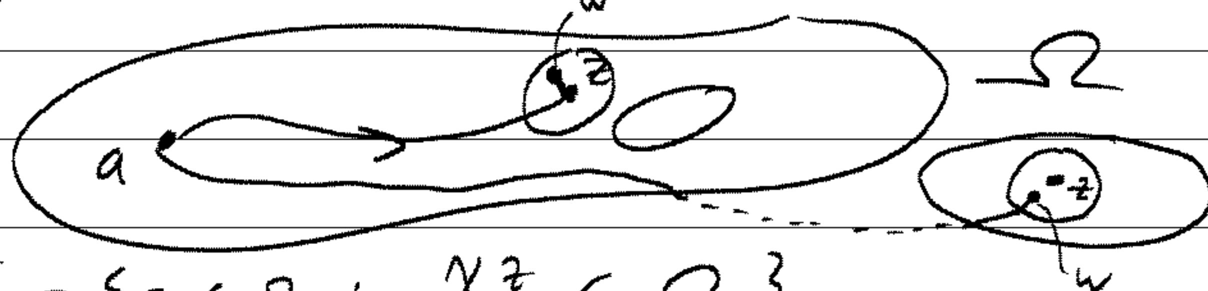
2/3/2010

Def^n: A domain is a connected open set in $\mathbb{C}$.
[domain = region]

Fact: Connected $\Longleftrightarrow$ Path Connected
for open sets in $\mathbb{C}$.

Why: Suppose Ω is connected. Pick $a \in \Omega$.



Let $U_1 = \{z \in \Omega : \gamma_a^z \subset \Omega\}$.

$\gamma_a^w = \gamma_a^z \cup L_z^w$. See U_1 open.

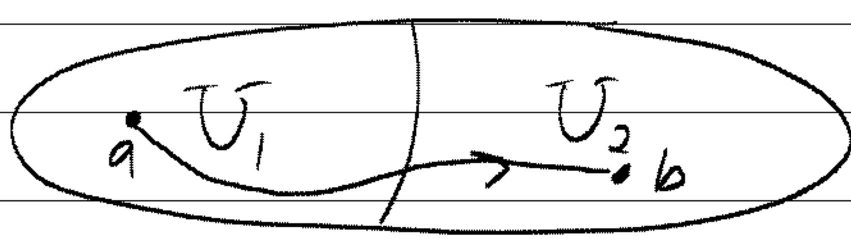
Let $U_2 = \Omega - U_1$. U_2 open too. $[\gamma_a^w \cup L_w^z]$.

Finally $a \in U_1 \Rightarrow \Omega = U_1$.

Other way: Suppose Ω path connected.

Assume $\Omega = U_1 \cup U_2$, two open sets, $U_1 \cap U_2 = \emptyset$.

Suppose $a \in U_1$.
$b \in U_2$.



$\gamma_a^b : z(t) : 0 \le t \le 1$. Let $t_M = \sup\{t : z(\tau) \in U_1,$
$0 \le \tau \le t\}$.

Where is $z(t_m)$? Easy to see $z(t_m) \notin U_1, \notin U_2$.

Baby fact: An analytic function f with $f' \equiv 0$ on a domain must be constant on Ω.

Path connected: $\gamma_a^z \subset \Omega$. $f(z) - f(a) = \int_{\gamma_a^z} f'(w)\,dw$

Exercise: A continuous function on a domain taking only integer values must be constant.

Last time: $\sum a_n (z-z_0)^n = a_N (z-z_0)^N + \cdots$

$= (z-z_0)^N \left[a_N + a_{N+1}(z-z_0) + \cdots\right]$

$= (z-z_0)^N H(z)$, $H(z_0) \neq 0$.

Theorem: If f is analytic on $D_R(z_0)$ and $f(z_0) = 0$, then either 1) $f \equiv 0$ on $D_R(z_0)$

2) the zero at z_0 is isolated.

Theorem: (Identity Theorem) Suppose f is analytic on a domain Ω. Let $Z_f = \{z \in \Omega : f(z) = 0\}$. Then either 1) $Z_f = \Omega$, or

2) Z_f has no limit points in Ω.

Proof: Suppose z_0 is a limit point of Z_f in Ω [meaning $\exists$ seq $\{z_n\}$ in Z_f with $z_n \to z_0$ as $n \to \infty$.] Note: continuity shows $z_0 \in Z_f$.

Let U_1 be equal to the set of all limit points of Z_f in Ω.

Step 1: Disc version of this theorem shows U_1 open.

[Why. $w_0 \in U_1$, $\exists$ $D_R(w_0) \subset \Omega$. Zeroes pile up at w_0. So it is not isolated. So $f \equiv 0$ on $D_R(z_0)$. So $D_R(w_0) \subset U_1$.

Step 2: Let $U_2 = \Omega - U_1$. Need to see U_2 open.

Suppose $w_0 \in U_2$.

Case 1: $f(w_0) \neq 0$. Then clear that w_0 is an interior point of U_2.

Case 2: $f(w_0) = 0$. w_0 not a limit point of Z_f. Hence w_0 is an isolated zero of f. So $\exists$ $D_\varepsilon(w_0) \subset \Omega$ such that w_0 is the only zero of f in $D_\varepsilon(w_0)$. So $D_\varepsilon(w_0) \subset U_2$. ✓

Step 3: $\Omega = U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$, open.

$z_0 \in U_1 \Rightarrow U_1 = \Omega$. [But $U_1 \subset Z_f$.]

So $Z_f = \Omega$.

We showed: If limit pt exists, $f \equiv 0$. Done.

Cor: If f and g are analytic on a domain Ω and agree on a set with a limit pt in the domain, then $f \equiv g$ on Ω.

Application: 1) only one way to extend e^x and trig functions to $\mathbb{C}$. $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

2) Any trig identity holds for <u>complex</u> angles.

$$\sin(\alpha+\beta) = \cos\alpha \sin\beta + \sin\alpha \cos\beta$$

z z z

$$\left[e^{i(\alpha+\beta)} = e^{i\alpha} e^{i\beta} \right]$$

Next, fix z, put w in place of α.

[every point of $\mathbb{R}$ is a limit pt of $\mathbb{R}$ in $\mathbb{C}$.]

<u>Homework hints:</u> $\Big(\underbrace{\sum a_n z^n}_{f}\Big)\Big(\underbrace{\sum b_n z^n}_{g}\Big) = \underbrace{\sum c_n z^n}_{h}$

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$$
$$= \sum_{k=0}^{n} a_k b_{n-k}$$

$$c_n = \left.\frac{\frac{d^n}{dz^n} h(z)}{n!}\right|_{z=0} = \left.\frac{\left(\text{Leibnitz Formula for } \frac{d^n}{dz^n}[fg]\right)}{n!}\right|_{z=0}$$

Use Taylor's Formula. Get identity.

$|f| \equiv c \implies f$ const.

$f = u + iv$

$$u^2 + v^2 = c^2 \qquad (*)$$

<u>Case $c=0$</u> : Easy.

<u>Case $c \neq 0$</u> :

$$\begin{cases} \frac{\partial}{\partial x}(*) & 2uu_x + 2vv_x = 0 \\ \frac{\partial}{\partial y}(*) & 2u\underbrace{u_y}_{-v_x} + 2v\underbrace{v_y}_{u_x} = 0 \end{cases}$$

$$\begin{bmatrix} u & v \\ v & -u \end{bmatrix} \begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\det = \underbrace{-u^2 - v^2}_{-c} \neq 0. \qquad \text{So } \nabla u \equiv 0.$$

Think $\rho(u,v) \sim u^2 + v^2$.