Practice Problems for Exam 2 Sol^{n}s.

1. a) $$\frac{s+1}{s^2+4s+3} = \frac{s+1}{(s+3)(s+1)} = \frac{A}{s+3} + \frac{B}{s+1}$$

Oops! Dumb problem. s+1 cancels.

$$\frac{s+1}{s^2+4s+3}e^{-2s} = \frac{e^{-2s}}{s+3} = e^{-2s}\mathcal{L}\left[e^{-3t}\right]$$

So $\mathcal{L}^{-1} = u(t-2)e^{-3(t-2)}$.

b) $$\underbrace{\frac{2s-1}{s^2+4s+13}}_{H(s)} = \frac{2s-1}{(s+2)^2+9} = \frac{2\left[(s+2)-2\right]-1}{(s+2)^2+3^2}$$

$$= 2\,\frac{s+2}{(s+2)^2+3^2} - \frac{5}{3}\,\frac{3}{(s+2)^2+3^2}$$

$$= \mathcal{L}\Big[\underbrace{2e^{-2t}\cos 3t - \tfrac{5}{3}e^{-2t}\sin 3t}_{h(t)}\Big]$$

Now $\mathcal{L}^{-1}\left[H(s)e^{-s}\right] = u(t-1)\,h(t-1) =$

$$u(t-1)\left(2e^{-2(t-1)}\cos 3(t-1) - \tfrac{5}{3}e^{-2(t-1)}\sin 3(t-1)\right)$$

2. a) $[u(t)-u(t-\pi)]\sin t + u(t-\pi)\cdot(t-\pi)$

b) $t^2 u(t-2) = [(t-2)+2]^2\, u(t-2)$

$$= \underbrace{\left[(t-2)^2+4(t-2)+4\right]}_{f(t-2)} u(t-2)$$

So $f(t) = t^2+4t+4$.

$$\mathcal{L}[t^2 u(t-2)] = \mathcal{L}[f(t-2)u(t-2)] = e^{-2s}F(s)$$

$$= e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right).$$

3. $(s^2 Y - 1\cdot s - 3) + 4(sY-1) + 5Y = e^{-s}$

$$Y = \frac{1}{s^2+4s+5}\left(s+7+e^{-s}\right)$$

$$= \frac{s+7}{(s+2)^2+1} + \frac{1}{(s+2)^2+1}e^{-s}$$

$$= \frac{(s+2)}{(s+2)^2+1^2} + 5\,\frac{1}{(s+2)^2+1^2} + \frac{1}{(s+2)^2+1^2}e^{-s}$$

So $y = e^{-2t}\cos t + 5e^{-2t}\sin t + u(t-1)\,5e^{-2(t-1)}\sin(t-1)$.

4. $$\begin{cases} x_1' = x_1 + 3x_2 + 1 \\ x_2' = x_2 - 2 \end{cases} \qquad \begin{cases} x_1(0) = 0 \\ x_2(0) = 0 \end{cases}$$

$$\begin{cases} sX_1 - 0 = X_1 + 3X_2 + 1/s \\ sX_2 - 0 = X_2 - 2/s \end{cases} \leftarrow X_2 = \frac{-2}{s(s-1)}$$

$$X_2 = \frac{A}{s} + \frac{B}{s-1} \qquad -2 = A(s-1) + Bs = \underbrace{(A+B)}_{0}s + \underbrace{(-A)}_{-2}$$

$A = 2$

$B = -A = -2$

$$X_2 = \frac{2}{s} - \frac{2}{s-1} \qquad \boxed{x_2 = 2 - 2e^{-t}}$$

$$(s-1)X_1 = 3X_2 + 1/s$$

$$X_1 = \frac{1}{s-1}\left[3\left(\frac{-2}{s(s-1)}\right) + \frac{1}{s}\right]$$

$$= \frac{1}{s-1}\left[\frac{-6}{s(s-1)} + \frac{s-1}{s(s-1)}\right]$$

$$= \frac{s-7}{s(s-1)^2} = \frac{A}{s} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)}$$

$$s-7 = A(s-1)^2 + Bs + Cs(s-1)$$

$$= \underbrace{(A+C)}_{0}s^2 + \underbrace{(-2A+B-C)}_{1}s + \underbrace{A}_{-7}$$

$A = -7$, $C = -A = 7$ $B = 1 + 2A + C = -6$

$$X_1 = \frac{-7}{s} - \frac{6}{(s-1)^2} + \frac{7}{s-1}$$

$$\boxed{x_1 = -7 - 6te^t + 7e^t}$$

5. $s^2 Y + 4Y = R(s)$

$$Y = \underbrace{\frac{1}{s^2+4}}_{F(s)} R(s)$$

$y = f * r$ where $f = \frac{1}{2} \sin 2t$.

So $y(t) = \int_0^t r(\tau) \frac{1}{2} \sin 2(t - \tau)\, d\tau$

6. See class notes from Lesson 35, 11/15/2010.

Got $\lambda = 0$, $y_0 = 1$.

$\lambda = \left(\frac{n\pi}{3}\right)^2$, $y_n = \cos \frac{n\pi}{3} x$ $n = 1, 2, 3, \ldots$

<u>For $\lambda < 0$ case:</u> write $\lambda = -\mu^2$. $r = \pm\mu$.

$$y = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

$$y' = c_1 \mu e^{\mu x} - c_2 \mu e^{-\mu x}$$

$y'(0) = c_1\mu - c_2\mu \stackrel{\text{want}}{=} 0.$ So $c_2 = c_1$ and

$$y'(3) = c_1\mu e^{\mu 3} - c_1\mu e^{-\mu 3} = 2c_1\mu \cdot \frac{e^{3\mu} - e^{-3\mu}}{2}$$

$$= 2c_1 \underset{\neq 0}{\mu} \underbrace{\sinh 3\mu}_{\text{not zero}} \quad \text{So } c_1 = 0. \quad c_2 = c_1 = 0.$$

Only get the zero sol^n. No negative e. vals.

7. $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{\pi} 1\,dx = \frac{1}{2}$

$= c_0$ too.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\sin nx\right]_0^{\pi} = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi} = \frac{1}{\pi n}(1 - \cos n\pi) = \frac{1-(-1)^n}{n\pi}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_0^{\pi} e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left[-\frac{1}{in}e^{-inx}\right]_0^{\pi} = \frac{1}{2\pi i n}\left(1 - e^{-in\pi}\right)$$

But $e^{-in\pi} = \underbrace{\cos(-n\pi)}_{=\cos(n\pi)=(-1)^n} + i\underbrace{\sin(-n\pi)}_{=0} = (-1)^n.$

So $c_n = \dfrac{1-(-1)^n}{2\pi i n}.$

Fourier Series: $\frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$

Complex Fourier Series: $\frac{1}{2} + \frac{1}{\pi i}\sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{e^{inx}}{n}.$

#11. See lecture notes for Lesson 35, 11/15/2010