

MA 527

Practice problems for Exam 2

Chapter 6: Laplace transforms. Definition, behavior under differentiation and integration, Laplace transforms of step and delta functions, t-shift, s-shift, convolution, periodic functions, and solving ODEs and systems of ODEs using Laplace transforms.

Chapter 5: Sturm-Liouville and orthogonal expansion. Definition of Sturm-Liouville problem, finding eigenvalues and eigenfunctions, orthogonality (and weight functions), series expansion in terms of eigenfunctions.

Chapter 11: Fourier series and transforms. Integral formula for Fourier coefficients, representation of even and odd functions, real and complex Fourier series, solving ODEs, approximation with trig polynomials, sine, cosine, and Fourier transforms and properties.

Sample problems. The problems below are similar to problems that you might find on the exam. You will be allowed one standard page of handwritten notes (both sides). A Laplace Transform table will be provided. The exam will be closed book and notes, no calculator or other computational aids.

1. a) Find the inverse Laplace transform of

$$\frac{s+1}{s^2+4s+3}e^{-2s}.$$

- b) Find the inverse Laplace transform of

$$\frac{2s-1}{s^2+4s+13}e^{-s}.$$

2. a) Write the function that is equal to $\sin t$ from $t = 0$ to $t = \pi$ and equal to $(t - \pi)$ for $t > \pi$ in terms of step functions.

- b) Find the Laplace transform of $t^2u(t-2)$.

3. Solve $y'' + 4y' + 5y = \delta(t-1)$, $y(0) = 1$, $y'(0) = 3$, where $\delta(t)$ is the Dirac delta function.

4. Solve the following system using the Laplace transform:

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \vec{x}(0) = 0.$$

5. The function $r(t)$ is given by numerical data. Use the Laplace Transform to write a solution to the problem $y'' + 4y = r(t)$ with $y(0) = 0$ and $y'(0) = 0$ as an integral involving $r(t)$.

6. Find the eigenvalues and eigenfunctions for the Sturm-Liouville Problem

$$y'' + \lambda y = 0 \quad \text{with } y'(0) = 0 \text{ and } y'(3) = 0.$$

7. Let $f(x)$ be a 2π periodic function with

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$$

Find the Fourier series for $f(x)$ both in the real trigonometric form and in the complex form.

8. Let

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute the Complex Fourier Transform of f .

9. Suppose $f(x)$ is continuous and differentiable on the x -axis, and that $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$ and $f'(x)$ is absolutely integrable on the x -axis.

Prove that $\mathcal{F}\{f'(x)\} = iw\mathcal{F}\{f(x)\}$.

10. Given that the Fourier Sine Series for $x(\pi - x)$ on $0 < x < \pi$ is

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \sin nx,$$

compute

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6}.$$

11. Let

$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 2\pi \\ 0 & \text{otherwise} \end{cases}$$

Compute the Fourier Cosine Transform of f . Use the result to compute

$$\frac{2}{\pi} \int_0^{\infty} \frac{\sin(2\pi w) \cos(\pi w)}{w} dw.$$