Exam 1 solutions

1.

$$\left[\begin{array}{ccc|c} 1 & 5 & 3 & 2 \\ 2 & 4 & 5 & 4 \\ -1 & 1 & -2 & k \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 5 & 3 & 2 \\ 0 & -6 & -1 & 0 \\ 0 & 6 & 1 & k+2 \end{array}\right] \begin{array}{l} \\ R2-2R1 \\ R3+R1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 5 & 3 & 2 \\ 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & k+2 \end{array}\right]$$

x_3 free   danger!

a) $k \neq -2$ no solutions

b) $k = -2$ ∞ many sol^{n}s

c) Never has a uniq. sol^n

2.

$$\det(A) = (-1)\cdot 1 \cdot \det\begin{bmatrix} 0 & 0 & 1 \\ 1 & 4 & 2 \\ a & -3 & 3 \end{bmatrix} \leftarrow \text{expand along row 1}$$

$$= -\left((+1)\cdot 1 \cdot \det\begin{bmatrix} 1 & 4 \\ a & -3 \end{bmatrix}\right) = -(-3-4a)$$

$= 4a+3$. $A\vec{x} = \vec{0}$ has non-zero solutions exactly when $\det(A) = 0$, i.e., when $a = -3/4$.

3. a.

$$A \sim \begin{bmatrix} 1 & 1 & -1 & -3 \\ 0 & 1 & 1 & -4 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{array}{l} \\ \\ R3-2R1 \end{array} \sim$$

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R1-R2 \\ \\ R3-R2 \end{array}$$

! Reduced row echelon form

Rank$(A) = 2$

Dim row space $= 2$

$\parallel$

Dim col space $= 2$

3.b. $\left[\begin{array}{cccc|c} 1 & -2 & 0 & -3 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]\begin{array}{l} E1 \\ E2 \\ \end{array}$ x_1, x_3 bound

x_2 free $\qquad$ x_4 free

Let $x_2 = t_1$, $x_4 = t_2$.

Solve E1 for bound $x_1 = 2x_2 + 3x_4 = 2t_1 + 3t_2$

Solve E2 for bound $x_3 = 5x_4 = 5t_2$

Make list: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2t_1 + 3t_2 \\ t_1 \\ 5t_2 \\ t_2 \end{pmatrix} = t_1\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_2\begin{pmatrix} 3 \\ 0 \\ 5 \\ 1 \end{pmatrix}$

$\text{Null}(\mathbb{B}) = \text{span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 5 \\ 1 \end{pmatrix} \right\}$

Basis vectors

4. e-vals: $\det\begin{bmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{bmatrix} = \lambda^2 - 1 = 0$ $\qquad \underline{\underline{\lambda = \pm 1}}$

<u>For $\lambda = 1$</u>: $(\mathbb{A} - \lambda \mathbb{I})\vec{a} = \vec{0}$ $\qquad$ $\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \rightsquigarrow$

$\lambda = 1$

$\rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$. e-vect $\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Normalize to have length one: $\vec{a} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

<u>For $\lambda = -1$</u>: $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\vec{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Unit length e-vect: $\vec{a} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

$$\mathbb{P} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \qquad \mathbb{D} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

e-vect. for $\lambda = -1$

e-vect for $\lambda = 1$

$\underbrace{\mathbb{P}^{-1} = \mathbb{P}^{T}}_{\text{orthog.}} = \mathbb{P}$ in this case!

5. $\lambda = \underbrace{-1}_{Re\,\lambda < 0} + i$ Spiral in, Asymptotically Stable

Complex sol^n: $\begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1+i)t} =$

$$= \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \left(e^{-t} \cos t + i e^{-t} \sin t\right)$$

$$= \Bigg[\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} \cos t \overset{i^2}{-} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} \sin t}_{\text{real sol}^n\ \vec{x}_1}\Bigg] + i \Bigg[\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} \sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} \cos t}_{\text{real sol}^n\ \vec{x}_2}\Bigg]$$

Gen^l Sol^n: $\vec{x} = c_1 e^{-t} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

6. Jacobian matrix $\mathbb{J} = \begin{bmatrix} \cos x & 1 \\ 4 & \cos y \end{bmatrix}$

Linearized system at $(0,0)$: $\vec{x}\,' = \mathbb{A}\vec{x}$ where

$$\mathbb{A} = J\Big|_{(0,0)} = \begin{bmatrix} \cos 0 & 1 \\ 4 & \cos 0 \end{bmatrix} = \underline{\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}}$$

e-vals: $\det \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 4 = 0$

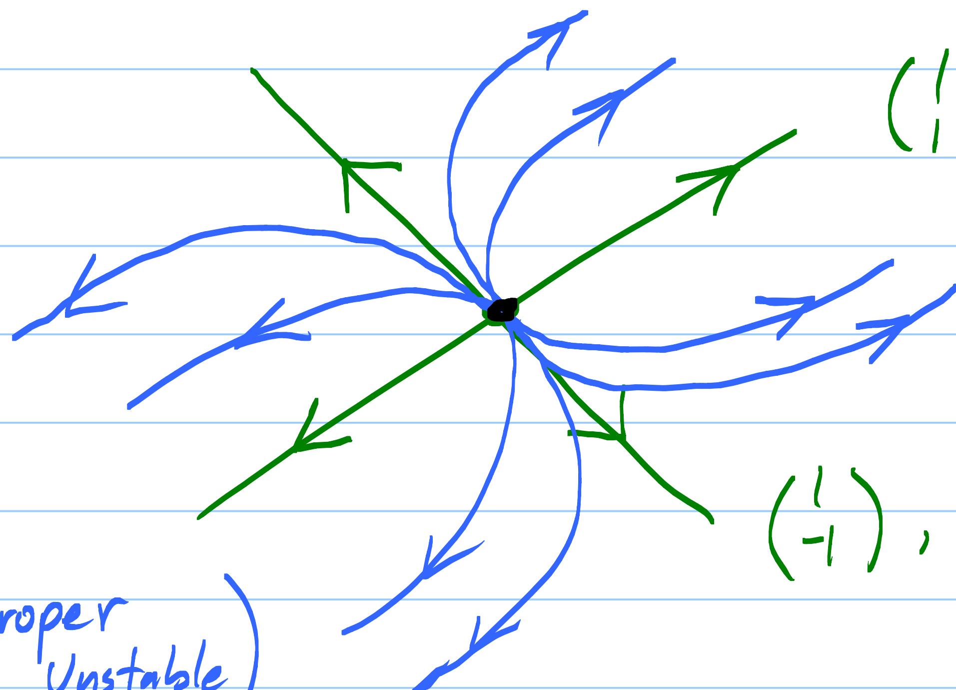
$$(1-\lambda)^2 = 4$$

$$1-\lambda = \pm 2$$

$$\underline{\lambda = -1, 3}$$

$\lambda = -1, 3$: <u>Saddle Point</u>, <u>Unstable</u>

7.



$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda = 3$ out

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\lambda = 2$ out

closest to zero

(Improper Node, Unstable)

<u>Node rule</u>: Near the origin, trajectories hug the eigenvector corresponding to the eigenvalue closest to zero. Far from the origin, the trajectories appear to flow parallel to the other eigenvector.