Course Evaluation at www.purdue.edu/eval

Final Exam: Wed., May 4, 3:20–5:20pm in LWSN B151

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

$$\sin \pi z = \pi z \prod_{n \neq 0}^{\infty} \left[ \left(1 - \frac{z}{n}\right) e^{z/n} \right] = \pi \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

<u>Infinite Products</u>: If $\prod_{n=1}^{\infty} a_n = L \neq 0$,

then $\lim_{n \to \infty} a_n = 1$.

<u>Def<sup>n</sup></u>: A product $\prod_{n=1}^{\infty} (1 + a_n)$

is said to converge <u>absolutely</u> if

$\sum_{n=1}^{\infty} |a_n| < \infty$.

<u>EX</u>: $\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$ converges absolutely

to an entire function.

<u>Theorem</u>: Suppose $u$ is a <u>continuous</u> real valued function on a domain $\Omega$ such that $u_x, u_{xx}, u_y, u_{yy}$ <u>exist</u> and $u_{xx} + u_{yy} \equiv 0$ on $\Omega$. Then $u$ is <u>harmonic</u> on $\Omega$.

<u>Pf</u>: <u>May</u> assume $\Omega = D_1(0)$ and $u$ is continuous on $\overline{D_1(0)}$ and $\Delta u \equiv 0$ in $D_1(0)$.

Define $\tilde{u}(z) = \int_0^{\infty} P(z,\theta)\, u(e^{i\theta})\, d\theta$. Want to see $u - \tilde{u} \equiv 0$. Suppose not. Then $\exists\, z_0 \in D_1(0)$ where $u(z_0) - \tilde{u}(z_0) \neq 0$. May assume $> 0$ (or switch $u$ and $\tilde{u}$). Say $u(z_0) - \tilde{u}(z_0) = K > 0$. Pick $\varepsilon$ with $0 < \varepsilon < K$. Let

$$V_\varepsilon = u - \tilde{u} + \varepsilon |z|^2$$

Note:
$$\begin{cases} \dfrac{\partial^2}{\partial x^2}(x^2+y^2) = 2 \\[2mm] \dfrac{\partial^2}{\partial y^2}(x^2+y^2) = 2 \end{cases} \qquad \text{So } \Delta |z|^2 = 4,$$

and $\Delta V_\varepsilon = 4\varepsilon > 0$.

Notice that $V_\varepsilon(e^{i\theta}) = \varepsilon < K$

$$V_\varepsilon(z_0) = \underbrace{u(z_0) - \tilde{u}(z_0)}_{K} + \varepsilon|z_0|^2 \geq K.$$

**Next:** $V_\varepsilon$ is cont on compact $\overline{D_1(0)}$, so it assumes a max, say at $w_0$. We have shown that $w_0 \in D_1(0)$ (not on $bD_1(0)$).

**Freshman Calculus Fact:** If $h: (a,b) \to \mathbb{R}$ is twice diff'ble and $h''(t_0) > 0$, $h'(t_0) = 0$, then $t_0$ is local minimum of $h$, strict min.

Hence $\frac{\partial^2}{\partial x^2} V_\varepsilon(w_0) \leq 0$, and

$$\frac{\partial^2}{\partial y^2} V_\varepsilon(w_0) \leq 0.$$

So $\Delta V_\varepsilon(w_0) \leq 0$. But $\Delta V_\varepsilon = 4\varepsilon > 0$. $\lightning$.

Hence $u \equiv \tilde{u}$, and $u$ is harmonic. ✓

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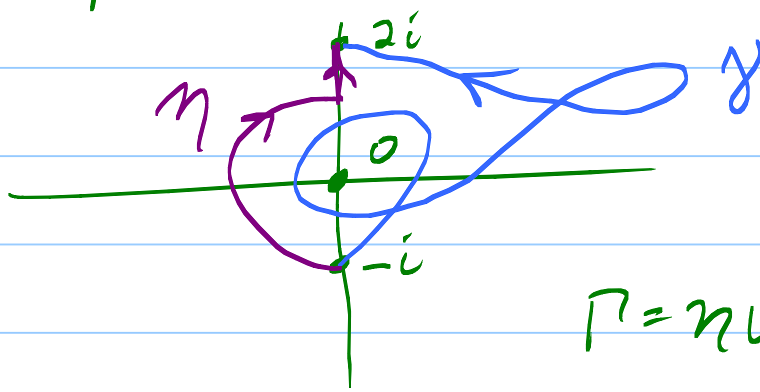
3. What are the possible values of

$$\int_\gamma \frac{1}{z}\, dz$$



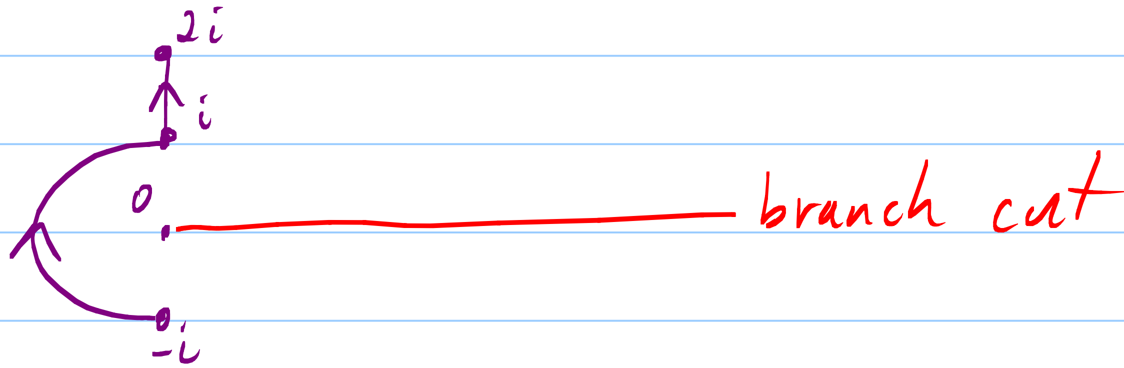

$\Gamma = \eta \cup (-\gamma)$

$\Omega = \mathbb{C}$

Gen$^{l}$ Residue: $\displaystyle\int_{\eta \cup (-\gamma)} \frac{1}{z}\, dz$

$$= 2\pi i \underbrace{\left(\mathrm{Ind}_0 \Gamma\right)}_{N} \underbrace{\mathrm{Res}_0 \frac{1}{z}}_{1} = 2\pi i N$$

So $\displaystyle\int_\eta \frac{1}{z}\, dz - \int_\gamma \frac{1}{z}\, dz = 2\pi i N$




$\log z = \mathrm{Ln}|z| + i\theta$, $\quad \theta \in \arg z \quad 0 < \theta < 2\pi$

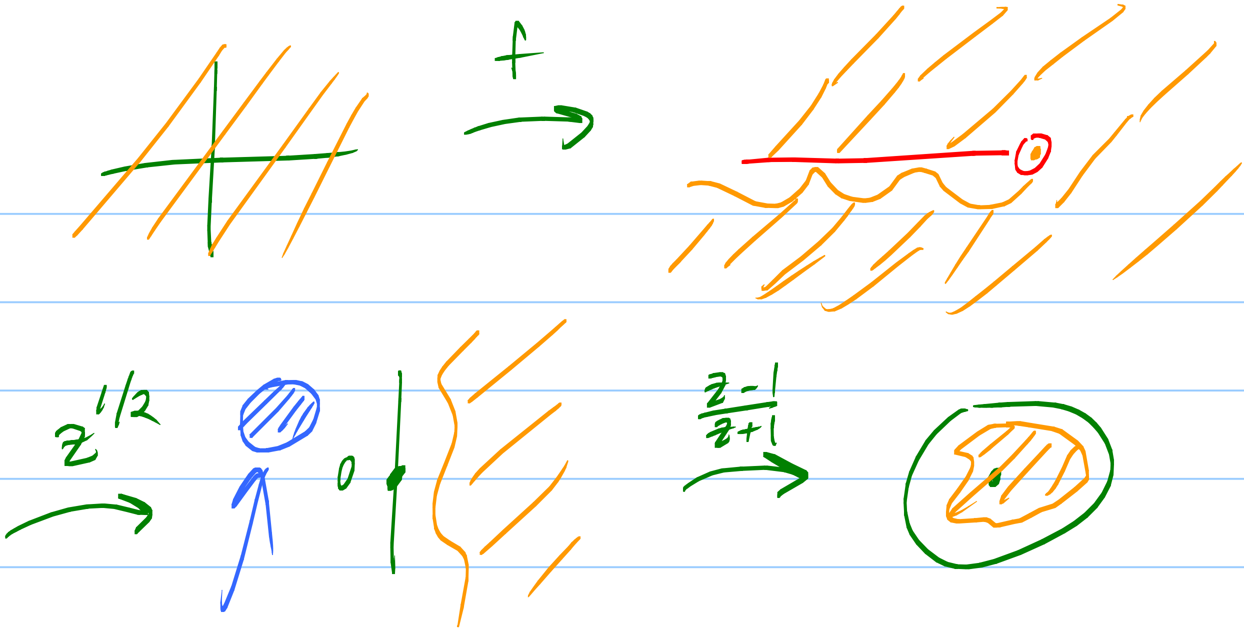
$$\int_\eta \frac{1}{z}\, dz = \int_\eta \frac{d}{dz}(\log z)\, dz = \log 2i - \log(-i)$$

$$= \left(\mathrm{Ln}|2i| + i\frac{\pi}{2}\right) - \left(\underbrace{\mathrm{Ln}|-i|}_{0} + i\frac{3\pi}{2}\right)$$

$$= \mathrm{Ln}\, 2 - i\pi$$

See $\displaystyle\int_\gamma \frac{1}{z}\, dz = \mathrm{Ln}\, 2 + i\pi\,(\text{odd integer})$

4.




misses open set.
So const.

8. $f$ meromorphic on simp. conn $\Omega$.

zeroes in $\Omega$: even multiplicity
poles in $\Omega$: even order.

Show that $f$ has a meromorphic square root on $\Omega$. Want $g$ with $g^2 = f$.

Think: $g = e^{\frac{1}{2}\log f} = \exp\left(\frac{1}{2}\int_{\gamma_a^z} \frac{f'}{f}\, dw\right)$

"Define" $g$ like this where $\gamma_a^z$ is a path in $\Omega$ that starts at a fixed $a \in \Omega$ and goes to $z$, missing zeroes and poles of $f$.

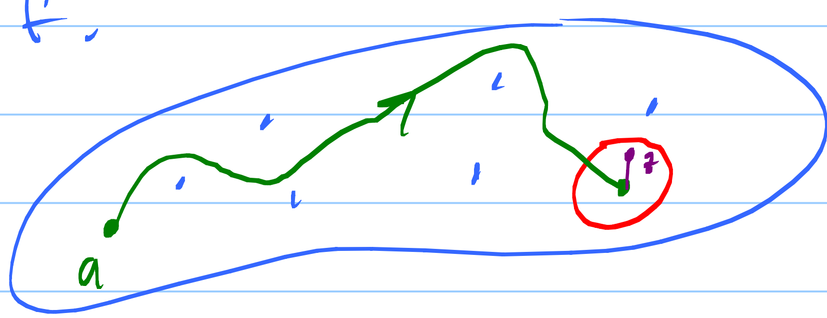
**Step 1**: $g$ is well defined.

$$\frac{\exp\left(\frac{1}{2}\int_{\gamma_a^z} \cdots\right)}{\exp\left(\frac{1}{2}\int_{\tilde{\gamma}_a^z} \cdots\right)} = \exp\left(\frac{1}{2}\int_{\underbrace{\gamma_a^z \cup (-\tilde{\gamma}_a^z)}_{\eta}} \frac{f'}{f}\,dw\right)$$

$$\exp\left(\frac{1}{2}\,2\pi i \sum_{\substack{\text{zeroes}\\ \text{and}\\ \text{poles } b\\ \text{of } f}} \underbrace{\mathrm{Ind}_\eta(b)}_{m}\ \underbrace{\mathrm{Res}_b \frac{f'}{f}}_{\pm 2n}\right)$$

$$= \exp\left(\pi i\ (\text{even integer})\right) = 1 \ \checkmark$$

**Step 2**: $g$ is analytic away from zeroes, poles of $f$.



**Step 3**: See

$$g' = \underbrace{\exp\left(\frac{1}{2}\int_{\gamma_a^z} \frac{f'}{f}\,dw\right)}_{g} \frac{1}{2}\frac{f'}{f}$$

$$g' = \frac{1}{2}\, g\, \frac{f'}{f}.$$

Step 4: $\frac{d}{dz}\left(\frac{g^2}{f}\right) \equiv 0$.

Correct by a constant to get $G$ with $G^2 = f$.

Last Step: Show zeroes of $f$ are removable for $G$, and $G$ has zeroes there. And $G$ has poles at poles of $f$. So $G$ is meromorphic.

M, T, W 1-2 pm. Office Hrs.