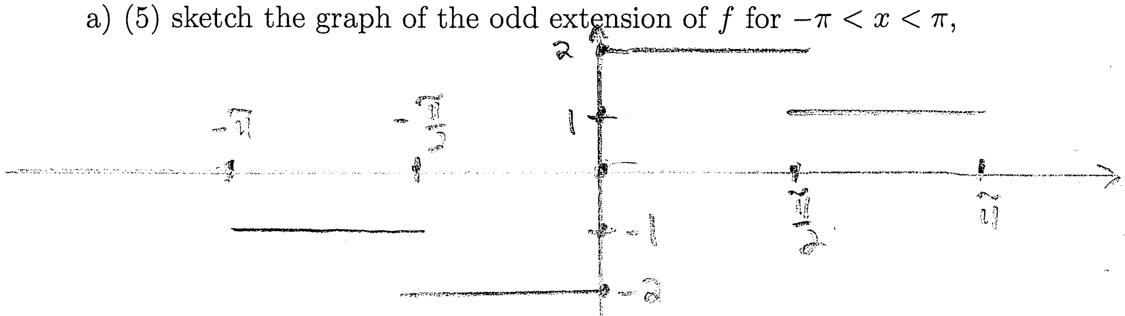
$$g(t) = e^{-3t}\cos 2t - \frac{3}{2} e^{-3t} \sin 2t$$

$$= u(t-1)\, g(t-1)$$

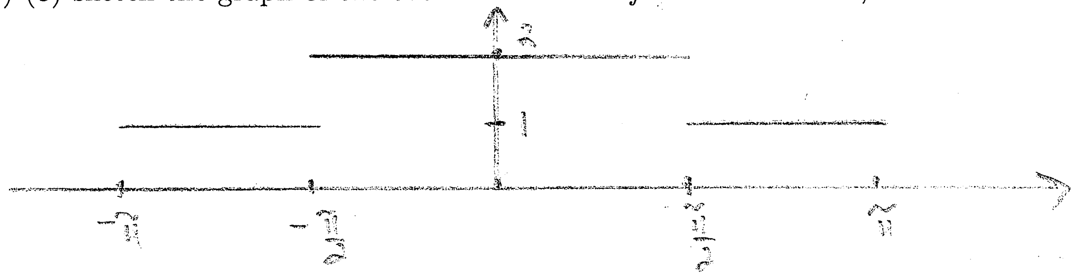
$$\boxed{f(t) = u(t-1)\left(e^{-3(t-1)} \cos 2(t-1) - \frac{3}{2} e^{-3(t-1)} \sin 2(t-1) \right)}$$

3. For the function $f(x)$ defined for $0 < x < \pi$ by $f(x) = 2$ if $0 < x < \pi/2$, and $f(x) = 1$ if $\pi/2 < x < \pi$,

a) (5) sketch the graph of the odd extension of f for $-\pi < x < \pi$,



b) (5) sketch the graph of the even extension of f for $-\pi < x < \pi$,



c) (20) compute the first two coefficients b_1 and b_2 of the sine series for f on $(0, \pi)$.

$$b_1 = \frac{2}{\pi}\left(\int_0^{\pi/2} 2\sin x\,dx + \int_{\pi/2}^{\pi} \sin x\,dx\right)$$

$$= \frac{2}{\pi}\left(-2\cos x\Big|_0^{\pi/2} - \cos x\Big|_{\pi/2}^{\pi}\right) = \frac{2}{\pi}\big((-0-(-2)) - (-1-0)\big) = \frac{6}{\pi}$$

$$b_2 = \frac{2}{\pi}\left(\int_0^{\pi/2} 2\sin 2x\,dx + \int_{\pi/2}^{\pi} \sin 2x\,dx\right)$$

$$= \frac{4}{\pi}\left[-\frac{1}{2}\cos 2x\right]_0^{\pi/2} + \left[-\frac{1}{2}\cos 2x\right]_{\pi/2}^{\pi}$$

$b_1 = \frac{6}{\pi}$ $\qquad$ $b_2 = \frac{2}{\pi}$

$$= \frac{4}{\pi}\left[-\frac{1}{2}(-1) - \left(-\frac{1}{2}\right)\right] + \frac{2}{\pi}\left[-\frac{1}{2}(1) - \left(-\frac{1}{2}(-1)\right)\right]$$

$$= \frac{4}{\pi} - \frac{2}{\pi} = \frac{2}{\pi}$$

4. (10) Given that the Fourier series for $f(x) = x$ for $-\pi < x < \pi$ is

$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$, compute the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}}{n}\right)^2 = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\| $$

$$\frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$\boxed{\pi^2/6}$$

$$\|$$

$$\frac{2}{\pi}\left[\frac{1}{3}x^3\right]_0^{\pi}$$

$$\|$$

$$\frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}\ ), \quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{4}\cdot\frac{2\pi^2}{3} = \frac{\pi^2}{6}$$

5. (5) Which of the following functions are eigenfunctions associated to the Sturm-Liouville problem

$$y''(x) + \lambda y(x) = 0 \qquad \text{with} \qquad y(0) = 0 \text{ and } y'(\pi) = 0?$$

Here, n denotes a positive integer.

A. $\sin nx$
B. $\cos nx$
C. $\sin(2n + \frac{1}{2})x$
D. $\cos(2n + \frac{1}{2})x$
E. 1

$y(0) = \sin 0 = 0$

$y'(\pi) = (2n+\frac{1}{2})\cdot\cos(2n\pi + \frac{\pi}{2}) = 0$

6. (15) Let

$$f(x) = \begin{cases} 1 & 0 < x < 3 \\ 0 & \text{otherwise.} \end{cases}$$

Compute the Fourier cosine transform $F(w)$ of f.

Fourier cosine transform $F(w) =$

$$\boxed{\sqrt{\frac{2}{\pi}}\ \frac{\sin 3w}{w}}$$

$$\sqrt{\frac{2}{\pi}} \int_0^3 1 \cdot \cos wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{w} \sin wx \right]_0^3$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\sin 3w}{w} - 0 \right)$$

7. (10) Let $F(w)$ denote your answer from problem ~~5~~ 6. Evaluate the two constants

$$a = \int_0^\infty F(w) \cos 2w \, dw \qquad \text{and} \qquad b = \int_0^\infty F(w) \cos 3w \, dw.$$

$$\sqrt{\frac{2}{\pi}}\, a = f(2) = 1$$

$$\sqrt{\frac{2}{\pi}}\, b = \frac{1}{2}(0+1) = \frac{1}{2}$$

$a = \sqrt{\frac{\pi}{2}}$ $\qquad$ $b = \frac{1}{2}\sqrt{\frac{\pi}{2}}$