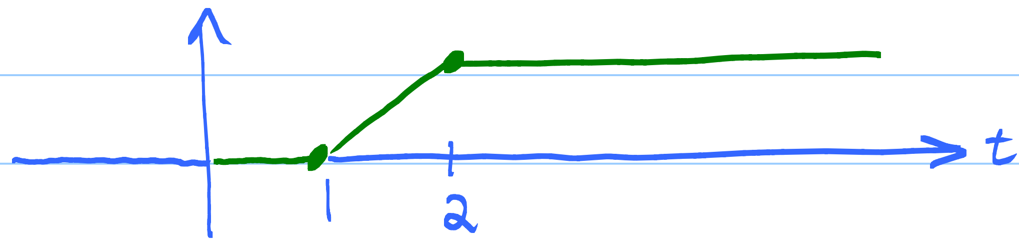
Exam 2 solutions

1. $2(s^2 Y - s\cdot 5 - 7) + 3(sY - 5) + Y = e^{-2s}$

$$Y = \frac{1}{2s^2+3s+1}\cdot\left(10s + 29 + e^{-2s}\right)$$

2.

a)



$$f(t) = [u(t-1) - u(t-2)](t-1) + [u(t-2)]\cdot 1$$

b) $t^2 u(t-3) = [(t-3)+3]^2 u(t-3)$

$$= u(t-3)(t-3)^2 + u(t-3)6(t-3) + 9u(t-3)$$

$$\mathcal{L}[t^2 u(t-3)] = e^{-3s}\mathcal{L}[t^2] + e^{-3s}\mathcal{L}[6t] + 9\frac{e^{-3s}}{s}$$

$$= e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$$

3. a) $\frac{3-s}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$

$$3 - s = A(s+1) + B(s+2) = \underbrace{(A+B)}_{-1}s + \underbrace{(A+2B)}_{3}$$

$A + B = -1$

$A + 2B = 3$

$B = 4, \quad A = -1 - B = -5$

$$\mathcal{L}^{-1}\left[\frac{3-s}{(s+2)(s+1)}\right] = \mathcal{L}^{-1}\left[\frac{-5}{s+2}+\frac{4}{s+1}\right] = \underline{-5e^{-2t}+4e^{-t}}$$

b) $$\mathcal{L}^{-1}\left[e^{-5s}\underbrace{\frac{1}{(s+1)^2}}_{F(s),\ f(t)=e^{-t}t}\right] = u(t-5)f(t-5) = \underline{u(t-5)e^{-(t-5)}(t-5)}$$

c) $$\frac{3-s}{(s+1)^2+4} = \frac{3-[(s+1)-1]}{(s+1)^2+2^2} = \frac{4}{(s+1)^2+2^2} - \frac{(s+1)}{(s+1)^2-2^2}$$

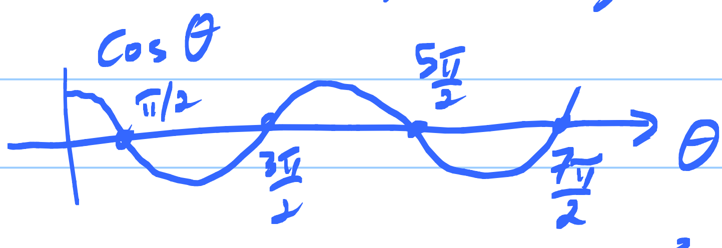
$$\mathcal{L}^{-1} = \underline{2e^{-t}\sin 2t - e^{-t}\cos 2t}$$

4. Let $\lambda = \mu^2$.

$$y = c_1\cos\mu x + c_2\sin\mu x$$
$$y' = -\mu c_1\sin\mu x + \mu c_2\cos\mu x$$

Want $y'(0) = -\mu c_1\cdot 0 + \mu c_2\cdot 1 \overset{\text{want}}{=} 0$ So $\underline{c_2 = 0}$.

So $y = c_1\cos\mu x$. Also need $y(\pi) = c_1\cos\mu\pi = 0$.
Don't want $c_1 = 0$ too, so need $\underline{\cos\mu\pi = 0}$.



Need $\mu\pi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$\mu = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

e-vals $\lambda = \mu^2 = \frac{1^2}{4}, \frac{3^2}{4}, \frac{5^2}{4}, \frac{7^2}{4}, \ldots$

or $\underline{\lambda = \left(\frac{1}{2}+n\right)^2,\ n = 0, 1, 2, \ldots}$

5.

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_{\pi/2}^{\pi} 1 \cdot \sin nx \, dx = \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\cos n\pi + \frac{1}{n}\cos\frac{n\pi}{2}\right]$$

Fourier Sine Series is $\sum_{n=1}^{\infty} b_n \sin nx$

6. $\langle 1, x^7 \rangle = \int_{-\pi}^{\pi} \underbrace{1 \cdot x^7}_{\text{odd}} \, dx = 0$

$$\langle 1, \cos x \rangle = \int_{-\pi}^{\pi} \cos x \, dx = \sin x \Big|_{-\pi}^{\pi} = 0$$

$$\langle x^7, \cos x \rangle = \int_{-\pi}^{\pi} \underbrace{\underset{\text{odd}}{x^7} \; \underset{\text{even}}{\cos x}}_{\text{odd}} \, dx = 0$$

If $f(x) = c_1 \cdot 1 + c_2 \cdot x^7 + c_3 \cos x$, then

$$f(x)x^7 = c_1 1 \cdot x^7 + c_2 x^7 \cdot x^7 + c_3 x^7 \cos x$$

Integrate: $\int_{-\pi}^{\pi} f(x)\, x^7 \, dx = 0 + c_2 \underbrace{\int_{-\pi}^{\pi} x^{14} dx}_{\frac{1}{15}x^{15}\Big|_{-\pi}^{\pi}} + 0$

$$c_2 = \frac{1}{\left(2 \cdot \frac{\pi^{15}}{15}\right)} \int_{-\pi}^{\pi} f(x) x^7 \, dx$$

7. a) $\tilde{\mathcal{F}}_s[f] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx$

$= \sqrt{\frac{2}{\pi}} \int_1^3 1 \cdot \sin wx \, dx = \sqrt{\frac{2}{\pi}} \left[-\frac{1}{w} \cos wx \right]_{x=1}^{3}$

$= \underline{\sqrt{\frac{2}{\pi}} \left[-\frac{1}{w} \cos 3w + \frac{1}{w} \cos w \right]} = F(w)$

b) $g(x) = \tilde{\mathcal{F}}_s[F]$ "=" $f(x)$.

$g\underline{(2) = f(2) = 1}$ because f is continuous there.

But $\underline{g(3) = \frac{1}{2}}$, the midpoint of the jump of f at $x=3$.